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INDEPENDENCE IN COMBINATORIAL GEOMETRIES OF RANK THREE

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1. **Introduction.** The class of all *combinatorial geometries of rank three* shall coincide with the class of all pairs (V, S) such that V is a set and S is a collection of non-empty subsets of V such that each pair of distinct elements of V belong to exactly one member of S. (See [3].)

Consider a combinatorial geometry (V, S) of rank three. The *line structure* for (V, S) is the function $P:2^V \to 2^V$ defined as follows: If $X \subseteq V$, then $P(X)=X$ if X is a singleton set, and $P(X)$ is the union of all $Y \in S$ such that Y contains at least two elements of X if X is not a singleton set. Associated with P is a sequence $\{P^n\}_{n\in N}$ of functions (with N being the set of all positive integers) and a function P^∞ defined as follows: $P^1=P$; if $n \in N$ then $P^{n+1}=PP^n$; and $P^\infty(X)=\cup\{P^n(X):n \in N\}$ for each $X \subseteq V$.

The notion of independence treated in this paper is the notion of independence with respect to the line structure P for a combinatorial geometry (V, S) of rank three. Consider an element n of $N \cup \{\infty\}$ and a subset X of V. By definition, X is *P^n-independent* if $x \notin P^n(X-\{x\})$ for each $x \in C$, where $S-T=\{y \in S: y \notin T\}$ if S is a set and T is a set. Also, if $Y \subseteq V$, then Y is *a P^n-generator of X* if $P^n(Y)=X$; and Y is *a P^n-basis of X* if Y is a P^n-independent P^n-generator of X.

This notion of independence is defined for an arbitrary *structure in a set*, that is, a function $P:2^V \to 2^V$ for some set V. (See [5] and [6].) The notion of P-independence is studied in [1], [2], [4] and [6], based on the following properties: P is *monotone* [if $X \subseteq Y \subseteq V$, then $P(X) \subseteq P(Y)$], P is *extensive* [if $X \subseteq V$, then $X \subseteq P(X)$], P is *idempotent* [if $X \subseteq V$, then $P(P(X))=P(X)$], P is *a closure structure* [P is monotone, extensive and idempotent], P has *the exchange property* [if $X \subseteq V$, $y \in V$, $x \in P(X \cup \{y\})$ and $x \notin P(X)$, then $y \in P(X \cup \{x\})$], P has *α-character* (with α being a cardinal number) [if $X \subseteq V$, then $P(X) \subseteq \cup\{P(Y): Y \subseteq X$ and $|Y| < \alpha\}$, where $|S|$ denotes the cardinal number of a set S], P is *finitary* [P has α-character with α being the first infinite cardinal number], P is *normal* [if G is a chain of subsets of V, then $P(\cup G) \subseteq \cup\{P(X):X \in G\}$], and P has *the equivalence covering property* [if $X \subseteq V$, $Y \subseteq V$ and $P(X)=P(Y)$, then $P(X) \subseteq \cup\{P(Z):Z \subseteq Y$ and $|Z| \leq |X|\}$]. (These definitions appear in [1], [2], [4], [5] and [6].)

In Section 2, some properties of the sequence $\{P^n\}_{n\in N}$ and the function P^∞ associated with the line structure P for a combinatorial geometry (V, S) of rank

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three are developed. A characterization of *line-closed subset of* V [a subset X of V such that $P(X)\subseteq X$] is given in terms of values of P^{∞}. Independence and generation with respect to P^{∞} and the terms of $\{P^n\}_{n\in N}$ is treated. Results from [2], [4], and [6] are applied. If $X\subseteq V$, then any two P^{∞}-bases of $P^{\infty}(X)$ are in one-to-one correspondence. Finally, in Section 3, a non-trivial vector space V over a division ring F is interpreted as a combinatorial geometry (V, S) of rank three. It follows that if p is the characteristic of F, α is the dimension of V and either $1<p<\alpha$ or $p=0$, then P^{∞} is the linear variety structure in V (and that the condition $p<\alpha$ is relevant).

2. **Main results.** Unless the contrary is indicated, the symbols V, S, P and L will be used with the understanding that (V, S) is a combinatorial geometry of rank three, P is the line structure for (V, S) and L is the function defined on V^2 such that if $x\in V$ and $y\in V$, then $L(x, y)=\{x\}$ if $x=y$ and $L(x, y)$ is the member of S which contains x and y if $x\neq y$. If Q_1 is a structure in a set W and Q_2 is a structure in W, then $Q_1\subseteq Q_2$ if $Q_1(X)\subseteq Q_2(X)$ for each $X\subseteq W$, and $Q_1=Q_2$ if $Q_1\subseteq Q_2\subseteq Q_1$. If S is a set, the symbol $|S|$ shall denote the cardinal number of S.

PROPOSITION 1. *Let P be a line structure.*

(a) *If $n\in N$, then P^n is extensive, monotone, normal and has* (2^n+1)*-character.*
(b) *P^{∞} is a finitary, normal closure structure.*
(c) *If $n\in N$, then $P^n\subseteq P^{n+1}\subseteq P^{\infty}$.*

Proof. It follows by induction that if $n\in N$, then P^n is extensive, monotone and has (2^n+1)-character (so that P^n is finitary). Hence, (c) follows, and it follows that P^{∞} is extensive, monotone, and finitary. It is easy to show that every finitary structure is normal. Therefore, P^{∞} and each P^n is normal. Since each P^n is normal, it follows from (c) that P^{∞} is idempotent. This completes a proof of the proposition.

COROLLARY. *A subset X of V is line-closed if and only if $X=P^{\infty}(Y)$ for some $Y\subseteq V$.*

PROPOSITION 2. *P has the exchange and equivalence covering properties.*

Proof. Assume that $X\subseteq V$, $y\in V$, $x\in P(X\cup\{y\})$ and $x\notin P(X)$. It is clear that $y\in P(X\cup\{x\})$ if $X=\varnothing$. Consider the case that $X\neq\varnothing$. Let z and w be distinct elements of $X\cup\{y\}$ such that $x\in L(z, w)$. One of z and w must be y, say $w=y$. If $x=y$, then $y\in P(X\cup\{x\})$ [since P is extensive]. If $x\neq y$, then $y\in L(z, x)$ [since (V, S) is a combinatorial geometry of rank three] while $L(z, x)\subseteq P(X\cup\{x\})$, so that $y\in P(X\cup\{x\})$. It follows that P has the exchange property. Since P has (2^1+1)-character, it follows that P has the equivalence covering property. The proposition follows.

The following proposition is an immediate consequence of (c) of Proposition 1, the definition of $\{P^n\}_{n\in N}$ and the definition of P^{∞}.

PROPOSITION 3. *If* $n \in (N \cup \{\infty\})$, *then* P^n *has the exchange property if and only if the following condition is satisfied:*

If $X \subseteq V$, $y \in V$ *and* $x \in [P^m(X \cup \{y\}) - P^n(X)]$ *for some* $m \in N$ *such that* $m \leq n$, *then* $y \in P^m(X \cup \{y\})$ *for some* $m \in N$ *such that* $m \leq n$.

PROPOSITION 4. *Suppose that* $n \in (N \cup \{\infty\})$.

(a) *If* $X \subseteq V$ *such that* $|X| \leq 2$ *or* $X \subseteq Y$ *for some* P^n*-independent subset* Y *of* V, *then* X *is* P^n*-independent.*

(b) *If* P^n *has the exchange property, then every* P^n*-independent subset* X *of* V *satisfies the following conditions:*

$x \in [V - P^n(X)]$ *implies that* $X \cup \{x\}$ *is* P^n*-independent.*

Proof. (a) is an immediate consequence of the definitions. Assume that P^n has the exchange property while X is a P^n-independent subset of V, $x \in V$ and $x \notin P^n(X)$. Suppose that $X \cup \{x\}$ is not P^n-independent. Choose an element y of $X \cup \{x\}$ such that $y \in P^n([X \cup \{x\}] - \{y\})$. Then $x \neq y$ [since $x \notin P^n(X)$ and P^n is extensive]. Hence, $y \in P^n([X - \{y\}] \cup \{x\})$ while P^n has the exchange property and $y \notin P^n(X - \{y\})$ [since X is P^n-independent and $y \in X$]. Therefore, $x \in P^n([X - \{y\}] \cup \{y\}) = P^n(X)$. But $x \notin P^n(x)$. It follows that $X \cup \{x\}$ is P^n-independent. (b) follows. The proof is complete.

Propositions 5 and 6 are immediate consequences of (c) of Proposition 1 and the definitions of $\{P^n\}_{n \in N}$ and P^∞.

PROPOSITION 5. *If* $n \in (N \cup \{\infty\})$, $m \in N$ *such that* $m < n$, *and* $X \subseteq V$, *then each* P^m*-generator of* $P^m(X)$ *is a* P^n*-generator of* $P^n(X)$.

PROPOSITION 6.

(a) *If* $n \in N$, $m \in N$ *and* $m < n$, *then each* P^n*-independent subset of* V *is* P^m*-independent.*

(b) *If* $n \in (N \cup \{\infty\})$, *then a subset* X *of* V *is* P^n*-independent if and only if* X *is* P^m*-independent for each* $m \in N$ *such that* $m \leq n$.

The converse of (a) of Proposition 6 is not a theorem. Let R be the residue class ring of integers modulo a prime number p, $W = R^2$ and T be the collection of all sets $\{r(x-y)+y : r \in R\}$. Then (W, T) is a combinatorial geometry of rank three. Let Q be the line structure for (W, T). Let $X = \{(0,0), (1,0), (0,1), (1,1)\}$. Then X is Q-independent. Also, $Q^2(X - \{(0,0)\})$ contains $(0,0)$, so that X is not Q^2-independent.

PROPOSITION 7. *Suppose that* $U \subseteq V$, *and that* $n \in (N \cup \{\infty\})$.

(a) *If* $X \subseteq V$ *such that* X *is a* P^n*-basis of* $P^n(U)$, *then* X *is a maximal* P^n*- independent subset of* U.

(b) *If* P^n *has the exchange property, then every maximal* P^n*-independent subset of* U *is a* P^n*-generator of* $P^n(U)$.

Proof. Since P^n is monotone, (a) follows. Assume that P^n has the exchange property while X is a maximal P^n-independent subset of U. It follows from (b) of Proposition 4 that $U \subseteq P^n(X)$ while $P^n(X) \subseteq P^n(U)$ [since P^n is monotone] and $P^\infty(U)$ is line-closed [corollary to Proposition 1]. Therefore, it follows that $P^\infty(X) = P^\infty(U)$. (b) follows. The proof is complete.

PROPOSITION 8. *Suppose that* $n \in (N \cup \{\infty\})$.

(a) *Every subset of* V *has a maximal* P^n*-independent subset.*

(b) *If* $U \subseteq V$, *then every* P^n*-independent subset of* U *can be extended to a maximal* P^n*-independent subset of* U.

Proof. Assume that $U \subseteq V$, and that X is a P^n-independent subset of U. Let F be the collection of all P^n-independent subsets of U, and let F_1 be the collection of all P^n-independent subsets Y of U such that $X \subseteq Y$. Then F contains ϕ and F_1 contains X. Since P^n is monotone and normal, it follows that each chain of F (ordered by set inclusion) has its union as an upper bound, and that each chain of F_1 has its union as an upper bound. Hence, it follows from Zorn's lemma that F and F_1 have maximal elements. The proposition follows.

Consider a structure Q in a set W. We shall say that Q has *the Steinitz exchange property* if the following condition is satisfied: If $X \subseteq W$, Y and Z are Q-bases of $Q(X)$ and A is a finite subset of X, then there is a finite subset B of Y such that $|B| = |A|$ and $(X - A) \cup B$ is a Q-basis of $Q(X)$. Also, we shall say that Q has *the dimension property* if the following condition is satisfied: If $X \subseteq W$, then any two Q-bases of $Q(X)$ are in one-to-one correspondence. It is known (See, e.g., [2] and [6]) that

I. If Q is a closure structure having the exchange property, then Q has the Steinitz exchange property.

It is known ([4] and [6]) that the following conditions are satisfied:

II. If Q is a closure structure having the equivalence covering property, $X \subseteq W$ and Y is a Q-basis of $Q(X)$, then $|Y| \leq |X|^2$.

III. If each subset X of W has a Q-basis of $Q(X)$, and if each two Q-independent subsets Y and Z of W such that Z is a Q-generator of $Q(Y)$ are in one-to-one correspondence, then Q has the equivalence covering property.

Therefore, it follows from I, II and III that

IV. If Q is a closure structure having the exchange property, then Q has the dimension property if and only if Q has the equivalence covering property.

PROPOSITION 9. *If* P^∞ *has the exchange property, then* P^∞ *has the equivalence covering property.*

Proof. Suppose that P^∞ has the exchange property, and that $X \subseteq V$. Recall that P^∞ is a closure structure [(b) of Proposition 1]. If $P^\infty(X)$ has a finite P^∞-basis, then it follows from I that any two P^∞-bases of $P^\infty(X)$ are in one-to-one correspondence. If $P^\infty(X)$ has an infinite P^∞-basis, then it follows from II that any two

P^∞-bases of $P^\infty(X)$ are in one-to-one correspondence. Therefore, P^∞ has the dimension property. It follows from IV that P^∞ has the equivalence covering property.

It was shown in [5] that monotonicity, extensiveness, idempotence, α-character (with α being a cardinal number), the exchange property and the equivalence covering property are independent. Therefore, Proposition 9 is not valid for all closure structures.

3. **Closing remarks.** Let V be a non-trivial vector space over a division ring F and S be the collection of all $\{r(x-y)+y : r \in F$ such that $x \in V, y \in V$ and $x \neq y\}$. Then (V, S) is a combinatorial geometry of rank three. Let P be the line structure for (V, S). Let $Q:2^V \to 2^V$ be the structure in V such that if $X \subseteq V$, then $Q(X)$ is the set of all finite linear combinations of elements of X whose coefficients sum to the multiplicative identity in F. Let p be the characteristic of F and α be the dimension of V. Then

IVa. If $p=0$, then $P^\infty = Q$. If $p \neq 0$, then $P^\infty(X)=Q(X)$ for each $X \subseteq V$ such that $|X|<p$; hence, if $1<\alpha<p$, then $P^\infty=Q$.

It is obvious that $P \subseteq Q$, and that $Q(X)$ is line-closed for each $X \subseteq V$. So $P^\infty \subseteq Q$. An inductive argument can be used to complete a proof of the assertion.

The following example shows that the condition $p<\alpha$ is needed. Let F be the residue class ring of integers modulo 2. Let $V=F^2$. Let $X=\{(0,0), (1,0), (0,1)\}$. Then $P^\infty(X)=P(X)=X$ while $Q(X)=V$.

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