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ON THE NONCENTRAL DISTRIBUTIONS OF THE SECOND LARGEST ROOTS OF THREE MATRICES IN MULTIVARIATE ANALYSIS[1]

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1. **Introduction and Summary.** The central distribution of the second largest (smallest) root following the Fisher-Girshick-Hsu-Roy distribution under certain null-hypothesis has been derived in series form by Pillai and Al-Ani [6]. In this paper the noncentral distributions of the second largest roots in the MANOVA situation, the canonical correlation, and equality of two covariance matrices are obtained. Further, the distribution of the second largest root of the covariance matrix is obtained as a limiting case. The largest root and its noncentral distributions have been considered already by Pillai and Sugiyama [7] for the situations stated above. However, in the present paper, the joint densities of the largest and the second largest roots are derived in all the above cases from which the distributions of the largest roots can be obtained, although in more elaborate forms.

2. **Noncentral distribution of the second largest root in the MANOVA case.** Let $\mathbf{X}$ be a $p\times n_1$ matrix variate ($p\leq n_1$) and $\mathbf{Y}$ a $p\times n_2$ matrix variate ($p\leq n_2$) and the columns be all independently normally distributed with covariance matrix $\mathbf{\Sigma}$, $E\mathbf{X}=\mathbf{M}$ and $E\mathbf{Y}=0$. Then it is known that $\mathbf{XX}'=\mathbf{S}_1$ is noncentral Wishart with n_1 degrees of freedom and $\mathbf{YY}'=\mathbf{S}_2$ is central Wishart with n_2 degrees of freedom and the covariance matrix $\mathbf{\Sigma}$, respectively.

Let $0<l_1<l_2<\cdots<l_p<1$ be the characteristic roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$; then the joint density function of $l_1,\ldots,l_p$ is given by Constantine [1],

$$c(p,n_1,n_2)\exp(\text{tr-}\mathbf{\Omega})|\mathbf{L}|^m|\mathbf{I}-\mathbf{L}|^n\prod_{i>j}(l_i-l_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{\Omega})c_\kappa(\mathbf{L})}{(n_{1/2})_\kappa c_\kappa(\mathbf{I})k!},\tag{2.1}$$

where $\mathbf{\Omega}$ is the noncentrality matrix, $\frac{1}{2}\mathbf{M}'\mathbf{\Sigma}^{-1}\mathbf{M}$, and $\mathbf{L}=\text{diag}(l_1,\ldots,l_p)$ and $c(p,n_1,n_2)=\pi^{p^2/2}\Gamma_p(\nu/2)/\Gamma_p(p/2)\Gamma_p(n_1/2)\Gamma_p(n_2/2)$, $m=\frac{1}{2}(n_1-p-1)$, $n=\frac{1}{2}(n_2-p-1)$, $\nu=\frac{1}{2}(n_1+n_2)$ and $c_\kappa(\mathbf{L})$ are zonal polynomials defined in [3]. Consider the transformation $q_i=l_i/l_{p-1}$, $i=1,\ldots,p-2$, and decompose $c_\kappa(\mathbf{L})=\sum_{\tau,\nu}a_{\tau,\nu}l_p^{k_1}c_\nu(\mathbf{L}_1)$ where $\mathbf{L}_1=\text{diag}(l_1,\ldots,l_{p-1})$ and the summation is over the partitions τ of k_1 and ν of k_2 such that $k_1+k_2=k$, and κ is the partition of k, and $a_{\tau,\nu}$ are constants defined

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in [2]. Then the joint distribution of $q_1, \ldots, q_{p-2}, l_{p-1}, l_p$ can be written in the form

$$q(l_{p-1}, l_p)|\mathbf{Q}|^m|\mathbf{I}-\mathbf{Q}|\ |\mathbf{I}-l_{p-1}\mathbf{Q}_1|^n|\mathbf{I}-(l_{p-1}\mid l_p)\mathbf{Q}_1| \tag{2.2}$$
$$\times \prod_{i>j}(q_i-q_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{\Omega})}{k!\,c_\kappa(\mathbf{I})(n_1/2)_\kappa}\sum_{\tau,\nu}a_{\tau,\nu}l_p^{k_1}l_{p-1}^{k_2}c_\nu(\mathbf{Q}_1),$$

where $\mathbf{Q}=\operatorname{diag}(q_1,\ldots,q_{p-2})$, $\mathbf{Q}_1=\operatorname{diag}(q_1,\ldots,q_{p-2},1)$, and $q(l_{p-1},l_p)=c(p,n_1,n_2)\exp \text{tr-}\mathbf{\Omega}\cdot l_{p-1}^{m(p-1)+1/2(p-2)(p+1)}l_p^{m+p-1}(1-l_p)^n$. By expanding $|\mathbf{I}-l_{p-1}\mathbf{Q}_1|^n$ as well as $|\mathbf{I}-(l_{p-1}\mid l_p)\mathbf{Q}_1|$ and using the results from Khatri and Pillai [5] for multiplication of zonal polynomials, we write (2.2) in the form

$$q(l_{p-1}, l_p)|\mathbf{Q}|^m|\mathbf{I}-\mathbf{Q}|\prod_{i>j}(q_i-q_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{\Omega})}{k!(n_1/2)_\kappa c_\kappa(\mathbf{I})}\sum_{s=0}^{\infty}\sum_{\eta}((-n)_\eta l_{p-1}^s/s!) \tag{2.3}$$
$$\times\sum_{l=0}^{p-2}(c(l)l_{p-1}^l/l!l_p^l)\sum_{\tau,\nu}a_{\tau,\nu}l_p^{k_1}l_{p-1}^{k_2}\sum_{\delta}g_{l\,(1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{Q}_1),$$

where η and δ are the partitions of s and $l+s+k_2$ respectively such that $\eta=(\eta_1,\ldots,\eta_p)$ and $\delta=(\delta_1,\ldots,\delta_p)$ where $s=\sum_{i=1}^p\eta_i$, $l+s+k_2=\sum_{i=1}^p\delta_i$, $g_{l\,(1,\eta,\nu)}^{\delta}$ are constants defined in [5] and $c(l)=(-1)^l(2l)!/[(l!)^2 2^l\chi_{[21^l]}(1)]$, where $\chi_{[21^l]}$ is the degree of the representation $[21^l]$ of the symmetric group on $2l$ symbols, and such that $\chi_{[\kappa]}(1)=k!\prod_{i<j}^p(k_i-k_j-i+j)/\prod_{i=1}^p(k_i+p-i)!$ and $\kappa=(k_1\ge k_2\ge\cdots\ge k_p\ge 0)$. Now integrate (2.3) with respect to $0\le q_1\le q_2\le\cdots\le q_{p-2}<1$ by the use of the lemma in [8], we get the joint density function of l_{p-1}, l_p in the form

$$(\Gamma_{p-1}((p-1)/2)/\pi^{(p-1)^2/2})\Gamma_{p-1}(p/2)q(l_{p-1},l_p)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{\Omega})}{k!(n_1/2)c_\kappa(\mathbf{I})}$$
$$\times\sum_{s=0}^{\infty}\sum_{\eta}\{(-n)_\eta/s!\}\sum_{l=0}^{p-2}\{c(l)/l!l_p^l\}\sum_{\tau,\nu}a_{\tau,\nu}l_p^{k_1}l_{p-1}^{s+l+k_2}\sum_{\delta}g_{l\,(1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{I}_{p-1}) \tag{2.4}$$
$$\times((n_1-1)(p-1)/2+s+l+k_2)(\Gamma_{p-1}((n_1-1)/2,\delta)/\Gamma_{p-1}((n_1+p-1)/2,\delta)).$$

Further, integrate (2.4) with respect to l_p, then the density function of l_{p-1} can be written

$$c_1(p,n_1,n_2)\exp(\text{tr-}\mathbf{\Omega})l_{p-1}^{m(p-1)+(p-2)(p+1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{\Omega})}{k!(n_1/2)_\kappa c_\kappa(\mathbf{I}_p)}$$
$$\times\sum_{s=0}^{\infty}\sum_{\eta}\{(-n)_\eta/s!\}\sum_{l=0}^{p-2}(c(l)/l!)\sum_{\tau,\nu}a_{\tau,\nu}l_{p-1}^{s+l+k_2}B(l_{p-1},1;m+p+k_1-l;n+1) \tag{2.5}$$
$$\times\sum_{\delta}g_{l\,(1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{I}_{p-1})((n_1-1)(p-1)/2+s+l+k_2)$$
$$\times(\Gamma_{p-1}((n_1-1)/2,\delta)/\Gamma_{p-1}((n_1+p-1)/2,\delta))$$

where

$$B(a,b;c,d) = \int_a^b x^{c-1}(1-x)^{d-1}dx;$$
$$c_1(p,n_1,n_2) = \pi^{p-1}\Gamma_p(\nu/2)\Gamma_{p-1}((p-1)/2)/\Gamma_p(n_1/2)\Gamma_p(n_2/2).$$

It may be pointed out that the density function of the largest root can be obtained from (2.4) by integrating it with respect to l_{p-1} over the range $0<l_{p-1}<l_p$, however a simpler form has been given in [7].

Let $\mathbf{\Omega}=\mathbf{0}$ in (2.5) then the central case is of the form

$$\begin{aligned} &c_1(p, n_1, n_2)l_{p-1}^{m(p-1)+(p-2)(p+1)/2} \sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta/s!\} \sum_{l=0}^{p-2} \{c(l)/l!\} l_{p-1}^{s+l} \\ (2.6) \qquad &\times B(l_{p-1}, 1; m+p-l; n+1) \sum_{\delta} g_{l\ (1,\eta)}^{\delta}\, c_\delta(\mathbf{I}_{p-1})((n_1-1)(p-1)/2+s+l) \\ &\times \Gamma_{p-1}((n_1-1)/2, \delta)/\Gamma_{p-1}((n_1+p-1)/2, \delta) \end{aligned}$$

where δ is the partition of $l+s$, (2.6) has been obtained by Pillai and Al-Ani [6].

3. **The distribution of the second largest root in the canonical correlation case.**

Let the columns of $\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix}$ be n independent normal $(p+q)$-dimensional variates $(p\leq q)$ with zero means and covariance matrix

$$\Sigma = \begin{pmatrix}\mathbf{\Sigma}_{11} & \mathbf{\Sigma}_{12}\\ \mathbf{\Sigma}_{12}' & \mathbf{\Sigma}_{22}\end{pmatrix}.$$

Let $\mathbf{R}=\operatorname{diag}(r_1, r_2, \ldots, r_p)$, where $r_1^2, \ldots, r_p^2$ are the characteristics roots of the equation

$$|\mathbf{X}_1\mathbf{X}_2'(\mathbf{X}_2\mathbf{X}_2')^{-1}\mathbf{X}_2\mathbf{X}_1' - r^2\mathbf{X}_1\mathbf{X}_1'| = 0$$

and also $\mathbf{P}=\operatorname{diag}(\rho_1, \rho_2, \ldots, \rho_p)$ where $\rho_1^2, \ldots, \rho_p^2$ are the characteristics roots of the equation

$$|\mathbf{\Sigma}_{12}\mathbf{\Sigma}_{22}^{-1}\mathbf{\Sigma}_{12}' - \rho^2\mathbf{\Sigma}_{11}| = 0.$$

Then, the density function of $r_1^2, \ldots, r_p^2$ is given by Constantine [1] in the following form

$$\begin{aligned} &c(n, p, q)|\mathbf{I}-\mathbf{P}^2|^{n/2}|\mathbf{R}^2|^{(q-p-1)/2}|\mathbf{I}-\mathbf{R}^2|^{(n-p-q-1)/2} \prod_{i>j} (r_i^2-r_j^2) \\ (3.1) \qquad &\qquad \times \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(n/2)_\kappa (n/2)_\kappa c_\kappa(\mathbf{R}^2) c_\kappa(\mathbf{P}^2)}{(q/2)_\kappa k!\, c_\kappa(\mathbf{I}_p)} \end{aligned}$$

where

$$c(n, p, q) = \frac{\Gamma_p(n/2)\pi^{p^2/2}}{\Gamma_p(q/2)\Gamma_p((n-q)/2)\Gamma_p(p/2)}.$$

By using the same transformation, namely $q_i=r_i^2/r_{p-1}^2$, $i=1, \ldots, p-2$ and the same method as in §2, the joint density function of r_{p-1}^2, r_p^2 can be shown to have the following form

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$$c_1(n,p,q)|\mathbf{I}-\mathbf{P}^2|^{n/2}(r_{p-1}^2)^{\{(q-p-1)(p-1)+(p-2)(p+1)\}/2}(r_p^2)^{(q+p-3)/2}$$

$$\times(1-r_p^2)^{(n-p-q-1)/2}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa c_\kappa(\mathbf{P}^2)}{(q/2)_\kappa k!\,c_\kappa(\mathbf{I}_p)}\sum_{s=0}^{\infty}\sum_{\eta}\frac{((p+q+1-n)2)_\eta}{s!}$$

(3.2)

$$\times\sum_{l=0}^{p-2}\{c(l)/l!(r_p^2)^l\}\sum_{\tau,\nu}a_{\tau,\nu}(r_p^2)^{k_1}(r_{p-1}^2)^{s+l+k_2}\sum_{\delta}g_{l\ (1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{I}_{p-1})$$

$$\times((q-1)(p-1)/2+s+l+k_2)(\Gamma_{p-1}((q-1)/2,\delta))/\Gamma_{p-1}((q+p-1)/2,\delta),$$

where $c_1(n,p,q)=\pi^{p-1}\Gamma_{p-1}((p-1)/2)\Gamma_p(n/2)/\Gamma_p(q/2)\Gamma_p((n-q)/2)$. Now, integrate (3.2) with respect to r_p^2 then the density function of r_{p-1}^2 can be written in the form

$$c_1(n,p,q)|\mathbf{I}-\mathbf{P}^2|^{n/2}(r_{p-1}^2)^{\{(q-p-1)(p-1)+(p-2)(p+1)\}/2}$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(n/2)_\kappa(n/2)_\kappa c_\kappa(\mathbf{P}^2)}{(q/2)_\kappa k!\,c_\kappa(\mathbf{I}_p)}\sum_{s=0}^{\infty}\sum_{\eta}\frac{(p+q+1-n)/2)_\eta}{s!}\sum_{l=0}^{p-2}\{c(l)/l!\}$$

(3.3)

$$\times\sum_{\tau,\nu}a_{\tau,\nu}(r_{p-1}^2)^{s+l+k_2}B(r_{p-1}^2,1;(q+p-1)/2+k_1-l;(n-p-q+1)/2)$$

$$\times\sum_{\delta}g_{l\ (1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{I}_{p-1})((q-1)(p-1)/2+s+l+k_2)(\Gamma_{p-1}((q-1)/2,\delta)/\Gamma_{p-1}$$

$$\times((q+p-1)/2,\delta)).$$

4. **Noncentral distribution of the second largest root of $\mathbf{S}_1\mathbf{S}_2^{-1}$.** Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be independently distributed as Wishart $W(n_1,p,\boldsymbol{\Sigma}_1)$ and $W(n_2,p,\boldsymbol{\Sigma}_2)$, respectively. Let the characteristics roots of $\mathbf{S}_1\mathbf{S}_2^{-1}$ and $\boldsymbol{\Sigma}_1\boldsymbol{\Sigma}_2^{-1}$ be denoted c_i and λ_i, $i=1,\ldots,p$ and such that $0<c_1<c_2<\cdots<c_p<\infty$ and $0<\lambda_1<\lambda_2<\cdots<\lambda_p<\infty$.

Let $g_i=\delta c_i/(1+\delta c_i)$, $i=1,\ldots,p$; $\delta>0$ and $\mathbf{G}=\operatorname{diag}(g_1,\ldots,g_p)$, and $\mathbf{W}=\operatorname{diag}(\lambda_1,\ldots,\lambda_p)$, then the distribution of $g_1,\ldots,g_p$ is given by Khatri [4] in the following form

$$c(p,m,n)|\delta\mathbf{W}|^{-\frac{1}{2}n_1}|\mathbf{G}|^m|\mathbf{I}-\mathbf{G}|^n\prod_{i>j}(g_i-g_j)\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{I}-(\delta\mathbf{W})^{-1})c_\kappa(\mathbf{G})}{k!\,c_\kappa(\mathbf{I})},\tag{4.1}$$

where m, n, and ν as defined in §2. Then, as before, we can obtain the joint density function of g_{p-1} and g_p in the following form

$$c_1(p,n_1,n_2)|\delta\mathbf{W}|^{-\frac{1}{2}n_1}g_{p-1}^{m(p-1)+\frac{1}{2}(p-2)(p+1)}g_p^{m+p-1}(1-g_p)^n$$

$$\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(\nu)_\kappa c_\kappa(\mathbf{I}-(\delta\mathbf{W})^{-1})}{k!\,c_\kappa(\mathbf{I})}\sum_{s=0}^{\infty}\sum_{\eta}\{(-n)_\eta/s!\}\sum_{l=0}^{p-2}c(l)/l!\,g_p^l$$

(4.2)

$$\times\sum_{\tau,\nu}a_{\tau,\nu}g_p^{k_1}g_{p-1}^{s+l+k_2}\sum_{\delta}g_{l\ (1,\eta,\nu)}^{\delta}\ c_\delta(\mathbf{I}_{p-1})((n_1-1)(p-1)/2+s+l+k_2)$$

$$\times(\Gamma_{p-1}((n_1-1)/2,\delta)/\Gamma_{p-1}((n_1+p-1)/2,\delta).$$

Now, integrate (4.2) with respect to g_p, the density function if g_{p-1} can be written in the following form

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$$c_1(p, n_1, n_2)|\delta \mathbf{W}|^{-\frac{1}{2}n_1} g_{p-1}^{m(p-1)+(p-2)(p+1)/2} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\nu)_\kappa c_\kappa(\mathbf{I}-(\delta\mathbf{W})^{-1})}{k!c_\kappa(I)}$$

$$\times \sum_{s=0}^{\infty} \sum_{\eta} \{(-n)_\eta/s!\} \sum_{l=0} (c(l)/l!) \sum_{\tau,\nu} a_{\tau,\nu} g_{p-1}^{s+l+k_2} B(g_{p-1}, 1;$$

$$\times m+p+k_1-l; n+1) \sum_{\delta} \underset{(1,\eta,\nu)}{g_l^\delta} c_\delta(\mathbf{I}_{p-1})((n_1-1)(p-1)/2+s+l+k_2)$$

$$\times (\Gamma_{p-1}((n_1-1)/2, \delta)/\Gamma((n_1+p-1)/2, \delta)).$$

5. **The distribution of the second largest root of a covariance matrix.** The distribution of the characteristics roots, $0<\omega_1\le\cdots\le\omega_p<\infty$, of $\mathbf{XX}'$ depends only upon the characteristic roots of $\mathbf{\Sigma}$ and can be given in the form (James [3])

$$k(p,n)|\mathbf{\Sigma}|^{-\frac{1}{2}n}|\mathbf{W}_1|^m\{\exp(-\tfrac{1}{2}\operatorname{tr}\mathbf{W}_1)\}\prod_{i>j}(\omega_i-\omega_j)\,{}_0F_0(\tfrac{1}{2}(\mathbf{I}_p-\mathbf{\Sigma}^{-1}), \mathbf{W}_1) \tag{5.1}$$

where $k(p,n)=\pi^{\frac{1}{2}p^2}/2^{\frac{1}{2}pn}\Gamma_p(n/2)\Gamma_p(p/2)$.

It may be pointed out that the form (5.1) can also be viewed as a limiting form of (4.1), when $n_2\to\infty$.

However, by methods similar to those in the previous sections, the density function of the second largest root $\gamma_{p-1}=\omega_{p-1}/2$ can be written in the form

$$\begin{aligned} &k_1(p,n)|\mathbf{\Sigma}|^{-\frac{1}{2}n}\gamma_{p-1}^{m(p-1)+(p-2)(p+1)/2}e^{-\gamma_{p-1}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{c_\kappa(\mathbf{I}-\mathbf{\Sigma}^{-1}}{k!c_\kappa(\mathbf{I})}\sum_{\tau,\nu}a_{\tau,\nu}\\ &\times\sum_{s=0}^{\infty}\sum_{\eta}\frac{(-1)^s}{s!}\sum_{l=0}^{p-2}c(l)\gamma_{p-1}^{s+l+k_2}\sum_{i=0}^{k_2}\sum_{\delta}b_{\delta,\nu}\sum_{\mu}\underset{(\delta,1,\eta)}{g_l^\mu}c_\mu(I_{p-2})[\Gamma_{p-2}\\ &\times((n-2)/2,\mu)/\Gamma_{p-2}((n+p)/2,\mu)][I(\gamma_{p-1},\infty;m+p+k_1-j)-\gamma_{p-1}\\ &\times I(\gamma_{p-1},\infty;m+p+k_1-j-1)],\end{aligned} \tag{5.2}$$

where $b_{\delta,\nu}$ are constants defined in [5], δ and μ are the partitions of i and $i+l+s$ respectively.

$$k_1(p,n) = k(p,n)\Gamma_{p-2}((p-2)/2)\Gamma_{p-2}((p+1)/2)/\pi^{(p-2)^2/2},$$

and

$$I(a,b;c) = \int_a^b x^{c-1}e^{-x}\,dx.$$

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