PAPER 44

ELECTROMAGNETIC ACCELERATION OF PARTICLES TO COSMIC RAY ENERGIES*

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ABSTRACT

It has been demonstrated that if a particle is accelerated in an electromagnetic field between two points designated by subscripts (1) and (2), the particle will gain energy in passing from (1) to (2) provided that the quantity J defined by

$$J = s\frac{\partial}{\partial s}(U_{s2} - U_{s1})^2 - 2\frac{\partial U_{s'}}{\partial t}\int \dot{S}\frac{\partial U_{s'}}{\partial s}\,dt$$

is positive, where ds is an element of path, U_{s1} and U_{s2} are the initial and final values of the vector potential along the path, and t is the time. Moreover, if the particle is at rest at the point (1), its energy W_2 at the point (2) is such that

$$W_2 > e\,|(U_{s2} - U_{s1})|,$$

here e is the charge on the particle.

A study has been made of the problem in which the motivating agency responsible for the electromagnetic field is a toroid with currents circulating in such fashion as they would circulate if the anchor ring of the toroid were wound with a wire in which a current decayed with time. The particular case studied is that where a particle moves along the axes of symmetry, and dimensions are chosen of astronomical size such as to make them apply to such phenomena as are observed in certain nebulae.

The magnitudes chosen are as follows:

$a \equiv$ Radius of cross section of the toroidal winding = 1 light year.
$r_0 \equiv$ Mean radius of toroid = 2000 light years.
$H_0 \equiv$ Initial field in the toroid = 10^{-3} gauss.
$v/\alpha \equiv$ Time for the current to decay to $1/e$ of its initial value = 1000 years.

With the above assumptions, a particle of electronic charge, starting from the center of the toroid and travelling along the axes of symmetry would acquire an energy in excess of 3×10^{14} eV.

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The general theory of acceleration of charged particles by magnetic induction invokes the application of Lagrange's equations to a Lagrangian function for a charge in an external electromagnetic field defined by a vector potential **U** and a scalar potential ϕ. In many problems ϕ is zero and the Lagrangian function becomes

$$L = -m_0 c^2 (1 - u^2/c^2)^{1/2} + \frac{e}{c}(\mathbf{U}.\mathbf{u}). \tag{1}$$

The magnetic field **H**, and the electric field **E** are given by

$$\mathbf{H} = \text{curl } \mathbf{U}; \quad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{U}}{\partial t}. \tag{2}$$

In many problems of changing magnetic fields it is possible, rather readily, to calculate line integrals of the electric field **E** along assigned paths, but such calculations are of no avail for calculating increase of particle energy unless we can show that the particles can describe paths such as to make use of the line integrals to the end of acquiring energy continually, at least over sufficiently long periods of time. The complexity of the particle motions is such that, usually it is not practicable to seek solutions for energy increase by calculating the path and calculating the increase of energy as the particle traverses it. In view of the foregoing considerations it is useful to develop criteria for continual increase of energy, and theorems which give information as to lower limits of energy attained in certain specified cases. A few of these matters are discussed in the following.

1. GENERAL CONSIDERATIONS PERTAINING TO THE CONTINUAL INCREASE OF ENERGY OF A PARTICLE STARTING FROM REST IN AN ELECTROMAGNETIC FIELD

It is clear that the particle, starting from rest, will move initially so as to make an acute angle with the electric field, i.e. with the vector

$$-(1/c)\,(\partial \mathbf{U}/\partial t).$$

As long as it continues to move so as to make an acute angle with the positive direction of **E**, the energy will continue to increase. It can only decrease by the motion developing a character in which the particle makes an obtuse angle with **E**, so that it has a component opposite to **E**. In order to acquire this condition, however, it would have to pass through a condition, at some point P, in which it moved perpendicular to **E** at the point. If there were no magnetic field, it certainly could not pass through this latter condition because, at the point P, the particle would have acting on it a field tending to increase the component velocity parallel to $\mathbf{E}$, and so to bring the particle's path back to the condition in which it made an acute angle with $\mathbf{E}$.

If there is a magnetic field when the particle is at P, with its path perpendicular to $\mathbf{E}$, there will arise from this magnetic field a force $\mathbf{v} \times \mathbf{H}/c$ perpendicular to $\mathbf{v}$ and parallel to $\mathbf{E}$. This force may be in the direction of $\mathbf{E}$ or in the opposite direction, depending on the circumstances. If, however, $|\mathbf{E}| > |\mathbf{H}|$, we shall certainly have $|\mathbf{E}| > |\mathbf{v} \times \mathbf{H}/c|$ so that even if $\mathbf{v} \times \mathbf{H}/c$ is in the opposite direction to $\mathbf{E}$, the resultant force will be in the direction of $\mathbf{E}$ and will bring the particle back to the condition in which its path makes an acute angle with $\mathbf{E}$, and so the particle gains energy at P.

The condition $|\mathbf{E}| > |\mathbf{H}|$ as a criterion for continual gain of energy is thus *sufficient** but not always a necessary condition for continual increase of energy.

There is one exception to the above theorem. It is to be found in the case where the particle passes through a place where $\mathbf{E}$ reverses sign. In this case, the argument fails. An example is to be found in the case of a particle traveling along the general direction of propagation of a plane wave. It will be acted on by the field of the wave, which will oscillate in sign along a direction perpendicular to the general direction of the particle, and indeed, in this case the particle will alternately gain and lose energy. Of course, in the plane wave we have cited we have $\mathbf{H} = \mathbf{E}$ at all times, so that, strictly speaking, the test of our theorem is too severe. However, we are certainly on the safe side if we exclude from the theorem cases where the particle passes through a place of reversal of $\mathbf{E}$.

Concerning the looping of a particle around a line

If $|\mathbf{E}| > |\mathbf{H}|$, or less stringently, if the path of the particle always makes an acute angle with $\mathbf{E}$, the particle can never describe an angle 2π around any line, OP, unless there is a finite line integral of $\mathbf{E}$ (at the particle) taken along the path of the particle projected in a plane perpendicular to OP. The reason is as follows: In the light of the hypothesis, there is a finite component of $\mathbf{E}$ in the direction of the projected path at each point thereof, and therefore, if the projected path curves through an angle 2π, $\mathbf{E}$ (at the particle) projected on that path will have a finite line integral taken over the range 2π.

* The sufficiency of the condition for an axially symmetrical field with the z and r components of $\mathbf{U}$ zero, was established in the writer's first paper on this subject[1]. A simplified version of the theory is given by the writer in [2]; also in a later paper, 'The Acquirement of Cosmic Ray energies by Electromagnetic Induction in Galaxies'[3].

The lower limit of the energy gained along a path

If T is the kinetic energy of the particle, and $W \equiv T + mc^2$, it is readily possible, by the application of Lagrange's equations, to show that

$$W_2^2 - W_1^2 = e^2(U_{s2} - U_{s1})^2 + \int \left[e^2 \dot{s} \frac{\partial}{\partial s} (U_s - U_{s1})^2 - 2e^2 \frac{\partial U_s}{\partial t} \int \dot{s} \frac{\partial U_s}{\partial s} \, dt \right] dt, \tag{3}$$

where U_s refers to the vector potential resolved along the direction of the path, at an arbitrary point on the path. $\dot{s}$ is the velocity along the path, subscripts (1) and (2) refer respectively to values at the point occupied at $t = 0$ and the point occupied at some later time. If we define J as

$$J \equiv \dot{s} \frac{\partial}{\partial s} (U_s - U_{s1})^2 - 2 \frac{\partial U_s}{\partial t} \int \dot{s} \frac{\partial U_s}{\partial s} \, dt \tag{4}$$

then, in cases where, at all instants, J is positive, we can write

$$W_2^2 - W_1^2 > e^2(U_{s2} - U_{s1})^2.$$

In cases where the particle starts from rest and where, for large kinetic energies, mc^2 is neglected, this expression assumes the simple form

$$| W_2 | > e \, | (U_{s2} - U_{s1}) |. \tag{5}$$

This relation is of value in certain cases.

2. CASE OF AN AXIALLY SYMMETRICAL FIELD, IN WHICH THE VECTOR POTENTIAL U HAS NO COMPONENT ALONG THE r OR z DIRECTIONS

The case cited has many interesting features, some of which have been developed by the writer in a paper [3]. We shall summarize a few of these. It results from Lagrange's equations that

$$\frac{1}{2} \frac{dW^2}{dt} = e^2 \left(U_\theta - \frac{r_0 U_0}{r} \right) \frac{\partial U_\theta}{\partial t}, \tag{6}$$

where $r_0 U_0$ apply at the instant and position when the particle commences to change its kinetic energy.

Criteria for continual increases of energy in the axially symmetrical case

Confining ourselves, without loss of generality, to the case where $\partial U_\theta / \partial t$ is positive, we see that the *necessary* condition for continued increase of energy is

$$U_\theta - \frac{r_0 U_0}{r} > 0 \tag{7}$$

except at $t = 0$.

A *sufficient*, but not always a necessary condition for (7) to hold is that, for all positions of the particle

$$\frac{d}{dt}(rU_\theta - r_0 U_0) > 0$$

or, since $r_0 U_0$ is a constant $\quad \frac{d}{dt}(rU_\theta) > 0. \qquad (8)$

It can readily be shown that

$$\frac{d}{dt}(rU_\theta) = [-E_\theta - (\mathbf{v} \times \mathbf{H})_\theta / c], \qquad (9)$$

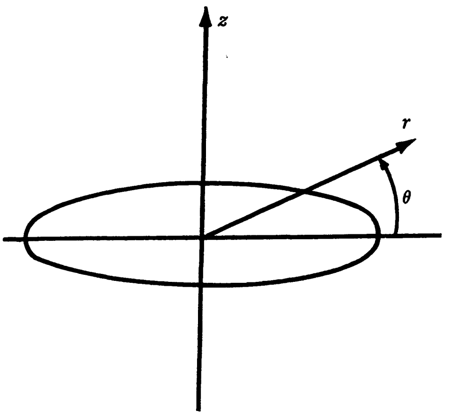

Fig. 1

where **v** is the velocity of the particle. Since $-E_\theta$ $(= \partial U_\theta / \partial t)$ is positive, a sufficient, but not always necessary, condition for (8) to be true is therefore

$$|E_\theta| > |\mathbf{H}|, \qquad (10)$$

which is a condition already cited for the more general case in which axial symmetry is not demanded. Thus conditions (7), (8) and (10) stand in order of stringency. Condition (10) is sufficient for (7) and (8). Condition (8) is sufficient for (7), while (7) is necessary and sufficient.

Case where, in axial symmetry, the particle starts from rest at the place and instant when $U_\theta = 0$

In this case, (6) becomes

$$\frac{dW^2}{dt} = e^2 \frac{\partial}{\partial t}(U_\theta)^2$$

so that the energy in this case increases continually under all circumstances.

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Criterion for absence of 'looping', in the case of axial symmetry

We have seen that (7) is the necessary and sufficient condition for continual increase of energy. Now it can readily be shown from Lagrange's equations that

$$rW\dot{\theta} = -ce(U_\theta - r_0 U_0/r).$$

Hence, if there is continual increase of energy, θ must always be of the same sign and finite. Thus, in such a case, the particle can never 'loop', except aroung the z-axis; for if the particle should loop in any other manner, there would have to be a place where $\dot{\theta}$ was zero.

Mechanisms in which there is no magnetic field in the space surrounding the motivating currents when those currents are steady

Tempting problems are presented by the discussion of an infinite solenoid, and by an anchor ring-winding. Here, there is no external magnetic field in the steady state, and even if the currents vary with the time, the magnetic field remains small for slow variations, in spite of the existence of a very definite electromotive force over a path encircling the solenoid outside thereof, or encircling the anchor ring so as to thread it.* If we could neglect the external magnetic field completely on such problems, we should always have $|\mathbf{E}| > |\mathbf{H}|$ and the sufficient criterion for continual gain of energy would be assured.

Although, in cases of the type cited, there is no magnetic field in the space surrounding the motivating currents when these currents are steady, there certainly is some magnetic field when the currents vary with the time. This can most readily be seen from the electromagnetic equations for free space, which demand a finite value for curl **H** and so for **H** if there is a finite value for $\partial\mathbf{E}/\partial t$. In free space we have in fact, all of the electromagnetic vectors, **E**, **H**, **U**, obeying the wave equation; and, as far as our interests are concerned, we need confine our attentions only to the wave equation for **U**.

$$\nabla^2\mathbf{U} - \frac{1}{c^2}\cdot\frac{\partial^2 U}{\partial t^2} = 0.$$

* In such problems, the role of the vector potential presents a more realistic picture of the origin of the electric field at a point than does the behavior of the magnetic field. Thus, in the steady state problem for a solenoid, there is no magnetic field at a point P outside the solenoid, but there is a very definite vector potential. It is true that the Faraday law survives to the extent of predicting that around a path encircling the solenoid there is an electromotive force equal to the magnetic flux through the path; but this magnetic flux is confined for the most part to the area inside the solenoid. As a matter of fact, even when, as in the case of a long finite solenoid, there is a small magnetic field outside the solenoid, that field is in a direction opposite to that of the flux in the solenoid.

For the case of axial symmetry exemplified in Fig. 1, this equation assumes the form

$$\frac{\partial^2 U_\theta}{\partial r^2}+\frac{\partial^2 U_\theta}{\partial z^2}+\frac{\partial}{\partial r}\left(\frac{U_\theta}{r}\right)=\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}. \qquad (11)$$

A useful problem which serves as a basis for the discussion of other problems is that of a circular current of small size, flowing around the axis of z at the origin. For the steady state case, this entity acts like a magnet of moment μ given by

$$\mu=\pi a^2 I,$$

where a is the radius and I the current flowing in the positive direction of θ. Thus μ is in the positive direction of the axis of z. It can easily be

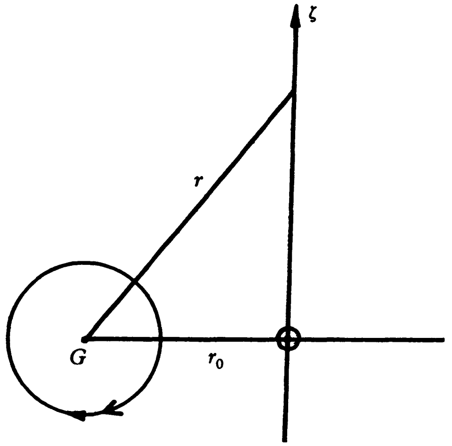

Fig. 2

verified that, for the case where I varies with the time according to the law $I=I_0 f(t)$, where $f(0)=1$, the appropriate solution of (11) is

$$U_\theta=\frac{\mu_0 r f(t-R/c)}{R^3}+\frac{\mu_0 r f'(t-R/C)}{CR^2}, \qquad (12)$$

where $\mu_0=\pi a^2 I_0$; $R^2=z^2+r^2$. The solution (12) can by the combination of a number of such circular current elements, serve as the basis for the solution of a solenoid, or of a toroidal winding in which the current varies with the time.

Case of a toroidal winding

It is convenient to transpose the current ring, for which (12) is the solution, to the position shown in Fig. 2, where it now forms an element of a toroidal winding, the anchor ring of the toroid having its plane of symmetry perpendicular to the axis of ζ.

Let the old axis of z corresponding to (12) be in the downward direction through the paper, so that the current is in the direction shown. The plane of the paper shall be the old plane $z=0$, and r shall continue to have its original meaning. The positive direction of θ is in the direction of the arrow and the value of U_θ at the point P is, from (12)

$$U_\theta = \frac{\mu_0}{r^2} f(t-r/c) + \frac{\mu_0 r_0}{cr} f'(t-r/c).$$

The component of $\mathbf{U}$ in the positive direction of ζ is thus

$$U_\zeta = -\frac{\mu_0 r_0}{r^3} f(t-r/c) - \frac{\mu_0 r_0}{cr^2} f'(t-r/c), \tag{13}$$

where r_0 is as indicated.

If now we consider the toroidal winding to be made up of a large number of circular currents like that shown, each of them will make a contribution to U_ζ. If I_0 as defined above, is taken to be the current per unit length measured around the solenoid, and if ϕ is measured around the ζ axis, the moment appropriate to the element of angle $d\phi$ is $\pi a^2 I_0 r_0 d\phi$ which replaces μ_0 in (13) for the contribution of $d\phi$ to the total vector potential A_ζ along the ζ direction. Integrating with respect to ϕ, we find for A_ζ itself the value

$$A_\zeta = -\frac{2\pi^2 a^2 I_0 r_0^2}{r^3}\left[f(t-r/c) + \frac{r}{c} f'(t-r/c)\right]. \tag{14}$$

It is convenient to express I_0 in terms of the magnetic field inside the toroid for the steady case. If H_0 is this field, we have

$$2\pi r_0 H_0 = 4\pi(2\pi r_0 I_0)$$

so that

$$H_0 = 4\pi I_0.$$

Writing

$$\frac{\pi a^2 H_0}{2} \equiv N_0 \tag{15}$$

we have

$$A_\zeta = -\frac{N_0 r_0^2}{r^3}\left[f(t-r/c) + \frac{r}{c} f'(t-r/c)\right]. \tag{16}$$

All components of the vector potential perpendicular to the ζ axis cancel, from symmetry, so that A_ζ, represents the complete vector potential along the ζ axis.

In the steady state, there is no magnetic field anywhere outside the toroid. Such is not the case for the non-steady state, however, for in general there is in such a case, a rate of change of vector potential and so an electric field at all points in the space around the toroid, and there is in general a rate of change of electric field which demands the existence of a magnetic

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field. However, along the ζ axis there is no magnetic field in a plane perpendicular to that axis; for if there were a field perpendicular to the axis at any part P, there would from symmetry be a magnetic flux towards or away from the axis at that point, and this is impossible. Any magnetic field in the vicinity of the ζ axis must take the form of circular lines around that axis, the situation conforming to

$$\frac{1}{c}\frac{\partial E_\zeta}{\partial t} = (\text{curl }\mathbf{H})_\zeta.$$

Applying this to a small tube of radius σ surrounding the axis we have

$$\frac{\pi\sigma^2}{c}\frac{\partial E_\zeta}{\partial t} = 2\pi\sigma H_\phi,$$

where H_ϕ is the magnetic field in the direction of increase of ϕ, thus,

$$H_\phi = \left(\frac{\sigma}{2c}\right)\frac{\partial E_\zeta}{\partial t}$$

and H_ϕ vanishes with the vanishing of σ for a finite value of $\partial E_\zeta/\partial t$.

In the light of the above we see that the only force on a charged particle on the axis of ζ is a force along that axis, and the force is $-(1/c)\,\partial A_\zeta/\partial t$ per unit charge.

We are now in a position to make use of (16). We shall choose $f(t)$ to qe of the form

$$f(t) = e^{-\alpha t}$$

so that $$f(t-r/c) = e^{-\alpha(t-r/c)} = e^{-\alpha t}e^{\alpha r/c}$$

and $$f'(t-r/c) = -\alpha a^{-\alpha(t-r/c)} = -\alpha e^{-\alpha t}e^{\alpha r/c}.$$

Thus $$A_\zeta = -\frac{N_0 r_0^2}{r^3}\left[1-\frac{\alpha r}{c}\right]e^{-\alpha t}e^{\alpha r/c}. \tag{17}$$

The electric field E_ζ along the ζ axis is given by

$$E_\zeta = -\frac{1}{c}\frac{\partial A_\zeta}{\partial t} = -\frac{N_0 r_0^2\alpha}{cr^3}\left[1-\frac{\alpha r}{c}\right]e^{-\alpha t}e^{\alpha r/c}. \tag{18}$$

The electric field reverses sign at the value of r given by $\alpha r/c = 1$. There are thus two categories of interest corresponding to $\alpha r/c < 1$ and $\alpha r/c > 1$. We shall call them cases A and B, respectively. The second contribution (involving $\alpha r/c$) on the right-hand side of (18) is of course a close mathematical relative of the radiation field from an electric dipole, which field varies less rapidly with the distance—to the extent of one power of r—than does the non-radiation field, whose counterpart, is the first term on the right-hand side of (18). At small distances the non-radiation term dominates, but at great distances the radiation term dominates.

Regarding the role played by the scale of the phenomena

For such phenomena as occur on stars, $(1/\alpha)$ may be expected to be of the order of a few days, as in the case of sunspots, for example. For these cases α may be taken to be of the order 10^{-6} or less, so that the quantity c/α is of the order 3×10^{16} cm. This is much larger than stellar dimensions* so that in general for particles accelerated in stars we shall be concerned with the case where $\alpha r/c \ll 1$.

For phenomena on a galactic scale such as we encounter in the nebulae, α may be expected to be much smaller than 10^{-6}. On the other hand, the dimensions available in a nebula are such that $\alpha r/c$ can approach or even exceed unity, and the scale of the space occupied by the motivating currents can afford to be correspondingly large so as to provide for significant acceleration at the great distances concerned. Thus, both cases *A* and *B* are likely to be of interest to us in cosmological speculations.

Case A, where $\alpha r/c < 1$

This corresponds to $(\zeta^2 + r_0^2) < c^2/\alpha^2$, i.e. to the region

$$(c^2/\alpha^2 - r_0^2)^{1/2} > \zeta > -(c^2/\alpha^2 - r_0^2)^{1/2},$$

i.e. to the region
$$\left(\frac{c^2}{\alpha^2 r_0^2} - 1\right)^{1/2} > \zeta/r_0 > -(c^2/\alpha^2 r_0^2 - 1)^{1/2}. \qquad (19)$$

For $\alpha = 10^{-6}$, and $r_0 \sim 10^{11}$ for stellar dimensions $c^2/\alpha^2 r_0^2 = 9 \times 10^{10}$. Thus (19) becomes

$$\frac{c}{\alpha r_0} > \frac{\zeta}{r_0} > -\frac{c}{\alpha r_0}$$

or, for the magnitudes cited

$$3 \times 10^5 > \frac{\zeta}{r_0} > -3 \times 10^5.$$

In this region, the field is always negative, and a positive particle starting in the region moves in the negative direction and gains energy continually.

Let us consider a case where $a = 10^9$ cm; $r_0 = 10^{10}$ cm; $\alpha = 10^{-5}$; $\mathbf{H}_0 = 10^4$, so that, from (15), $N_0 = 1{\cdot}5 \times 10^{22}$.

Let us calculate the gain in energy of a proton in traveling from $-\zeta = 0$ to $-\zeta = r_0 = 10^{10}$ cm. The quantity $\alpha r/c$ will, over the whole path,

* Of course for special phenomena in which $(1/\alpha)$ might be of the order of 1 sec, $\alpha r/c$ would be comparable with unity even for stellar dimensions.

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always be less than $0{\cdot}5 \times 10^{-5}$, so that, replacing the factor $e^{-\alpha t}e^{\alpha r/c}$ by unity, as will subsequently be justified,

$$\mathbf{E}_\zeta = -\frac{N_0 r_0^2 \alpha}{c r^3}.$$

The energy gained will be given by

$$W = -\frac{N_0 r_0^2 \alpha e}{c}\int \frac{d\zeta}{r^3}.$$

Writing $-\zeta/r_0 = \tan\lambda$, we have $\pi/4$ for the upper limit of λ, and

$$W = \frac{N_0 \alpha e}{c}\int_0^{\pi/4} \frac{\sec^2\lambda\, d\lambda}{(1+\tan^2 r)^{\frac{3}{2}}} = \frac{N_0 \alpha e}{c}\int_0^{\pi/4} \cos\lambda\, d\lambda.$$

Thus

$$W = \frac{N_0 \alpha e}{c}\sin\frac{\pi}{4} = \frac{N_0 \alpha e}{1{\cdot}4c} = \frac{300\, N_0 \alpha}{1{\cdot}4c}\,\text{eV.},$$

$$W = 10^9\ \text{eV}.$$

If $\mathbf{E}_m$ is the field at the end of the above path, the time taken to describe the path is less than τ as given by $r_0 = \mathbf{E}_m e\tau^2/2m$, where m is the relativistic mass at the end of the path. m is not greatly different from the rest mass, $1{\cdot}6 \times 10^{-24}$, so that it results that $r_0 = N_0 \alpha r_0^2 e \tau^2 / 2 m r^3 c$ and, on inserting the values with $r/r_0 = 2^{\frac{1}{2}}$, we find τ of the order 1 second. Thus the replacement of $e^{-\alpha t}e^{\alpha r/c}$ by unity as above is justified.

Case B where $\alpha r/c > 1$ over the path

Let us write $\alpha r_0/c \equiv \eta$, and let us consider a case where $a = 10^{18}$ cm ($=1$ light year); $\mathbf{H}_0 = 10^{-3}$ gauss, (so that $N_0 = 1{\cdot}5 \times 10^{33}$); $1/\alpha = 3 \times 10^{10}$, (corresponding to the decay of the motivating currents to $1/e$ of their initial values in 1000 years); $\eta = 2$, so that $r_0 = 2 \times 10^{21}$ cm ($=$about 2000 light years).

The exponential factor in the expression for $\mathbf{E}_\zeta$ is composed of two factors $\exp \alpha r/c$ and $\exp(-\alpha t)$. We shall first examine the consequences of neglecting the second factor, so that

$$\mathbf{E}_\zeta = \frac{N_0 r_0^2 \alpha}{c r^3}\left[\frac{\alpha r}{c} - 1\right] e^{\alpha r/c}.$$

It is easy to show that $\mathbf{E}_\zeta$ increases continually with r. We have

$$W > \frac{N_0 r_0^2 \alpha e}{c}\int \left[\frac{\alpha}{c r^2} - \frac{1}{r^3}\right] d\zeta$$

over the range of integration concerned. Writing $\zeta/r_0 = \tan\lambda$

$$W > \frac{N_0 \alpha e}{c}\left[\frac{\alpha r_0}{c}\int_0^\lambda d\lambda - \int_0^\lambda \cos\lambda\, d\lambda\right],$$

$$W > \frac{300\ N_0 \eta}{r_0}\ [\eta\lambda - \sin\lambda] \quad \text{electron volts.}$$

Now a proton with a velocity 99 per cent of the velocity of light has an energy of 6×10^9 eV. It has a mass m equal to $7\ m_0$. For such a proton, and with $\eta = 2$

$$2\lambda - \sin\lambda < \frac{2 \times 10^7 r_0}{2 \times N_0} < 1{\cdot}5 \times 10^{-5}$$

so that $\lambda = 1{\cdot}5 \times 10^{-5}$ and the corresponding value of ζ is $1{\cdot}5 \times 10^{-5} r_0$. Since the minimum value of $\mathbf{E}_\zeta$ occurs at $r = r_0$, and in this case is given by $\mathbf{E}_{\text{min.}} = (N_0/r_0^2)\ (\alpha r_0/c)\ (\alpha r_0/c - 1)\ \exp\,(\alpha r_0/c)$, we have, for $\eta = 2$, $\mathbf{E}_{\text{min.}} = 2(7{\cdot}3)\ N_0/r_0^2$, and if τ is the time for the particle to reach the point $\zeta = 1{\cdot}5 \times 10^{-5} r_0$, we have, with $m = 7 \times 1{\cdot}6 \times 10^{-24}$

$$1{\cdot}5 \times 10^{-5} r_0 > \frac{14{\cdot}6\ N_0 e \tau^2}{2 r_0^2 m}.$$

Hence $\tau < 4 \times 10^5$ sec and $\alpha\tau < 1{\cdot}3 \times 10^{-5}$. Thus, the neglect of the factor $\exp\,(-\alpha t)$ in the expression for $\mathbf{E}_\zeta$ is valid for the above calculation, in which λ is limited to the small value $1{\cdot}5 \times 10^{-5}$.

We may now evaluate the total situation as follows: The least value of $\alpha r/c$ is $\alpha r_0/c$. The total time to travel a distance ζ larger than the value $1{\cdot}5 \times 10^{-5} r_0$ considered above will be less than $\zeta/v + \tau$, where $v = 99c/100$, and $\tau = 4 \times 10^5$ sec. Thus, if ζ is such that $\alpha\zeta/v + \alpha\tau$ is not greater than $\alpha r_0/c$, the exponent $(\alpha r/c - \alpha t)$ in the general expression for $\mathbf{E}_\zeta$ will always be positive. This gives

$$\zeta \leqslant r_0 v/c - v\tau \leqslant (99/100)\ (r_0 - 4 \times 10^5 c).$$

Since $r_0 = 2 \times 10^{21}$, the quantity $4 \times 10^5 c$ may be neglected and ζ may be permitted a value sensibly as large as r_0, so that the upper limit $\lambda = \pi/4$ may be used in the expression for W. We thus find that, in travelling over the said range, the particle gains energy W such that

$$W > \frac{300\ N_0 \eta}{r_0}\left[\frac{\pi}{4}\eta - \frac{1}{1{\cdot}4}\right].$$

Putting $\eta = 2$, $N_0 = 1{\cdot}5 \times 10^{33}$ and $r_0 = 2 \times 10^{21}$

$$W > 3 \times 10^{14}\ \text{eV}.$$

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Of course, the numbers here cited are susceptible of enormous variations without exceeding the realm of reason, and our example is taken simply as an illustration.

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Discussion

Alfvén: My only remark is that I think we all are very glad that the senior of all accelerating processes has worked very well!

Singer: Would your theory be applicable to all values of $(dH/dt)/H$, say, when the line integral cannot be defined?

Swann: Yes, the mechanism works for all values of $(dH/dt)/H$. This indeed is my quantity $-\alpha$.

Singer: Could the process work statistically, i.e. the particle gains and loses energy, but on the average gets accelerated?

Swann: Yes, it could work under suitable circumstances, but I have confined my attention to a problem where the energy increases continually.

Schlüter: Has your theory been worked out solely for the case of vacuum, without electrical conductivity?

Swann: Yes.

Bunemann: What gauge of potentials has one to use in order to make the formula $W > e \mid U_{s2} - U_{s1} \mid$ right? You seem to have used a particular gauge, the retarded potentials.

Swann: The potentials used are the retarded potentials defined by

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = -\rho u/c; \quad \Delta^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\rho.$$

However, any equivalent pair of potentials could give equivalent results but in general with great analytical difficulty.