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RESEARCH ARTICLE

Non-abelian Mellin transformations and applications

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Abstract
We study non-abelian versions of the Mellin transformations, originally introduced by Gabber-Loeser on complex affine tori. Our main result is a generalisation to the non-abelian context and with arbitrary coefficients of the t-exactness of Gabber-Loeser's Mellin transformation. As an intermediate step, we obtain vanishing results for the Sabbah specialisation functors. Our main application is to construct new examples of duality spaces in the sense of Bieri-Eckmann, generalising results of Denham-Suciu.

Contents

1. Introduction

The geometry and topology of a complex algebraic or analytic variety can be studied via the cohomology groups of its coherent and constructible sheaves. The Fourier-Mukai transformation on the coherent side and the Mellin transformation on the constructible side are functors that allow one to compute the cohomology groups of a coherent or constructible sheaf twisted by all (topologically trivial) line bundles or local systems. As the Fourier-Mukai transformation has become an essential tool in birational geometry, the Mellin transformation has also proved useful in the study of perverse sheaves, especially on complex affine tori [9, 14], abelian varieties [20, 3] and, more generally, semi-abelian varieties [12, 13, 15]. In particular, whether a constructible complex is a perverse sheaf can be completely determined by its Mellin transformation.

In this paper, we establish non-abelian generalisations of the t-exactness result of Gabber-Loeser [9] to certain families of Stein manifolds, such as complements of essential hyperplane arrangements. In this general setting, we reduce the global t-exactness of the Mellin transformation to a certain local vanishing result for the multivariable Sabbah specialisation functor [19]. In fact, we show the t-exactness of the multivariable Sabbah specialisation functor, generalising the well-known result about the t-exactness of the nearby cycle functor. As a special case of this local vanishing result, we prove a local version of the t-exactness result of Gabber-Loeser. Using the t-exactness of the non-abelian Mellin transformations, we construct new families of duality spaces that generalise those of Denham-Suciu [7].

Let A be a Noetherian commutative ring of finite cohomological dimension. Let U be a complex analytic variety with fundamental group G. Let $\mathcal{L}_U$ be the universal $A[G]$-local system on U (as defined in Section 2.1). Denote by $q : U \to \mathrm{pt}$ the projection to a point space. For an A-constructible complex $\mathcal{F}$ on U, we define its *Mellin transformations* by

$$\mathfrak{M}_*^U(\mathcal{F}) \coloneqq Rq_*(\mathcal{F} \otimes_A \mathcal{L}_U) \quad \text{and} \quad \mathfrak{M}_!^U(\mathcal{F}) \coloneqq Rq_!(\mathcal{F} \otimes_A \mathcal{L}_U).$$

We omit the uppercase U when there is no risk of confusion. These are non-abelian counterparts of similar transformations introduced by Gabber-Loeser [9] on complex affine tori.

Our main result is a generalisation to the non-abelian context of Gabber-Loeser's t-exactness of the Mellin transformation $\mathfrak{M}_*$ (see [9, Theorem 3.4.1], [14, Theorem 3.2]).

Theorem 1.1. *Let U be a complex manifold with a smooth compactification $U \subset X$, such that the boundary divisor $E = \bigcup_{1\le k\le n} E_k$ is a simple normal crossing divisor. Assume that the following properties hold:*

1. *For any subset $I \subset \{1, \dots, n\}$, $E_I^\circ \coloneqq \bigcap_{k\in I} E_k \setminus \bigcup_{l\notin I} E_l$ is either empty or a Stein manifold. When $I = \emptyset$, this means $U = X \setminus E$ is a Stein manifold.*
2. *For any point $x \in E$, the local fundamental group of U at x maps injectively into the fundamental group of U: that is, the homomorphism $\pi_1(U_x) \to \pi_1(U)$ induced by inclusion is injective, where $U_x = B_x \cap U$ with B_x a small enough complex ball in X centred at x.*

Then for any A-perverse sheaf $\mathcal{P}$ on X, the Mellin transformation $\mathfrak{M}_*(\mathcal{P}|_U)$ is concentrated in degree zero. In other words, the functor

$$\mathcal{F} \mapsto \mathfrak{M}_*(\mathcal{F}|_U) : D_c^b(X, A) \to D^b(A[G])$$

is t-exact with respect to the perverse t-structure on $D_c^b(X, A)$ and the standard t-structure on $D^b(A[G])$.

Remark 1.2. If U is algebraic, the conclusion of the above theorem can be reformulated as the assertion that the functor

$$\mathfrak{M}_* : D_c^b(U, A) \to D^b(A[G])$$

is t-exact. Indeed, if we choose an algebraic compactification $j : U \hookrightarrow X$ satisfying the properties of the theorem, then a constructible complex $\mathcal{F}$ in U is the restriction of the constructible complex $Rj_*\mathcal{F}$ on X. A similar statement holds in the analytic category, provided that one works with a fixed Whitney stratification $\mathcal{S}$ of the pair (X, E) and constructibility is taken with respect to $\mathcal{S}$ (e.g., see [18, Theorem 2.6(c)]).

Examples of varieties $U \subset X$ satisfying the above conditions include complements of essential hyperplane arrangements, toric arrangements and elliptic arrangements in their respective wonderful compactifications, as well as complements of at least $n + 1$ general hyperplane sections in a projective manifold of dimension n (see [7, Section 2.3]).

Notice that the standard inclusion $(\mathbb{C}^*)^n \hookrightarrow \mathbb{P}^n$ satisfies both conditions in Theorem 1.1. Thus we get the following generalisation of Gabber-Loeser's t-exactness theorem from field coefficients to more general ring coefficients, where constructibility is taken in the algebraic sense. We also remark that

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the original proof of Gabber-Loeser does not apply to this general setting since it uses in an essential way the artinian and noetherian properties of the category of perverse sheaves with field coefficients. Moreover, our result cannot be directly deduced from that of Gabber-Loeser. Indeed, given an A-perverse sheaf $\mathcal{P}$, it is possible that $\mathcal{P} \overset{L}{\otimes}_A A/m$ is not perverse for some maximal ideal m of A.

Corollary 1.3. *The Mellin transformation*

$$\mathfrak{M}_* : D^b_c((\mathbb{C}^*)^n, A) \to D^b(A[\mathbb{Z}^n])$$

is t-exact with respect to the perverse t-structure on $D^b_c((\mathbb{C}^*)^n, A)$ *and the standard t-structure on* $D^b(A[\mathbb{Z}^n])$.

As another application of Theorem 1.1, we obtain new examples of duality spaces (in the sense of Bieri-Eckmann [4]) that are non-affine or singular varieties. In particular, we recast the fact, initially proved by Denham-Suciu [7] by different methods, that linear, toric and elliptic arrangement complements are duality spaces.

As a key step in proving Theorem 1.1, we obtain local vanishing results about Sabbah's specialisation functors (Theorem 4.2). As a special case, we obtain a local version of the t-exactness result of Gabber-Loeser, where constructibility is taken in the analytic category.

Theorem 1.4. *Let* $\mathcal{P}$ *be an A-perverse sheaf defined in a neighbourhood of* $0 \in \mathbb{C}^n$. *Let* $B \subset \mathbb{C}^n$ *be a small ball centred at the origin, and let* B° *be the complement of all coordinate hyperplanes in B. Let* $\mathcal{L}_{B^\circ}$ *be the universal* $\pi_1(B^\circ)$*-local system on* B°. *Then*

$$H^k\left(B^\circ, \mathcal{P}|_{B^\circ} \otimes_A \mathcal{L}_{B^\circ}\right) = 0 \quad \textit{for any } k \neq 0.$$

In this paper, we make essential use of the language of derived categories and perverse sheaves (see, e.g., [11], [21], [8], [17] and [18] for comprehensive references).

2. Preliminaries

2.1. Universal local system

Let X be a connected locally contractible topological space with base point x. Let $G = \pi_1(X, x)$, and let $p : \tilde{X} \to X$ be the universal covering map. We regard the universal cover $\tilde{X}$ as the space of homotopy classes of paths from the base point x to a variable point in X, with the natural action of G on the right. Let

$$\mathcal{L}_X := p_! A_{\tilde{X}},$$

where $A_{\tilde{X}}$ is the constant sheaf with stalk A on $\tilde{X}$. Then $\mathcal{L}_X$ is a local system of rank-one free right $A[G]$-modules. Equivalently, $\mathcal{L}_X$ can be defined as the rank-one $A[G]$-local system such that the stalk at x is equal to $A[G]$ and the monodromy action is defined as the left multiplication of G on $A[G]$.

Remark 2.1. We call $\mathcal{L}_X$ the *universal local system* of X for the following reason. Given any A-module representation $\rho : G \to \mathrm{Aut}_A(V)$, we can regard V as a left $A[G]$-module. Then we have an A-local system $\mathcal{L}_X \otimes_{A[G]} V$ whose monodromy action is precisely ρ. Moreover, every A-local system on X can be obtained uniquely in this way.

Lemma 2.2. *Let* (Y, y) *and* (Z, z) *be two path-connected locally contractible pointed topological spaces. Let* $\mathcal{L}_Y$ *and* $\mathcal{L}_Z$ *be the universal* $A[\pi_1(Y, y)]$*- and* $A[\pi_1(Z, z)]$*-local systems on Y and Z, respectively. Let* $g : Y \to Z$ *be a continuous map with* $g(y) = z$. *If* $g_* : \pi_1(Y, y) \to \pi_1(Z, z)$ *is injective, then as an* $A[\pi_1(Y, y)]$*-local system,* $g^*(\mathcal{L}_Z)$ *is a direct sum of copies of* $\mathcal{L}_Y$ *indexed by the right cosets* $g_*\pi_1(Y)\backslash\pi_1(Z)$.

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Proof. By definition, the local system $g^*(\mathcal{L}_Z)$ has stalk $A[\pi_1(Z,z)]$ at y, and the monodromy action of $\alpha \in \pi_1(Y,y)$ is equal to the left multiplication of $g_*\alpha$. As a left $A[\pi_1(Y,y)]$-module, $A[\pi_1(Z,z)]$ is free, and the summands are parametrised by the right cosets $g_*\pi_1(Y)\backslash\pi_1(Z)$. Thus the assertion of the lemma follows. □

2.2. (Weakly) constructible complexes, perverse sheaves and Artin's vanishing.

Recall that a sheaf $\mathcal{F}$ of A-modules on a complex algebraic, respectively, analytic variety X is said to be weakly constructible if there is an algebraic, respectively, analytic Whitney stratification $\mathcal{S}$ of X so that the restriction $\mathcal{F}|_S$ of $\mathcal{F}$ to every stratum $S \in \mathcal{S}$ is an A-local system. We say that $\mathcal{F}$ is constructible if, moreover, the stalks $\mathcal{F}_x$ for all $x \in X$ are finitely generated A-modules. Let $D^b(X,A)$ be the bounded derived category of complexes of sheaves of A-modules on X. A bounded complex $\mathcal{F} \in D^b(X,A)$ is called (weakly) constructible if all its cohomology sheaves $\mathcal{H}^j(\mathcal{F})$ are (weakly) constructible. Let $D^b_{(w)c}(X,A)$ be the full triangulated subcategory of $D^b(X,A)$ consisting of (weakly) constructible complexes.

The category $D^b_{(w)c}(X,A)$ is endowed with the *perverse t-structure*: that is, two strictly full subcategories ${}^pD^{\leq 0}_{(w)c}(X,A)$ and ${}^pD^{\geq 0}_{(w)c}(X,A)$ defined by stalk and, respectively, costalk vanishing conditions; see [2] or [11, Chapter X] for details. The heart of the perverse t-structure is the category of (weakly) *A-perverse sheaves* on X.

Artin's vanishing theorem for perverse sheaves is a key ingredient of both the local and global vanishing results in this paper. We recall here the version for weakly constructible complexes.

Theorem 2.3 ([11, Theorem 10.3.8]). *Let X be a Stein manifold.*

1. *For any* $\mathcal{F} \in {}^pD^{\leq 0}_{wc}(X,A)$, $H^k(X,\mathcal{F}) = 0$ *for* $k > 0$.
2. *For any* $\mathcal{F} \in {}^pD^{\geq 0}_{wc}(X,A)$, $H^k_c(X,\mathcal{F}) = 0$ *for* $k < 0$.

2.3. Sabbah specialisation complex

In this subsection, let X be a connected complex manifold. For $1 \leq k \leq n$, let $f_k : X \to \mathbb{C}$ be holomorphic functions, and let $D_k = f_k^{-1}(0)$ be the corresponding divisors. Set $\bigcup_{1\leq k\leq n} D_k = D$, with complement $X \setminus D = U$. Let

$$F = (f_1, \ldots, f_n)\colon X \to \mathbb{C}^n,$$

and denote by $F_U : U \to (\mathbb{C}^*)^n$ the restriction of F to U. Let $i : D \hookrightarrow X$ and $j : U \hookrightarrow X$ be the closed and open embeddings, respectively. Let $\mathcal{L}_{(\mathbb{C}^*)^n}$ be the universal local system on $(\mathbb{C}^*)^n$, and let

$$\mathcal{L}^F_U = F_U^*(\mathcal{L}_{(\mathbb{C}^*)^n}).$$

We make the following.

Definition 2.4. The Sabbah specialisation functor is defined as

$$\Psi_F : D^b_{(w)c}(X,A) \to D^b_{(w)c}(D,R), \quad \mathcal{F} \mapsto i^*Rj_*\big(\mathcal{F}|_U \otimes_A \mathcal{L}^F_U\big),$$

where $R = A[\pi_1((\mathbb{C}^*)^n)]$.

Remark 2.5. (i) The above definition is similar to that of [6, Definition 3.2], and it differs slightly from Sabbah's initial definition [19, Definition 2.2.7], where one has to restrict further to $\cap_k D_k$. Both [6] and [19] assume $A = \mathbb{C}$.

(ii) When $n = 1$ and A is a field, the functor Ψ_F is noncanonically isomorphic to Deligne's (shifted/perverse) nearby cycle functor $\psi_F[-1]$. In this case, Ψ_F is an exact functor with respect to the perverse t-structures (see [5, Theorem 1.2], as well as the discussion in [19, Section 2.2.9]).

3. *t*-exactness of the Sabbah specialisation functor

As already mentioned in Remark 2.5, it is known that when defined over a field, the univariate Sabbah specialisation functor is exact with respect to the perverse t-structures. The proof uses the (stalkwise) isomorphism between the univariate Sabbah specialisation functor and the perverse nearby cycle functor to conclude the right t-exactness and then uses Verdier duality to deduce the left t-exactness. However, this proof does not work when the ground field is replaced by a general ring since the Verdier duality may no longer exchange the subcategories ${}^pD^{\leq 0}$ and ${}^pD^{\geq 0}$. Instead, one needs to also compute costalks. In this section, we prove the t-exactness of the univariate Sabbah specialisation functor over a general ring (Theorem 3.1) by showing a costalk formula (Proposition 3.3). Consequently, we prove that the Sabbah specialisation functor in any number of variables is also t-exact (Corollary 3.4). Throughout this section, we work with bounded weakly constructible complexes.

Let X be a complex manifold, and let $f\colon X \to \mathbb{C}$ be a holomorphic function. Let $D = f^{-1}(0)$, and set $U = f^{-1}(\mathbb{C}^*)$. Let $i\colon D \hookrightarrow X$ and $j\colon U \hookrightarrow X$ be the closed and open embeddings, respectively.

Fixing, as before, a commutative Noetherian ring A of finite cohomological dimension, we let $\mathcal{L}_{\mathbb{C}^*}$ be the universal $A[\pi_1(\mathbb{C}^*)]$-local system on $\mathbb{C}^*$ and denote its pullback to U by

$$\mathcal{L}_U^f := f^*\mathcal{L}_{\mathbb{C}^*}.$$

Given a weakly A-constructible complex $\mathcal{F}$ on X, the univariate Sabbah specialisation complex of $\mathcal{F}$ is the following object in $D^b_{wc}(D, A[\pi_1(\mathbb{C}^*)])$:

$$\Psi_f(\mathcal{F}) = i^*Rj_*\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^f\big).$$

Theorem 3.1. *The Sabbah specialisation functor*

$$\Psi_f : D^b_{wc}(X, A) \to D^b_{wc}\big(D, A[\pi_1(\mathbb{C}^*)]\big)$$

is t-exact with respect to the perverse t-structures.

Fixing a chart of X near $x \in D$, we consider two real-valued functions on this chart: r is the Euclidean distance to x, and d is the function given by $d(y) = |f(y) - f(x)|$.

Lemma 3.2. *Let $\mathcal{F}$ be a weakly A-constructible complex on X. Choose $0 < \epsilon << \delta << 1$. Define*

$$\Pi_{\epsilon,\delta} := \{y \in X \mid r(y) < \delta, 0 < d(y) < \epsilon\} \quad \textit{and} \quad \Delta_\epsilon^* := \{z \in \mathbb{C}^* \mid |z| < \epsilon\}.$$

Let $f' : \Pi_{\epsilon,\delta} \to \Delta_\epsilon^*$ be the restriction of f, which is the Milnor fibration of f at x. Then $Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}})$ is locally constant on Δ_ϵ^*: that is, it has locally constant cohomology sheaves.

Proof. It is a well-known fact that (see, e.g., [21, Corollary 4.2.2])

$$Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}}) \cong \mathbb{D}\, Rf'_*\, \mathbb{D}(\mathcal{F}|_{\Pi_{\epsilon,\delta}}),$$

where $\mathbb{D}$ denotes the Verdier dualising functor. By [21, Definition 5.1.1 and Example 5.1.4], $Rf'_*\,\mathbb{D}(\mathcal{F}|_{\Pi_{\epsilon,\delta}})$ is locally constant. Hence, $Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}})$ is also locally constant. □

The following result shows that the Sabbah specialisation functor Ψ_f has the same costalks as Deligne's perverse nearby cycle functor $\psi_f[-1]$. In view of Remark 2.5(2), one would expect that there is a noncanonical isomorphism between these two functors. However, this fact, if true, is not needed in this paper.

Proposition 3.3. *Let $\mathcal{F}$ be a weakly A-constructible complex on X. Then*

$$H^k_x\big(D, \Psi_f(\mathcal{F})\big) = H^{k-1}_c\big(M_f, \mathcal{F}|_{M_f}\big), \tag{3.1}$$

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where M_f is a local Milnor fibre of f at $x \in D$ and $H^*_x(-)$ denotes the local cohomology at $x \in X$. Here, M_f can be identified with the fibre of f' from the preceding lemma.

Proof. By applying the attaching triangle

$$j_! j^! \to id \to i_* i^* \xrightarrow{+1}$$

to $Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U)$, we get the distinguished triangle

$$j_!(\mathcal{F}|_U \otimes \mathcal{L}^f_U) \to Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U) \to i_* i^* Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U) \xrightarrow{+1}. \tag{3.2}$$

Let $i_x : \{x\} \hookrightarrow X$ and $k_x : \{x\} \hookrightarrow D$ be the inclusion maps with $i_x = i \circ k_x$. Applying the functor $i^!_x$ to the triangle (3.2) and using the fact that $i^!_x Rj_* = 0$, we get from the corresponding cohomology long exact sequence that for any $k \in \mathbb{Z}$,

$$H^k\big(i^!_x i_* i^* Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U)\big) \cong H^{k+1}\big(i^!_x j_!(\mathcal{F}|_U \otimes \mathcal{L}^f_U)\big).$$

Since

$$i^!_x i_* i^* Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U) = k^!_x i^! i_* i^* Rj_*(\mathcal{F}|_U \otimes \mathcal{L}^f_U) = k^!_x \Psi_f(\mathcal{F}),$$

we get, for any $k \in \mathbb{Z}$, an isomorphism

$$H^k_x\big(D, \Psi_f(\mathcal{F})\big) := H^k(k^!_x \Psi_f(\mathcal{F})) \cong H^{k+1}\big(i^!_x j_!(\mathcal{F}|_U \otimes \mathcal{L}^f_U)\big). \tag{3.3}$$

For $0 < \epsilon << \delta << 1$, we have

$$\begin{aligned} H^{k+1}\big(i^!_x j_!(\mathcal{F}|_U \otimes \mathcal{L}^f_U)\big) &\cong H^{k+1}_c\big(\{y \in X \mid r(y) < \delta, d(y) < \epsilon\}, j_!(\mathcal{F}|_U \otimes \mathcal{L}^f_U)\big) \\ &\cong H^{k+1}_c\big(\{y \in X \mid r(y) < \delta, 0 < d(y) < \epsilon\}, \mathcal{F}|_U \otimes \mathcal{L}^f_U\big) \\ &\cong H^{k+1}_c\big(\Delta^\circ_\epsilon, Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}}) \otimes \mathcal{L}_{\mathbb{C}^*}|_{\Delta^*_\epsilon}\big), \end{aligned} \tag{3.4}$$

where the first isomorphism can be deduced, for example, from [17, Proposition 7.2.5], and the last follows from the projection formula.

Let $\mathrm{Exp} : \mathbb{C} \to \mathbb{C}^*$ be the universal covering map, and let $\mathrm{Exp}_\epsilon : \mathrm{Exp}^{-1}(\Delta^\circ_\epsilon) \to \Delta^\circ_\epsilon$ be its restriction. Then $\mathcal{L}_{\mathbb{C}^*} \cong \mathrm{Exp}_! \underline{A}_{\mathbb{C}}$. By the projection formula, we have

$$\begin{aligned} H^{k+1}_c\big(\Delta^\circ_\epsilon, Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}}) \otimes \mathcal{L}_{\mathbb{C}^*}|_{\Delta^*_\epsilon}\big) &\cong H^{k+1}_c\big(\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon), \mathrm{Exp}^*_\epsilon Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}}) \otimes \underline{A}_{\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon)}\big) \\ &\cong H^{k+1}_c\big(\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon), \mathrm{Exp}^*_\epsilon Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}})\big). \end{aligned} \tag{3.5}$$

Since $\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon)$ is a real two-dimensional contractible manifold, $H^2_c(\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon), A) \cong A$ and $H^k_c(\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon), A) = 0$ for $k \neq 2$. Since $Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}})$ is locally constant and its stalk is isomorphic to $R\Gamma_c(M_f, \mathcal{F}|_{M_f})$, we get by the Künneth formula that

$$H^{k+1}_c\big(\mathrm{Exp}^{-1}(\Delta^\circ_\epsilon), \mathrm{Exp}^*_\epsilon Rf'_!(\mathcal{F}|_{\Pi_{\epsilon,\delta}})\big) \cong H^{k-1}_c(M_f, \mathcal{F}|_{M_f}). \tag{3.6}$$

Now the desired formula given by equation (3.1) follows from equations (3.3), (3.4), (3.5) and (3.6). □

Proof of Theorem 3.1. Since $\mathcal{L}^f_U$ is a local system of free A-modules, the tensor product $\otimes_A \mathcal{L}^f_U$ is t-exact. Since j is an open embedding of a hypersurface complement, it is a quasi-finite Stein morphism, and hence Rj_* is also t-exact (see, e.g., [18, Proposition 3.29, Example 3.67, Theorem 3.70]). The right t-exactness of Ψ_f follows now from the right t-exactness of i^*.

For the left t-exactness of Ψ_f, we need to check the costalk vanishing conditions on each stratum. By Proposition 3.3, we notice that Ψ_f has the same costalks as the Deligne perverse nearby cycle functor $\psi_f[-1]$; see, for example, [21, Lemma 5.4.2] or [18, Proposition 4.11] for the costalk calculation of the latter. Then the assertion follows from the corresponding result for the (left) t-exact functor $\psi_f[-1]$; see, for example, [21, Theorem 6.0.2] or [18, Theorem 4.22(2)]. □

Next, we show how to derive the *t*-exactness of the multivariate Sabbah specialisation functor from the univariate case of Theorem 3.1.

Corollary 3.4. *Under the notations of Definition 2.4, the multivariate Sabbah specialisation functor* $\Psi_F : D^b_{wc}(X, A) \to D^b_{wc}(D, R)$ *is t-exact with respect to the perverse t-structures.*

Proof. Without loss of generality, we assume that $n \geq 2$. Let $f = f_1 \cdots f_n$, and let $f_U : U \to \mathbb{C}^*$ be the restriction of f to U. As before, denote $f_U^*(\mathcal{L}_{\mathbb{C}^*})$ and $F_U^*(\mathcal{L}_{(\mathbb{C}^*)^n})$ by $\mathcal{L}_U^f$ and $\mathcal{L}_U^F$, respectively.

Using the natural isomorphisms $\pi_1((\mathbb{C}^*)^n) \cong \mathbb{Z}^n$ and $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$, the holomorphic map

$$\Pi : (\mathbb{C}^*)^n \to \mathbb{C}^*, \quad (z_1, \ldots, z_n) \mapsto z_1 \cdots z_n$$

induces the homomorphism

$$\xi : \mathbb{Z}^n \to \mathbb{Z}, \quad (a_1, \ldots, a_n) \mapsto a_1 + \cdots + a_n$$

on the fundamental groups. Then we have a natural isomorphism of rank one $A[\mathbb{Z}]$-local systems,

$$\mathcal{L}_U^f \cong \mathcal{L}_U^F \otimes_{A[\mathbb{Z}^n]} A[\mathbb{Z}],$$

where the $A[\mathbb{Z}^n]$-module structure on $A[\mathbb{Z}]$ is induced by ξ.

Fix a splitting $\mathbb{Z}^n = \mathrm{Ker}(\xi) \oplus \mathbb{Z}$ of the short exact sequence

$$0 \to \mathrm{Ker}(\xi) \to \mathbb{Z}^n \xrightarrow{\xi} \mathbb{Z} \to 0,$$

which induces a splitting of the short exact sequence of affine tori

$$1 \to \mathrm{Ker}(\Pi) \to (\mathbb{C}^*)^n \xrightarrow{\Pi} \mathbb{C}^* \to 1.$$

By the definition of the universal local system, the above splitting of affine tori induces an isomorphism of A-local systems

$$\mathcal{L}_{(\mathbb{C}^*)^n} \cong \mathcal{L}_{\mathrm{Ker}(\Pi)} \otimes_A \mathcal{L}_{\mathbb{C}^*}.$$

We denote the pullback $F^*\mathcal{L}_{\mathrm{Ker}(\Pi)}$ by $\mathcal{L}'_U$. Then taking the pullback of the above isomorphism, we have

$$\mathcal{L}_U^F \cong \mathcal{L}'_U \otimes_A \mathcal{L}_U^f$$

as A-local systems.

Since $\mathcal{F}$ is a weakly A-constructible complex on X and $\mathcal{L}'_U$ is a local system of free A-modules, $Rj_*(\mathcal{F}|_U \otimes_A \mathcal{L}'_U)$ is a weakly A-constructible complex on X (see [18, Theorem 2.6(c)]). Therefore, considering $\Psi_F(\mathcal{F})$ as an object in $D^b_{wc}(D, A[\mathbb{Z}])$ under the functor between group algebras induced by $\xi : \mathbb{Z}^n \to \mathbb{Z}$, we have

$$\Psi_F(\mathcal{F}) \cong i^*Rj_*(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F) \cong i^*Rj_*((\mathcal{F}|_U \otimes_A \mathcal{L}'_U) \otimes_A \mathcal{L}_U^f) \cong \Psi_f\left(Rj_*(\mathcal{F}|_U \otimes_A \mathcal{L}'_U)\right).$$

Since $\mathcal{F}$ is a weakly A-perverse sheaf on X and $\mathcal{L}'_U$ is a local system of free A-modules, the tensor product $\mathcal{F}|_U \otimes_A \mathcal{L}'_U$ is a weakly A-perverse sheaf on U. Since $j : U \hookrightarrow X$ is an open embedding whose complement is a divisor, j is a quasi-finite Stein mapping, and hence the pushforward $Rj_*(\mathcal{F}|_U \otimes_A \mathcal{L}'_U)$ is a weakly A-perverse sheaf on X. By Theorem 3.1, $\Psi_f(Rj_*(\mathcal{F}|_U \otimes_A \mathcal{L}'_U))$ is a weakly A-perverse sheaf. Since the definition of the perverse t-structure does not involve the ring of coefficients, we conclude that $\Psi_F(\mathcal{F})$ is also perverse as a (weakly) R-constructible complex. □

4. Local vanishing of the multivariate Sabbah specialisation functor

Let X be a complex manifold. For $1 \leq k \leq n$, let $f_k : X \to \mathbb{C}$ be holomorphic functions as in Section 2.3, with $D_k = f_k^{-1}(0)$ the corresponding divisors. Set

$$F = (f_1, \dots, f_n) : X \to \mathbb{C}^n.$$

For any subset $I \subset \{1, \dots, n\}$, let

$$D_I = \bigcap_{k \in I} D_k \quad \text{and} \quad D_I^\circ = D_I \setminus \bigcup_{m \notin I} D_m.$$

For a subset $J \subset \{1, \dots, n\}$, we let $D_{>J} = \bigcup_{I \supsetneq J} D_I$. We also let

$$D_{\geq m} = \bigcup_{|I|=m} D_I \quad \text{and} \quad D_{\geq m}^\circ = D_{\geq m} \setminus D_{\geq m+1}.$$

Let $D = D_{\geq 1}$ and $U = X \setminus D$. Let $F_U : U \to (\mathbb{C}^*)^n$ be the restriction of F to U.

If S is an open submanifold of X, we denote the open embedding by $j_S : S \hookrightarrow X$. If S is a locally closed, but not open, subvariety of X, we denote the inclusion map by $i_S : S \hookrightarrow X$.

Remark 4.1. Here we do not assume that the divisors D_k define a local complete intersection. So the codimension of D_I° is only $\leq |I|$. In fact, D_I° may not be equidimensional.

Given any weakly A-constructible complex $\mathcal{F}$ on X and any nonempty subset $I \subset \{1, \dots, n\}$, we define

$$\Psi_{D_I^\circ}(\mathcal{F}) := i_{D_I^\circ}^* Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F),$$

that is, the restriction of the Sabbah specialisation complex $\Psi_F(\mathcal{F})$ to D_I°. Here, $\mathcal{L}_U^F$ is defined as in Section 2.3. While, of course, $\Psi_{D_I^\circ}(\mathcal{F})$ also depends on F, we drop this dependence from our notation to avoid clutter.

In this section, we prove the following.

Theorem 4.2. *Let $R = A[\pi_1((\mathbb{C}^*)^n)]$. Then the functor*

$$\Psi_{D_I^\circ} : D^b_{wc}(X, A) \to D^b_{wc}(D_I^\circ, R)$$

is t-exact with respect to the perverse t-structures.

First, we show that the functor $\Psi_{D_I^\circ}$ is right t-exact.

Lemma 4.3. *The functor $\Psi_{D_I^\circ}$ is right t-exact: that is, it maps ${}^pD^{\leq 0}_{wc}(X, A)$ to ${}^pD^{\leq 0}_{wc}(D_I^\circ, R)$.*

Proof. By definition, for any weakly constructible complex $\mathcal{F}$ on X, we have $\Psi_{D_I^\circ}(\mathcal{F}) \cong i_{D_I^\circ, D}^* \Psi_F(\mathcal{F})$, where $i_{D_I^\circ, D} : D_I^\circ \hookrightarrow D$ is the inclusion map. The right t-exactness of $\Psi_{D_I^\circ}$ follows from Corollary 3.4, together with the fact that the pullback functor $i_{D_I^\circ, D}^*$ is right t-exact (here we write $i_{D_I^\circ, D}$ as a composition of the closed inclusion of D_I into D, followed by the open inclusion of D_I° into D_I); see [2, 1.4.10, 1.4.12] □

The left t-exactness of $\Psi_{D_I^\circ}$ along zero-dimensional strata can be formulated as the following local vanishing result. In particular, the proposition is weaker than Theorem 4.2.

Proposition 4.4. *Let $x \in D_I^\circ$ be an arbitrary point, and denote the closed embedding by $i_x : \{x\} \hookrightarrow D_I^\circ$. Then for any constructible complex $\mathcal{F}$ in ${}^pD^{\geq 0}_{wc}(X, A)$, we have*

$$H^k\big(i_x^! \Psi_{D_I^\circ}(\mathcal{F})\big) = 0 \quad \text{for any} \quad k < 0. \tag{4.1}$$

Before proving the remaining left t-exactness of the functor $\Psi_{D_I^\circ}$, we need the following lemma, which is a local cohomology version of [1, spectral sequence (10)]. We give here a different proof, and we will use similar arguments later to show Lemma 5.2. Here, we use local cohomology instead of the cohomology of the exceptional pullback to avoid introducing more notations for the inclusion maps from x to various spaces.

Lemma 4.5. *Assume $|I| = m$. There is a spectral sequence*

$$E_1^{pq} = \begin{cases} H_x^{p+q+1}\Big(X, (i_{D^\circ_{\geq -q}})_! i^*_{D^\circ_{\geq -q}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)\Big), & \text{when } 1-m \leq q \leq -1 \\ H_x^p\Big(X, i_{D*} i_D^* Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)\Big), & \text{when } q = 0 \\ 0, & \text{otherwise} \end{cases}$$
$$\Longrightarrow H_x^{p+q}\big(D_I^\circ, \Psi_{D_I^\circ}(\mathcal{F})\big).$$

Proof. Consider the double complex $\mathcal{A}^{\bullet,\bullet}$ of weakly constructible sheaves on X defined by

$$\mathcal{A}^{p,q} = (i_{D_{\geq p+1}})_* i^*_{D_{\geq p+1}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)$$

when $0 \leq p = -q \leq m-1$ and when $0 \leq p = -1-q \leq m-2$. For other p, q, we let $\mathcal{A}^{p,q} = 0$.

By base change, we have a natural isomorphism

$$(i_{D_{\geq p+1}})_* i^*_{D_{\geq p+1}} (i_{D_{\geq p}})_* i^*_{D_{\geq p}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big) \cong (i_{D_{\geq p+1}})_* i^*_{D_{\geq p+1}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big),$$

and hence the adjunction distinguished triangle can be written as

$$\begin{aligned}(i_{D^\circ_{\geq p}})_! i^*_{D^\circ_{\geq p}} Rj_{U*}\big(\mathcal{F} \otimes_A \mathcal{L}_U^F\big) &\to (i_{D_{\geq p}})_* i^*_{D_{\geq p}} Rj_{U*}\big(\mathcal{F} \otimes_A \mathcal{L}_U^F\big) \\ &\to (i_{D_{\geq p+1}})_* i^*_{D_{\geq p+1}} Rj_{U*}\big(\mathcal{F} \otimes_A \mathcal{L}_U^F\big) \xrightarrow{+1}.\end{aligned} \tag{4.2}$$

Now we define all the horizontal differentials d' to be zero except $1 \leq p \leq m-1$, where we let $d' : \mathcal{A}^{p-1,-p} \to \mathcal{A}^{p,-p}$ be the second map in equation (4.2). Similarly, we define all the vertical differentials d'' to be zero except $0 \leq p \leq m-2$, when we let $d'' : \mathcal{A}^{p,-p-1} \to \mathcal{A}^{p,-p}$ be the identity maps.

Since all column complexes $(\mathcal{A}^{p,\bullet}, d'')$ are exact except $p = m-1$, and the $(m-1)$th column is equal to $(i_{D_{\geq m}})_* i^*_{D_{\geq m}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)[m-1]$, we have an isomorphism

$$\mathrm{tot}(\mathcal{A}^{\bullet,\bullet}) \cong (i_{D_{\geq m}})_* i^*_{D_{\geq m}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)$$

in $D^b_{w-c}(X, R)$, where $\mathrm{tot}(\mathcal{A}^{\bullet,\bullet})$ is the total complex of $\mathcal{A}^{\bullet,\bullet}$ considered an object in $D^b_{w-c}(X, R)$.

Consider the filtration $F_q := \mathcal{A}^{\bullet,\leq q}$ of $\mathcal{A}^{\bullet,\bullet}$ by row truncations. The graded pieces of the filtration are the rows in $\mathcal{A}^{\bullet,\bullet}$. Using the adjunction distinguished triangle, we have

$$\mathrm{tot}\big(\mathrm{Gr}_q(\mathcal{A}^{\bullet,\bullet})\big) \cong \begin{cases} (i_{D^\circ_{\geq -q}})_! i^*_{D^\circ_{\geq -q}} Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big)[1] & \text{when } 1-m \leq q \leq -1 \\ i_{D*} i_D^* Rj_{U*}\big(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F\big) & \text{when } q = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4.3}$$

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where $\mathrm{tot}(\mathrm{Gr}_q(\mathcal{A}^{\bullet,\bullet}))$ is the total complex of $\mathrm{Gr}_q(\mathcal{A}^{\bullet,\bullet})$ viewed as an object in $D^b_{w-c}(X,R)$. Since $x \in D_I^\circ$ and D_I° is open in $D_{\geq m}$, the complexes $R(i_{D_{\geq m}})_* i^*_{D_{\geq m}} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F)$ and $R\big(i_{D_I^\circ}\big)_* i^*_{D_I^\circ} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F)$ are quasi-isomorphic in a neighbourhood of x. Hence we have isomorphisms

$$\begin{aligned} H_x^{p+q}\Big(X, (i_{D_{\geq m}})_* i^*_{D_{\geq m}} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F)\Big) &\cong H_x^{p+q}\Big(X, R(i_{D_I^\circ})_* i^*_{D_I^\circ} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F)\Big) \\ &\cong H_x^{p+q}\big(D_I^\circ, \Psi_{D_I^\circ}(\mathcal{F})\big). \end{aligned} \tag{4.4}$$

Taking local cohomology of the filtered complex $\mathrm{tot}(\mathcal{A}^{\bullet,\bullet})$, we have a spectral sequence

$$E_1^{pq} = H_x^{p+q}\big(X, \mathrm{tot}\big(\mathrm{Gr}_q(\mathcal{A}^{\bullet,\bullet})\big)\big) \to H_x^{p+q}\big(X, \mathrm{tot}(\mathcal{A}^{\bullet,\bullet})\big).$$

The isomorphisms given by equations (4.3) and (4.4) yield the spectral sequence in the lemma. □

Proof of Theorem 4.2 and Proposition 4.4. We simultaneously prove the theorem and the proposition using induction on $|I|$, the cardinality of I. When $|I| = 1$, Theorem 4.2, and hence Proposition 4.4, follow from Corollary 3.4 (since restriction to opens is t-exact). Fixing an integer $m \geq 2$, we assume that Theorem 4.2, and hence Proposition 4.4, hold for all I with $|I| < m$, and we want to show that they both hold for I with $|I| = m$.

To prove Proposition 4.4, we notice that the statement can be reduced to the case when D is a simple normal crossing divisor. In fact, consider the multivariate graph embedding

$$F^\dagger : X \to X \times \mathbb{C}^n, \quad x \mapsto (x, F(x)),$$

which restricts to a closed embedding $U \to X \times (\mathbb{C}^*)^n$. The local vanishing in equation (4.1) of Sabbah's specialisation functor for $F : X \to \mathbb{C}^n$ can be reduced to the local vanishing of the Sabbah specialisation functor for the projection $p_2 : X \times \mathbb{C}^n \to \mathbb{C}^n$. This is due to the following natural isomorphism

$$R\hat{F}^\dagger_* \Psi_F(\mathcal{F}) \cong \Psi_{p_2}(RF^\dagger_* \mathcal{F}),$$

where $\hat{F}^\dagger$ is the restriction of $F^\dagger$ to D.

Now we prove Proposition 4.4, assuming that D is a simple normal crossing divisor. We identify $A[\pi_1((\mathbb{C}^*)^n)]$ with $A[t_1^\pm, \dots, t_n^\pm]$ using the standard isomorphisms

$$A[\pi_1((\mathbb{C}^*)^n)] \cong A[\mathbb{Z}^n] \cong A[t_1^\pm, \dots, t_n^\pm].$$

Let B_x be a small polydisc in X centred at x, and let $U_x = B_x \cap U$. Without loss of generality, we assume that $I = \{1, \dots, m\}$. Since D is a normal crossing divisor, we have a natural isomorphism $A[\pi_1(U_x)] \cong A[t_1^\pm, \dots, t_m^\pm]$. Let $\mathcal{L}_{U_x}$ be the universal $A[t_1^\pm, \dots, t_m^\pm]$-local system on U_x. As $A[t_1^\pm, \dots, t_m^\pm]$-local systems on U_x, we have a noncanonical isomorphism

$$\mathcal{L}_U^F|_{U_x} \cong \mathcal{L}_{U_x} \otimes_A A[t_{m+1}^\pm, \dots, t_n^\pm].$$

Therefore,

$$\begin{aligned} H^k\big(i_x^! \Psi_{D_I^\circ}(\mathcal{F})\big) &= H^k\big(i_x^! i^*_{D_I^\circ} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^F)\big) \\ &\cong H_x^k\big(B_x, i^*_{D_I^\circ} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_{U_x})\big) \otimes_A A[t_{m+1}^\pm, \dots, t_n^\pm]. \end{aligned}$$

Thus it suffices to prove the vanishing in equation (4.1) under the following assumption, which we will make for the remainder of the proof of Proposition 4.4.

Assumption. *The space $X = B_x = \Delta^l$ is a small polydisc in $\mathbb{C}^l$ centred at the origin x, and $n = m$ and $f_1, \dots, f_m$ are the first m coordinate functions. In particular, $\mathcal{L}_U^F = \mathcal{L}_U$ and $U = U_x$.*

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We claim that

$$H_x^{p+q+1}\Big(X, (i_{D^\circ_{\geq -q}})_! i^*_{D^\circ_{\geq -q}} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)\Big) = 0 \tag{4.5}$$

for $1-m \leq q \leq -1$ and $p+q<0$. In fact, let $J \subset \{1,\dots,m\}$ with $|J| = -q$. To show equation (4.5), it suffices to show that for $1-m \leq q \leq -1$ and $p+q<0$,

$$H_x^{p+q+1}\Big(X, (i_{D^\circ_J})_! i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)\Big) = 0. \tag{4.6}$$

Without loss of generality, we assume that $J = \{1,\dots,-q\}$. By the above assumptions, $U = (\Delta^\circ)^m \times \Delta^{l-m}$, where Δ is a small disc in $\mathbb{C}$ centred at the origin, and $\Delta^\circ = \Delta \setminus \{0\}$. We further decompose U as $U = (\Delta^\circ)^{-q} \times (\Delta^\circ)^{m+q} \times \Delta^{l-m}$, and we let $\mathcal{L}_U^J$ and $\mathcal{L}_U^{J^c}$ be the pullback of the universal local systems $\mathcal{L}_{(\Delta^\circ)^{-q}}$ and $\mathcal{L}_{(\Delta^\circ)^{m+q}\times\Delta^{l-m}}$ to U, respectively. Then as $A[t_1^\pm,\dots,t_m^\pm]$-local systems,

$$\mathcal{L}_U \cong \mathcal{L}_U^J \otimes_A \mathcal{L}_U^{J^c},$$

where the $A[t_1^\pm,\dots,t_m^\pm]$-module structures on $\mathcal{L}_U^J$ and $\mathcal{L}_U^{J^c}$ are induced by the natural projections $\pi_1(U) \to \pi_1((\Delta^\circ)^{-q})$ and $\pi_1(U) \to \pi_1((\Delta^\circ)^{m+q})$, respectively. Thus, we have

$$\begin{aligned} i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U) &\cong i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^J \otimes_A \mathcal{L}_U^{J^c}) \\ &\cong i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^J) \otimes_A \mathcal{L}_{D^\circ_J}, \end{aligned} \tag{4.7}$$

where $\mathcal{L}_{D^\circ_J}$ is the universal local system on D°_J, and the second isomorphism follows from the fact that $\mathcal{L}_U^{J^c}$ extends as an $A[t^\pm_{-q+1},\dots,t^\pm_m]$-local system to $X \setminus (D_{-q+1} \cup \cdots \cup D_m)$, and the restriction of the extension to D°_J is isomorphic to $\mathcal{L}_{D^\circ_J}$.

Applying the inductive hypothesis in Theorem 4.2 to the space $X \setminus (D_{-q+1} \cup \cdots \cup D_m)$ and functions $f_1,\dots,f_{-q}$, it follows that

$$\mathcal{G} := i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U^J) \in {}^pD^{\geq 0}_{w-c}(D^\circ_J, A[t_1^\pm,\dots,t^\pm_{-q}]).$$

By equation (4.7), we have

$$H_x^{p+q+1}\Big(X, (i_{D^\circ_J})_! i^*_{D^\circ_J} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)\Big) \cong H_x^{p+q+1}\big(X, (i_{D^\circ_J})_!(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J})\big). \tag{4.8}$$

Consider the distinguished triangle

$$(i_{D^\circ_J})_!(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J}) \to R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J}) \to (i_{D_{>J}})_* i^*_{D_{>J}} R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J}) \xrightarrow{+1}.$$

Since $i^!_x R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J}) = 0$, the local cohomology long exact sequence implies that

$$\begin{aligned} H_x^{p+q+1}\big(X, (i_{D^\circ_J})_!(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J})\big) &\cong H_x^{p+q}\big(X, (i_{D_{>J}})_* i^*_{D_{>J}} R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J})\big) \\ &\cong H_x^{p+q}\big(D_{>J}, i^*_{D_{>J}} R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J})\big). \end{aligned} \tag{4.9}$$

Notice that the last term of the above isomorphism is equal to the $(p+q)$th cohomology of the costalk at x of the multivariate Sabbah specialisation functor applied to $\mathcal{G}$ with respect to the holomorphic functions $f_i|_{D_J}$ on D_J for $i \in \{1,\dots,m\} \setminus J$. Thus, by Corollary 3.4,

$$H_x^{p+q}\big(X, (i_{D_{>J}})_* i^*_{D_{>J}} R(i_{D^\circ_J})_*(\mathcal{G} \otimes_A \mathcal{L}_{D^\circ_J})\big) = 0 \tag{4.10}$$

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when $p+q<0$. Combining equations (4.8), (4.9) and (4.10), we have

$$H_x^{p+q+1}\Big(X,(i_{D_J^\circ})_! i^*_{D_J^\circ} Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)\Big)=0 \tag{4.11}$$

when $p+q<0$.

Notice that

$$H_x^{p+q}(X, i_{D*}i_D^* Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)) \cong H_x^{p+q}(D, i_D^* Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U)),$$

and the right-hand side is equal to the $(p+q)$th cohomology of the costalk at x of the Sabbah specialisation $\Psi_F(\mathcal{F})$. Thus, by Corollary 3.4, we have

$$H_x^{p+q}(X, i_{D*}i_D^* Rj_{U*}(\mathcal{F}|_U \otimes_A \mathcal{L}_U))=0 \tag{4.12}$$

when $p+q<0$.

Therefore, the vanishing in equation (4.1) follows from Lemma 4.5 and equations (4.11), (4.12). We have finished the proof of Proposition 4.4.

We now complete the proof of Theorem 4.2. In Lemma 4.3, we have proved the right t-exactness of $\Psi_{D_I^\circ}$. By Proposition 4.4, we know the left t-exactness of the functor $\Psi_{D_I^\circ}$ at zero-dimensional strata. The proof of the left t-exactness at higher-dimensional strata can be reduced to the case of zero-dimensional strata by using the standard normal slice arguments (see, e.g., [21, page 427]). □

The next corollary shows that Theorem 4.2 can be considered a generalisation of the t-exactness of the Sabbah specialisation functor.

Corollary 4.6. *Let $R=A[\pi_1((\mathbb{C}^*)^n)]$, and define the functor Ψ_{D_I} by*

$$\Psi_{D_I}: D^b_{wc}(X,A)\to D^b_{wc}(D_I,R),\quad \mathcal{F}\mapsto i^*_{D_I}Rj_{U*}(\mathcal{F}|_U\otimes_A\mathcal{L}_U).$$

Then Ψ_{D_I} is t-exact with respect to the perverse t-structures.

Proof. It suffices to show that if $\mathcal{P}$ is a weakly constructible A-perverse sheaf on X, then $\Psi_{D_I}(\mathcal{P})$ is perverse. Notice that $\Psi_{D_I}(\mathcal{P})$ is equal to an iterated extension of constructible complexes $(i_{D_J,D_I})_*(i_{D_J^\circ,D_J})_!\Psi_{D_J^\circ}(\mathcal{P})$ for $J\supset I$. Since D_J° is a hypersurface complement in D_J and D_J is closed in D_I, Theorem 4.2 implies that $(i_{D_J,D_I})_*(i_{D_J^\circ,D_J})_!\Psi_{D_J^\circ}(\mathcal{P})$ is a perverse sheaf supported on D_J. Since extensions of perverse sheaves are also perverse, $\Psi_{D_I}(\mathcal{P})$ is a perverse sheaf on D_I. □

5. Non-abelian Mellin transformations

First, we recall and extend the notations of Theorem 1.1.

Let X be a compact complex manifold. Let $E=\bigcup_{1\le k\le d}E_k$ be a normal crossing divisor on X, and let $U=X\setminus E$ with inclusion map $j: U\hookrightarrow X$. For any nonempty subset $I\subset\{1,\dots,d\}$, let $E_I=\bigcap_{i\in I}E_i$ and $E_I^\circ=E_I\setminus\bigcup_{j\notin I}E_j$. Let $E_{\ge m}=\bigcup_{|I|=m}E_I$, and let $E^\circ_{\ge m}=E_{\ge m}\setminus E_{\ge m+1}$.[1] For any open submanifold S of X, we denote the open embedding by $j_S: S\hookrightarrow X$. For a locally closed, but not open, submanifold S of X, we denote the inclusion map by $i_S: S\hookrightarrow X$. Let $\mathcal{L}_U$ be the universal $A[\pi_1(U)]$-local system on U.

Proposition 5.1. *Under the above notations, let $\mathcal{P}\in D^b_c(X,A)$ be an A-perverse sheaf. Given a nonempty subset $I\subset\{1,\dots,d\}$, assume that at every point $x\in E_I^\circ$, the local fundamental group maps injectively to the global fundamental group: that is, the condition (2) in Theorem 1.1 holds. Then*

$$i^*_{E_I^\circ}Rj_*(\mathcal{P}|_U\otimes\mathcal{L}_U)$$

is a weakly constructible A-perverse sheaf on E_I°.

[1] Here we denote divisors by E, as opposed to D in the earlier sections, since no defining equations are present in this case.

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Proof. It suffices to check the statement locally on E_I°. For an arbitrary point $x \in E_I^\circ$, let B_x be a small ball in X centred at x, and let $U_x = B_x \cap U$. By Lemma 2.2, $\mathcal{L}_U|_{U_x}$ is a direct sum of possibly infinitely many copies of $\mathcal{L}_{U_x}$. Let $i_{E_I^\circ \cap B_x, B_x}\colon E_I^\circ \cap B_x \to B_x$ and $j_{U_x,B_x}\colon U_x \to B_x$ be the closed and open embeddings, respectively. The restriction of the complex $i^*_{E_I^\circ} Rj_*(\mathcal{P}|_U \otimes \mathcal{L}_U)$ to $E_I^\circ \cap B_x$ is equal to $(i_{E_I^\circ \cap B_x, B_x})^* R(j_{U_x,B_x})_*(\mathcal{P}|_{U_x} \otimes \mathcal{L}_U|_{U_x})$, which is just a direct sum of copies of $(i_{E_I^\circ \cap B_x, B_x})^* R(j_{U_x,B_x})_*(\mathcal{P}|_{U_x} \otimes \mathcal{L}_{U_x})$. Hence, by Theorem 4.2, the weakly constructible complex $(i_{E_I^\circ \cap B_x, B_x})^* R(j_{U_x,B_x})_*(\mathcal{P}|_{U_x} \otimes \mathcal{L}_U|_{U_x})$ is an A-perverse sheaf on $E_I^\circ \cap B_x$. □

Before proving Theorem 1.1, we need the following lemma, similar to Lemma 4.5.

Lemma 5.2. *There exists a spectral sequence*

$$E_1^{pq} = \begin{cases} H^{p+q}\Big(X, (i_{E^\circ_{\geq -q+1}})_! i^*_{E^\circ_{\geq -q+1}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U)\Big) & \text{when } q \leq 0 \\ 0 & \text{when } q > 0 \end{cases}$$
$$\Longrightarrow H^{p+q}\big(X, i_* i^* Rj_*(\mathcal{P}|_U \otimes_A \mathcal{L}_U)\big).$$

Here, if $E^\circ_{\geq -q+1} = \emptyset$, our convention is that both $\big(i_{E^\circ_{\geq -q+1}}\big)_!$ and $i^*_{E^\circ_{\geq -q+1}}$ are zero functors.

Proof. As in the proof of Lemma 4.5, we define a double complex $\mathcal{B}^{\bullet,\bullet}$ by

$$\mathcal{B}^{p,q} = (i_{E_{\geq p}})_* i^*_{E_{\geq p}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U)$$

when $p = -q \geq 0$ or $p - 1 = -q \geq 0$. For other values of p, q, we let $\mathcal{B}^{p,q} = 0$. Consider the adjunction distinguished triangle

$$\begin{aligned} &\big(i_{E^\circ_{\geq p}}\big)_! i^*_{E^\circ_{\geq p}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \to \big(i_{E_{\geq p}}\big)_* i^*_{E_{\geq p}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \\ &\qquad \to \big(i_{E_{\geq p+1}}\big)_* i^*_{E_{\geq p+1}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \xrightarrow{+1}. \end{aligned} \tag{5.1}$$

We define all horizontal differentials d' to be zero except $p \geq 0$, in which case we let $d' : \mathcal{B}^{p,-p} \to \mathcal{B}^{p+1,-p}$ be the second map in equation (5.1). We define all vertical differential d'' to be zero except $p \geq 1$, when we let $d'' : \mathcal{B}^{p,-p} \to \mathcal{B}^{p,-p+1}$ be the identity map. For the rest of the proof, we can use the same arguments as in the proof of Lemma 4.5, with local cohomology replaced by hypercohomology. □

We now have all the ingredients for proving our main result, Theorem 1.1.

Proof of Theorem 1.1. Given any A-perverse sheaf $\mathcal{P}$ on X, we need to show that

$$H^k(\mathfrak{M}_*(\mathcal{P}|_U)) = 0 \text{ for } k \neq 0.$$

By assumption, U is a Stein manifold. Since $\mathcal{P}|_U \otimes_A \mathcal{L}_U$ is a weakly constructible A-perverse sheaf, by Artin's vanishing Theorem 2.3, we get

$$H^k(\mathfrak{M}_*(\mathcal{P})) \cong H^k\big(U, \mathcal{P}|_U \otimes_A \mathcal{L}_U\big) = 0 \quad \text{for } \ k > 0.$$

To show the vanishing in negative degrees, we consider the following distinguished triangle:

$$j_!(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \to Rj_*(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \to i_* i^* Rj_*(\mathcal{P}|_U \otimes_A \mathcal{L}_U) \xrightarrow{+1},$$

where $i\colon E \hookrightarrow X$ and $j\colon U \hookrightarrow X$ are the closed and open embeddings, respectively. Since X and E are compact, the associated hypercohomology long exact sequence reads as

$$\cdots \to H^k_c\big(U, \mathcal{P}|_U \otimes_A \mathcal{L}_U\big) \to H^k\big(U, \mathcal{P}|_U \otimes_A \mathcal{L}_U\big) \to H^k\big(E, i^* Rj_*(\mathcal{P}|_U \otimes_A \mathcal{L}_U)\big) \to \cdots \tag{5.2}$$

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By Proposition 5.1, as a weakly constructible complex on $E^{\circ}_{\geq -q+1}$, $i^*_{E^{\circ}_{\geq -q+1}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U)$ is perverse. By assumption, $E^{\circ}_{\geq -q+1}$ is a disjoint union of Stein manifolds. Thus, by Artin's vanishing Theorem 2.3, we have that

$$H^{p+q}\Big(X, (i_{E^{\circ}_{\geq -q+1}})_! i^*_{E^{\circ}_{\geq -q+1}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U)\Big) \cong H^{p+q}_c\Big(E^{\circ}_{\geq -q+1}, i^*_{E^{\circ}_{\geq -q+1}} Rj_{U*}(\mathcal{P}|_U \otimes_A \mathcal{L}_U)\Big)$$

vanishes for $q \leq 0$ and $p + q < 0$. Therefore, by Lemma 5.2, we have

$$H^k(X, i_* i^* Rj_*(\mathcal{P}|_U \otimes_A \mathcal{L}_U)) = 0, \quad \text{for } k < 0. \tag{5.3}$$

Furthermore, since $\mathcal{P}|_U \otimes_A \mathcal{L}_U$ is a weakly constructible A-perverse sheaf on the Stein manifold U, we get by Artin's vanishing Theorem 2.3 that

$$H^k_c(U, \mathcal{P}|_U \otimes_A \mathcal{L}_U) = 0, \quad \text{for } k < 0. \tag{5.4}$$

By plugging equations (5.3) and (5.4) into the long exact sequence (5.2), we conclude that

$$H^k(\mathfrak{M}_*(\mathcal{P})) \cong H^k(U, \mathcal{P}|_U \otimes_A \mathcal{L}_U) = 0, \quad \text{for } k < 0,$$

thus completing the proof of Theorem 1.1. □

6. Some applications

One of our motivations for studying the t-exactness of the non-abelian Mellin transformation is to extend results of Denham-Suciu [7] concerning duality spaces (in the sense of Bieri and Eckmann [4]). Let us first recall the following definition.

Definition 6.1. Let U be a topological space with fundamental group G, which is homotopy equivalent to a connected, finite-type CW-complex. Let $\mathcal{L}_U$ be the universal $\mathbb{Z}[G]$-local system on U. We say that U is a *duality space of dimension n* if $H^k(U, \mathcal{L}_U) = 0$ for $k \neq n$ and $H^n(U, \mathcal{L}_U)$ is a torsion-free $\mathbb{Z}$-module.

It is proved in [7] that for any smooth, connected, complex quasi-projective variety U satisfying the conditions of Theorem 1.1, the topological space U is a duality space of dimension $\dim_{\mathbb{C}} U$. In particular, complements of essential hyperplane arrangements, elliptic arrangements or toric arrangements are examples of duality spaces. The aim of this section is to construct new examples of duality spaces that are non-affine or singular varieties (see Proposition 6.4 and Corollary 6.6). In fact, our results apply more generally to complex analytic varieties.

We begin with the following general result.

Proposition 6.2. *Let $U \subset X$ be complex manifolds satisfying the conditions of Theorem 1.1. Let $f : Y \to U$ be a proper map from a pure d-dimensional complex analytic variety Y, satisfying the following assumptions:*

(i) *f induces an isomorphism on fundamental groups.*
(ii) *$Rf_*(\mathbb{Q}_Y[d])$ and $Rf_*(\mathbb{F}_Y[d])$ are perverse sheaves on U, where $\mathbb{F} = \mathbb{F}_p$ is a field of prime characteristic p.*
(iii) *$Rf_*(\mathbb{Z}_Y[d]) \in {}^pD^{\leq 0}_c(U, \mathbb{Z})$ is semi-perverse.*

Then Y is a duality space of dimension $d = \dim_{\mathbb{C}} Y$. In particular, when f is the identity map of U, we get that U itself is a duality space of dimension $\dim_{\mathbb{C}} U$.

Proof. We follow similar arguments as in the proof of [14, Theorem 4.11(1)]. First, by definition, we have that

$$H^k(Y, \mathcal{L}_Y) \cong H^k(\mathfrak{M}^Y_*(\mathbb{Z}_Y)).$$

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By (i), we get that $\mathcal{L}_Y \cong f^*\mathcal{L}_U$ with coefficients in $A = \mathbb{Z}, \mathbb{Q}$ or $\mathbb{F}$. By the properness of f and projection formulas, we have

$$\mathfrak{M}_*^Y(\mathbb{Z}_Y) \cong \mathfrak{M}_*^U(Rf_*\mathbb{Z}_Y)$$

as well as

$$\mathfrak{M}_*^Y(\mathbb{Z}_Y) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathfrak{M}_*^Y(\mathbb{Q}_Y) \cong \mathfrak{M}_*^U(Rf_*\mathbb{Q}_Y) \tag{6.1}$$

and

$$\mathfrak{M}_*^Y(\mathbb{Z}_Y) \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{F} \cong \mathfrak{M}_*^Y(\mathbb{F}_Y) \cong \mathfrak{M}_*^U(Rf_*\mathbb{F}_Y). \tag{6.2}$$

By Theorem 1.1, the assumptions (ii) and (iii) and the isomorphisms in equations (6.1) and (6.2), the complexes $\mathfrak{M}_*^Y(\mathbb{Z}_Y[d]) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathfrak{M}_*^Y(\mathbb{Z}_Y[d]) \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{F}$ are concentrated in degrees zero and the cohomology of $\mathfrak{M}_*^Y(\mathbb{Z}_Y[d])$ vanishes in positive degrees. Thus, by Lemma 6.3 below, the complex $\mathfrak{M}_*^Y(\mathbb{Z}_Y[d])$ is also concentrated in degree zero, and its cohomology in degree zero is a torsion-free $\mathbb{Z}$-module. In other words, Y is a duality space of dimension d. □

Lemma 6.3. *Let $N^\bullet$ be a bounded complex of free $\mathbb{Z}$-modules. Suppose that*

1. $H^k(N^\bullet \otimes_{\mathbb{Z}} \mathbb{Q}) = H^k(N^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p) = 0$ *for any* $k \neq 0$ *and for any prime number* p*;*
2. $H^k(N^\bullet) = 0$ *for* $k > 0$*.*

Then $H^k(N^\bullet) = 0$ for $k \neq 0$, and $H^0(N^\bullet)$ is a torsion-free $\mathbb{Z}$-module.

Proof. By (2), it suffices to show that $H^k(N^\bullet) = 0$ for $k < 0$ and $H^0(N^\bullet)$ is torsion-free. For any $k < 0$, since $H^k(N^\bullet \otimes_{\mathbb{Z}} \mathbb{Q}) = 0$, $H^k(N^\bullet)$ is a torsion $\mathbb{Z}$-module. Thus it suffices to show that $H^k(N^\bullet)$ is torsion free for all $k \leq 0$.

Suppose that for some $k \leq 0$, $H^k(N^\bullet)$ has nonzero torsion elements. Let k_0 be the smallest such k, and let p be a prime number such that $H^{k_0}(N^\bullet)$ has nonzero p-torsion elements. Here, notice that if $\eta \in H^{k_0}(N^\bullet)$ has order $m > 0$, and if p is a prime divisor of m, then $\frac{m}{p}\eta$ is a p-torsion element. The p-torsion element in $H^{k_0}(N^\bullet)$ induces a short exact sequence

$$0 \to \mathbb{F}_p \to H^{k_0}(N^\bullet) \to H^{k_0}(N^\bullet)/\mathbb{F}_p \to 0.$$

Then as part of the associated long exact sequence, we have

$$0 = \mathrm{Tor}_2^{\mathbb{Z}}\big(H^{k_0}(N^\bullet)/\mathbb{F}_p, \mathbb{F}_p\big) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{F}_p, \mathbb{F}_p) = \mathbb{F}_p \to \mathrm{Tor}_1^{\mathbb{Z}}(H^{k_0}(N^\bullet), \mathbb{F}_p),$$

which implies that $\mathrm{Tor}_1^{\mathbb{Z}}(H^{k_0}(N^\bullet), \mathbb{F}_p) \neq 0$. By the universal coefficient theorem, there is a noncanonical isomorphism

$$H^{k_0-1}(N^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p) \cong H^{k_0-1}(N^\bullet) \otimes_{\mathbb{Z}} \mathbb{F}_p \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H^{k_0}(N^\bullet), \mathbb{F}_p).$$

Thus $H^{k_0-1}(N^\bullet \otimes_{\mathbb{Z}} \mathbb{F}_p) \neq 0$. Since $k_0 \leq 0$, this contradicts our assumption (1). □

The following are consequences of Proposition 6.2.

Corollary 6.4. *Assume $U \subset X$ are complex manifolds satisfying the conditions of Theorem 1.1. Let $f: Y \to U$ be a proper birational semi-small map from a pure d-dimensional complex manifold Y. Then Y is a duality space of dimension d. In particular, blowing up U along any codimension-two submanifold gives rise to a duality space.*

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Proof. It suffices to check that the map f satisfies the assumptions of Proposition 6.2.

First, since f is a birational and proper map between the complex manifolds U and Y, f induces an isomorphism between their fundamental groups (see, e.g., [10, page 494]).

Secondly, since Y is smooth and f is semi-small, by [21, Example 6.0.9], we have:

1. $Rf_*(\mathbb{Q}_Y[d])$ and $Rf_*(\mathbb{F}_Y[d])$ are perverse sheaves on U.
2. $Rf_*(\mathbb{Z}_Y[d]) \in {}^pD_c^{\leq 0}(U, \mathbb{Z})$ is semi-perverse.

Hence the assertion follows from Proposition 6.2. □

Corollary 6.5. *Let U be a complex manifold with a compactification X satisfying the conditions in Theorem 1.1, and let $Z \subset U$ be a connected closed analytic subvariety, which is also locally closed in X. Assume that Z is a local complete intersection, and the inclusion $Z \hookrightarrow U$ induces an isomorphism on fundamental groups. Then Z is a duality space of dimension* $\dim_{\mathbb{C}} Z$.

Proof. First, since Z is locally closed in X, $R(j \circ i)_* \mathbb{Z}_Z \cong Rj_* j^* \bar{i}_* \mathbb{Z}_{\bar{Z}}$ is constructible on X, where $\bar{Z}$ is the closure of Z in X, and $i : Z \to U$, $\bar{i} : \bar{Z} \to X$ and $j : U \to X$ are the inclusion maps (see, e.g., [18, Theorem 2.5]).

Let $d = \dim_{\mathbb{C}} Z$. Since Z is a local complete intersection, $\mathbb{Z}_Z[d]$, $\mathbb{Q}_Z[d]$ and $\mathbb{F}_Z[d]$ are perverse sheaves on Z. Since i is a closed embedding, $i_* \mathbb{Z}_Z[d]$, $i_* \mathbb{Q}_Z[d]$ and $i_* \mathbb{F}_Z[d]$ are perverse sheaves on U. Since $i : Z \to U$ is also assumed to induce an isomorphism on fundamental groups, the assertion follows from Proposition 6.2. □

Corollary 6.6. *Let $Y \subset \mathbb{P}^n$ be a hypersurface such that the singular locus Y_{sing} has codimension at least 3. Let $D_1, \ldots, D_m \subset \mathbb{P}^n$ be smooth hypersurfaces in general position and transversal to Y such that $Y \cap D_1 \cap \cdots \cap D_m = \emptyset$. Then $Y \setminus (D_1 \cup \cdots \cup D_m)$ is a duality space.*

Proof. Set $U = \mathbb{P}^n \setminus (D_1 \cup \cdots \cup D_m)$. It is clear that U with compactification $\mathbb{P}^n$ satisfies the conditions in Theorem 1.1. Note that $Y \setminus (D_1 \cup \cdots \cup D_m)$ is a hypersurface in U. By the above corollary, we only need to show that the inclusion map $Y \setminus (D_1 \cup \cdots \cup D_m) \to U$ induces an isomorphism on the fundamental groups. By the Lefschetz hyperplane section theorem, after intersecting with a generic projective linear space $L \subset \mathbb{P}^n$ with $\dim L = 3$, we can assume that Y is a smooth hypersurface Y in $\mathbb{P}^3$ intersecting $D_1 \cup \cdots \cup D_m$ transversally. Using the Lefschetz hyperplane section theorem, we get that

$$Y \setminus (D_1 \cup \cdots \cup D_m) \to \mathbb{P}^3 \setminus (D_1 \cup \cdots \cup D_m)$$

induces an isomorphism on the fundamental groups. □

We regard a complex manifold U as in Theorem 1.1 as the affine/Stein counterpart of a complex projective aspherical manifold. The Mellin transformations of certain projective aspherical manifolds are discussed in [16], where, under certain assumptions, we show that the Mellin transformation of a nontrivial constructible complex is nonzero (see [16, Proposition 3.3 and Proposition 5.6]).

We conjecture that this fact remains true in the more general setting of Theorem 1.1.

Conjecture 6.7. *Assume that $U \subset X$ are complex manifolds satisfying the conditions in Theorem 1.1. Let $\mathcal{F} \in D_c^b(X, A)$ be a constructible complex such that $\mathcal{F}|_U \neq 0$. Then $\mathfrak{M}_*(\mathcal{F}|_U) \neq 0$.*

Remark 6.8. If $U \subset X$ are algebraic varieties, and if U admits a quasi-finite map to some semi-abelian variety, then the conjecture holds; for more details, see [16, Remark 5.15]. Such examples include complements of essential linear hyperplane arrangements, complements of toric arrangements and complements of elliptic arrangements.

A particular consequence of Conjecture 6.7 is that the perverse t-structure on $D_c^b(U, A)$ can be completely detected by the Mellin transformation.

Proposition 6.9. *Assume that the above conjecture holds for complex manifolds $U \subset X$ satisfying the conditions in Theorem 1.1. Then for an A-constructible complex $\mathcal{F}$ on X, $\mathcal{F}|_U$ is perverse if and only if $\mathfrak{M}_*(\mathcal{F}|_U)$ is concentrated in degree zero.*

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Proof. The 'only if' part is exactly Theorem 1.1. To show the converse, suppose that $\mathcal{F}|_U$ is not a perverse sheaf. Then there exists $k \neq 0$ such that ${}^p\mathcal{H}^k(\mathcal{F}|_U) \neq 0$. It follows from Theorem 1.1 that $H^k(\mathfrak{M}_*(\mathcal{F}|_U)) \cong H^0(\mathfrak{M}_*({}^p\mathcal{H}^k(\mathcal{F}|_U)))$, which is nonzero by Conjecture 6.7. This is a contradiction to the assumption that $\mathfrak{M}_*(\mathcal{F}|_U)$ is concentrated in degree zero. □

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Conflicts of Interest. The authors have no conflict of interest to declare.

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References

[1] D. Arapura, 'The Leray spectral sequence is motivic', *Invent. Math.* **160**(3) (2005), 567–589.

[2] A. A. Beilinson, J. Bernstein and P. Deligne, '*Faisceaux pervers:* Analysis and topology on singular spaces, I (Luminy, 1981)', *Astérisque* **100** (1982), 5–171.

[3] B. Bhatt, C. Schnell and P. Scholze, 'Vanishing theorems for perverse sheaves on abelian varieties, revisited', *Selecta Math.* **24**(1) (2018), 63–84.

[4] R. Bieri and B. Eckmann, 'Groups with homological duality generalizing Poincaré duality', *Invent. Math.* **20** (1973), 103–124.

[5] J.-L. Brylinski, 'Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques', *Astérisque* **140–141** (1986), 3–134, 251.

[6] N. Budur, 'Bernstein-Sato ideals and local systems', *Ann. Inst. Fourier (Grenoble)* **65**(2) (2015), 549–603.

[7] G. Denham and A. Suciu, 'Local systems on arrangements of smooth, complex algebraic hypersurfaces', *Forum Math. Sigma* **6** (2018), e6.

[8] A. Dimca, *Sheaves in Topology*, *Universitext* (Springer-Verlag, Berlin, 2004.)

[9] O. Gabber and F. Loeser, '*Faisceaux perversl-adiques sur un tore*', *Duke Math. J.* **83**(3) (1996), 501–606.

[10] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, *Pure and Applied Mathematics* (Wiley-Interscience, New York, 1978.)

[11] M. Kashiwara and P. Schapira, *Sheaves on Manifolds. With a chapter in French by Christian Houzel*, *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Vol. 292 (Springer-Verlag, Berlin, 1990.)

[12] T. Krämer, 'Perverse sheaves on semiabelian varieties', *Rend. Semin. Mat. Univ. Padova* **132** (2014), 83–102.

[13] Y. Liu, L. Maxim and B. Wang, 'Generic vanishing for semi-abelian varieties and integral Alexander modules', *Math. Z.* **293**(1–2) (2019), 629–645.

[14] Y. Liu, L. Maxim and B. Wang, 'Mellin transformation, propagation, and abelian duality spaces', *Adv. Math* **335** (2018), 231–260.

[15] Y. Liu, L. Maxim and B. Wang, 'Perverse sheaves on semi-abelian varieties', *Selecta Math.***27**(2) (2021).

[16] Y. Liu, L. Maxim and B. Wang, 'Aspherical manifolds, Mellin transformation and a question of Bobadilla-Kollár', *J. Reine Angew. Math.* **781** (2021), 1–18.

[17] L. Maxim, *Intersection Homology & Perverse Sheaves, with Applications to Singularities, Graduate Texts in Mathematics*, Vol. 281 (Springer, 2019).

[18] L. Maxim and J. Schürmann, *Constructible Sheaf Complexes in Complex Geometry and Applications*, Handbook of Geometry and Topology of Singularities, Vol. III, 679–791 (Springer, Cham, 2022).

[19] C. Sabbah, 'Modules d'Alexander et D-modules', *Duke Math. J.* **60**(3) (1990), 729–814.

[20] C. Schnell, 'Holonomic D-modules on abelian varieties', *Publ. Math. Inst. Hautes Études Sci.* **121** (2015), 1–55.

[21] J. Schürmann, *Topology of Singular Spaces and Constructible Sheaves*, *Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne*, Vol. 63 (Birkhäuser Verlag, Basel, 2003).