HOMOMORPHISMS AND CONGRUENCES ON ω^α-BISIMPLE SEMIGROUPS

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Let S be a bisimple semigroup, let E_S denote the set of idempotents of S, and let $\leqq$ denote the natural partial order relation on E_S. Let $\leqq^*$ denote the inverse of $\leqq$. The idempotents of S are said to be well-ordered if $(E_S, \leqq^*)$ is a well-ordered set.

The structure of bisimple semigroups with idempotents well-ordered is described modulo group theory and ordinal arithmetic in [4]. These semigroups are precisely the ω^α-bisimple semigroups where α is an ordinal number and ω is the order type of the natural numbers under the usual order.

In this paper we describe the homomorphisms of an ω^α-bisimple semigroup into and onto an ω^β-bisimple semigroup and we describe the congruence relations on an ω^α-bisimple semigroup. Any congruence relation R on an ω^α-bisimple semigroup S is a group congruence (S/R is a group), an ω^β-bisimple congruence (S/R is an ω^β-bisimple semigroup) for some ordinal number β such that $0 < \beta < \alpha$, an ω^α-bisimple congruence which is idempotent separating (S/R is an ω^α-bisimple semigroup and each R-class contains at most one idempotent), or an ω^α-bisimple congruence which is not idempotent separating.

We generally follow the notation and terminology of [1] and [2]. Extensive use is made of the results in [4].

1. Introduction

In this section we present some introductory items and review pertinent material.

Let X be a right cancellative semigroup with identity and let $P(X)$ denote the collection of principal left ideals of X. Let $a, b \in X$ and let (a) and (b) denote the principal left ideals of X generated by a and b respectively. We choose to

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write $(a) \leqq (b)$ if and only if $(a) \supseteq (b)$ and to call the partially ordered set $(P(X), \leqq)$ the *ideal structure* of X. If U denotes the group of units of X, a (normal) subgroup V of U is called a *right normal divisor* of X if $aV \subseteq Va$ for every $a \in X$.

Let α be any order type. A right cancellative semigroup with identity whose ideal structure has order type α is called an *α-right cancellative semigroup* and a bisimple semigroup S such that $(E_S, \leqq^*)$ has order type α is called an *α-bisimple semigroup* [10].

Let α be any ordinal number. Let ω denote the order type of the positive integers with the usual order. Let H_α be the set of ordinal numbers less than ω^α and let $+$ denote usual ordinal addition. For ordinal numbers β and γ we define $\odot$ by $\gamma \odot \beta = \beta + \gamma$. Then, $(H_\alpha, \odot)$ is an ω^α-right cancellative semigroup [4, Example]. Using this operation and [5, Theorem 2, p. 323], each non-zero element β of H_α may be uniquely expressed in the *normal form*

$$\beta = b_k\omega^{\beta_k} \odot \cdots \odot b_2\omega^{\beta_2} \odot b_1\omega^{\beta_1}$$

where k and $b_1, b_2, \cdots, b_k$ are positive integers and $\beta_1, \beta_2, \cdots, \beta_k$ is a decreasing sequence of ordinal numbers with $\beta \geqq \omega^{\beta_1} \geqq \beta_1$. Note that we use usual exponential notation for ordinal numbers and that we write $\delta\omega^\gamma$ in place of the usual $\omega^\gamma\delta$. The semigroup $(H_\alpha, \odot)$ is called *the semigroup of ordinal numbers* less than ω^α. We have displayed the normal form of $\gamma \odot \beta$ for $\beta, \gamma \in H_\alpha$ [4, (1.1)]. For non-zero elements β and γ of H_α having normal forms

$$\beta = b_k\omega^{\beta_k} \odot \cdots \odot b_2\omega^{\beta_2} \odot b_1\omega^{\beta_1} \text{ and } \gamma = c_m\omega^{\gamma_m} \odot \cdots \odot c_2\omega^{\gamma_2} \odot c_1\omega^{\gamma_1},$$

we make the following definitions:

$$(1.1) \qquad N(\gamma,\beta) = \begin{cases} 0 \text{ if } \beta_1 < \gamma_1, \\ \sup\{j \colon \beta_j \geqq \gamma_1\} \text{ if } \beta_1 \geqq \gamma_1. \end{cases}$$

$$M(\gamma,\beta) = k - N(\gamma,\beta) - 1.$$

$$D(\gamma) = \text{degree } \gamma = \gamma_1.$$

Now let $W_\alpha = H_\alpha \times H_\alpha$. Define an operation on W_α by

$$(1.2) \qquad (\beta,\gamma)(\delta,\eta) = (\beta + (\max\{\gamma,\delta\} - \gamma), \eta + (\max\{\gamma,\delta\} - \delta)).$$

Then, under this operation, W_α is a semigroup which we call *the α-bicyclic semigroup*. It is not difficult to show that W_α is a bisimple inverse semigroup with identity $(0,0)$ whose right unit subsemigroup is isomorphic to $(H_\alpha, \odot)$. Thus, W_α is an ω^α-bisimple semigroup. Note that W_1 is the bicyclic semigroup, and for positive integers n, W_n is isomorphic to the $2n$-cyclic semigroup of Warne [10].

If G is a group, X is an element of G, and θ is an endomorphism of G, we write θ^0 indicating the identity mapping and we write $YC_X = XYX^{-1}$ for $Y \in G$.

THEOREM 1.1. (Hogan [4]) *Let P be a right cancellative semigroup with identity whose ideal structure is well-ordered. Then P is an ω^α-right cancellative semigroup for some ordinal number α.*

THEOREM 1.2. (Hogan [4]) *P is an ω^α-right cancellative semigroup if and only if $P \cong G \times H_\alpha$, where G is a group and $(H_\alpha, \odot)$ is the semigroup of ordinal numbers less than ω^α, under the multiplication given as follows:*

For $(X,\gamma), (Y,\beta) \in G \times H_\alpha$,

$$(1.3)\quad (X,\gamma)(Y,\beta) = (X, c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1})(Y, b_k\omega^{\beta_k} \odot \cdots \odot b_1\omega^{\beta_1})$$

$$= \begin{cases} \left(X\left(\left((Y\theta_{\gamma_1})\left(\prod_{j=0}^{M(\gamma,\beta)} z^{b_{k-j}}(\gamma_1, \beta_{k-j})\right)\right)\theta_{\gamma_1}{}^{c_1-1}\prod_{i=2}^{m}\theta_{\gamma_i}^{c_i}\right), \gamma \odot \beta\right) \\ \qquad\qquad \text{if } \gamma \neq 0,\ \beta \neq 0, \text{ and } M(\gamma,\beta) \geqq 0, \\ \left(X\left(Y \prod_{i=1}^{m} \theta_{\gamma_i}^{c_i}\right), \gamma \odot \beta\right) \text{ otherwise,} \end{cases}$$

where $\{\theta_\sigma: 0 \leqq \sigma < \alpha\}$ is a collection of endomorphisms of G and $\{z(\delta,\sigma): 0 \leqq \sigma < \delta < \alpha\}$ is a collection of elements of G such that

$$(1.4)\qquad z(\delta,\sigma)\theta_\rho = z(\rho,\sigma)C_{z(\rho,\delta)} \text{ for } 0 \leqq \sigma < \delta < \rho < \alpha$$

and

$$(1.5)\qquad \theta_\delta\theta_\sigma = \theta_\sigma C_{z(\sigma,\delta)} \text{ for } 0 \leqq \delta < \sigma < \alpha.$$

COROLLARY 1.3. (Hogan [4]) *Let P be a right cancellative semigroup with identity. Then the following are equivalent:*

(1.6) *P has well-ordered ideal structure.*

(1.7) *P is an ω^α-right cancellative semigroup for some ordinal number α.*

(1.8) *$\mathscr{L}$ is a congruence on P and $P/\mathscr{L} \cong H_\alpha$ for some ordinal number α.*

Moreover, $P \cong H_\alpha$ for some ordinal number α if and only if P has trivial unit group and satisfies one of (1.6), (1.7), *and* (1.8).

As a consequence of Theorem 1.2, we use the notation

$$P = (G, H_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$$

to denote an ω^α-right cancellative semigroup. The collection $\{\theta_\sigma: 0 \leqq \sigma < \alpha\}$ of endomorphisms of G is called the *collection of structure endomorphisms* of P, and the collection

$$\{z(\delta,\sigma): 0 \leqq \sigma < \delta < \alpha\}$$

of elements of G is called the *collection of distinguished elements.* A normal subgroup N of G is called $\{\theta_\sigma\}$*-invariant* if $N\theta_\sigma \subseteq N$ for each structure endomorphism θ_σ.

Note that P is a Schreier extension [7, p. 1117] of G by $P/\mathscr{L} \cong H_\alpha$ and that the function pair of this extension is given by

$$(1.9)\qquad |\gamma^\beta| = \begin{cases} \left(\prod_{j=0}^{M(\gamma,\beta)} z^{b_{k-j}}(\gamma_1, \beta_{k-j})\right)\theta_{\gamma_1}^{c_1-1} \prod_{i=2}^{m} \theta_{\gamma_i}^{c_i} \\ \qquad\qquad \text{if } \gamma \neq 0,\ \beta \neq 0, \text{ and } M(\gamma,\beta) \geqq 0, \\ E, \text{ the identity of } G, \text{ otherwise,} \end{cases}$$

and

$$(1.10)\qquad A^\gamma = A\left(\prod_{i=1}^{m} \theta_{\gamma_i}^{c_i}\right) \text{ for } A \in G$$

where γ and β have the normal forms

$$\gamma = c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1} \text{ and } \beta = b_k\omega^{\beta_k} \odot \cdots \odot b_1\omega^{\beta_1}.$$

Let us now review the main result of Clifford [3]. Let P be a right cancellative semigroup with identity such that the intersection of two principal left ideals is a principal left ideal. For each class of $\mathscr{L}$-equivalent elements of P pick a fixed representative. Define $a \vee b$ to be the representative of the class containing c where $Pa \cap Pb = Pc$. Define the operation $*$ for $a, b \in P$ by

$$(1.11)\qquad (a * b)b = a \vee b.$$

Let $P^{-1} \circ P$ denote the set of ordered pairs (a, b) of elements of P. Define equality on $P^{-1} \circ P$ by

(1.12) $(a, b) = (a', b')$ if $a' = ua$ and $b' = ub$ where u is a unit of P.

Define a product in $P^{-1} \circ P$ by

$$(1.13)\qquad (a, b)(c, d) = ((c * b)a,\ (b * c)d).$$

THEOREM 1.4. (Clifford [3]) *Starting with a right cancellative semigroup with identity P having the property that the intersection of two principal left ideals is a principal left ideal, equations* (1.11), (1.12), *and* (1.13) *define a semigroup $P^{-1} \circ P$ which is a bisimple semigroup with identity in which any two idempotents commute.*

Conversely, if S is a bisimple semigroup with identity in which any two idempotents commute, then its right unit subsemigroup P is a right cancellative semigroup with identity having the property that the intersection of two principal left ideals is a principal left ideal and S is isomorphic with $P^{-1} \circ P$. The semi-lattice of principal left ideals of P under intersection is isomorphic with the semi-lattice of idempotent elements of S.

As a result of Theorem 1.4, we may identify S and $P^{-1} \circ P$ if S is a bisimple semigroup with identity in which any two idempotents commute.

THEOREM 1.5. Hogan [4]) *Let S be a bisimple semigroup with idempotents well-ordered. Then S is a bisimple inverse semigroup with identity and S is an ω^α-bisimple semigroup for some ordinal number α.*

THEOREM 1.6. (Hogan [4]) *A bisimple semigroup S with idempotents well-ordered is an ω^α-bisimple semigroup if and only if $\mathscr{H}$ is a congruence on S and $S/\mathscr{H} \cong W_\alpha$.*

COROLLARY 1.7. (Hogan [4]) *A bisimple semigroup S with trivial unit group and idempotents well-ordered is an ω^α-bisimple semigroup if and only if $S \cong W_\alpha$.*

In the material which follows, we observe the notational conventions of $N(\beta, \delta)$ being replaced by 0 if $\delta = 0$ and of a product of the form $\Pi_{j=0}^{M(\beta,\delta)}$ being replaced by the identity of G if $\delta = 0$ or if $M(\beta, \delta) < 0$.

THEOREM 1.8. (Hogan [4]) *S is an ω^α-bisimple semigroup if and only if $S \cong G \times W_\alpha$, where G is a group and W_α is the α-bicyclic semigroup, under the multiplication*

$$(1.14)\quad (X,(\gamma,\beta))(Y,(\delta,\rho)) = (X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, b_k\omega^{\beta_k} \odot \cdots \odot b_1\omega^{\beta_1}))$$
$$(Y,(d_n\omega^{\delta_n} \odot \cdots \odot d_1\omega^{\delta_1}, r_q\omega^{\rho} \odot \cdots \odot r_1\omega^{\rho_1})) = (Z,(\gamma,\beta)(\delta,\rho))$$

where Z is given as follows:

$$Z = XY \text{ if } \beta = \delta,$$

$$Z = X\left\{\left[\left(\prod_{j=0}^{M(\sigma,\delta)} z^{d_{n-j}}(\sigma_1,\delta_{n-j})\right)^{-1}(Y\theta_{\sigma_1})\left(\prod_{j=0}^{M(\sigma,\rho)} z^{r_{q-j}}(\sigma_1,\rho_{q-j})\right)\right]\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}\right\}$$

if $\beta > \delta$ and $\sigma = \beta - \delta$ has normal form

$$s_p\omega^{\sigma_p} \odot \cdots \odot s_1\omega^{\sigma_1},$$

$$Z = \left\{\left[\left(\prod_{j=0}^{M(\sigma,\gamma)} z^{c_{m-j}}(\sigma_1,\gamma_{m-j})\right)^{-1}(X\theta_{\sigma_1})\left(\prod_{j=0}^{M(\sigma,\beta)} z^{b_{k-j}}(\sigma_1,\beta_{k-j})\right)\right]\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}\right\}Y$$

if $\beta < \delta$ and $\sigma = \delta - \beta$ has normal form

$$s_p\omega^{\sigma_p} \odot \cdots \odot s_1\omega^{\sigma_1}.$$

Juxtaposition denotes multiplication in G and W_α, $\{\theta_\sigma: 0 \leqq \sigma < \alpha\}$ is a collection of endomorphisms of G, and $\{z(\delta,\sigma): 0 \leqq \sigma < \delta < \alpha\}$ is a collection of elements of G such that

$$z(\delta,\sigma)\theta_\rho = z(\rho,\sigma)C_{z(\rho,\delta)} \text{ for } 0 \leqq \sigma < \delta < \rho < \alpha \tag{1.15}$$

and

$$\theta_\delta\theta_\sigma = \theta_\sigma C_{z(\sigma,\delta)} \text{ for } 0 \leqq \delta < \sigma < \alpha. \tag{1.16}$$

As a consequence of Theorem 1.8, we use the notation $S = (G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$ to denote an ω^α-bisimple semigroup. The collection $\{\theta_\sigma: 0 \leqq \sigma < \alpha\}$ of endomorphisms is called *the collection of structure endomorphisms* and the collection $\{z(\delta,\sigma): 0 \leqq \sigma < \delta < \alpha\}$ is called *the collection of distinguished elements.*

2. The Homomorphism Theory

In this section we describe the homomorphisms from an ω^α-bisimple semigroup into and onto an ω^β-bisimple semigroup.

Let $S = (G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$ be an ω^α-bisimple semigroup. The right unit subsemigroup of S is

$$P = (G, H_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\}).$$

In the proof of Theorem 1.8, we have shown that the mapping Φ_S from $P^{-1} \circ P$ to S given by

$$((E,\gamma),(X,\delta))\Phi_S = (X,(\gamma,\delta)), \tag{2.1}$$

where E is the identity of G, is an isomorphism.

THEOREM 2.1. *Let $S = (G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma\})$ be an ω^α-bisimple semigroup and let*

$$S^* = (G^*, W_\beta, \{\Psi_\sigma\}, \{t(\delta,\sigma)\})$$

be an ω^β-bisimple semigroup. Let μ be any ordinal number such that $0 \leqq \mu \leqq \alpha \leqq \mu + \beta$ and let $\{\lambda_\sigma: 0 \leqq \sigma < \alpha - \mu\}$ be any subset of H_β with

$$\lambda_0 > 0 \text{ and} \tag{2.2}$$

$$0 \leqq \sigma < \rho < \alpha - \mu \text{ implies } D(\lambda_\sigma) < D(\lambda_\rho) \text{ satisfied.} \tag{2.3}$$

Let $\{U_\delta: 0 \leqq \delta < \alpha\}$ be elements of G^*, let τ be an element of H_β, and let f be a homomorphism of G into G^* such that

$$fC_{U_\sigma} = \theta_\sigma f \text{ for } 0 \leqq \sigma < \mu, \tag{2.4}$$

$$U_\sigma(Af)^{(\lambda_\sigma - \mu)} = (A\theta_\sigma f)U_\sigma \text{ for } 0 \leqq \mu \leqq \sigma < \alpha \text{ and } A \in G, \tag{2.5}$$

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(2.6) $\quad U_\sigma(U_\delta^{(\lambda_\sigma-\mu)})\left|\lambda_{\sigma-\mu}^{\lambda_\delta-\mu}\right| = (z(\sigma,\delta)f)U_\sigma$ *for* $0 \leqq \mu \leqq \delta < \sigma < \alpha$,

(2.7) $\quad U_\sigma(U_\delta^{(\lambda_\sigma-\mu)}) = (z(\sigma,\delta)f)U_\sigma$ *for* $0 \leqq \delta < \mu \leqq \sigma < \alpha$,

and

(2.8) $\quad U_\sigma U_\delta = (z(\sigma,\delta)f)U_\sigma$ *for* $0 \leqq \delta < \sigma < \mu \leqq \alpha$.

For each

$$(X,(\gamma,\delta)) = (X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1})) \in S,$$

let

(2.9) $\quad (X,(\gamma,\delta))F =$

$$((E^*,(\gamma g) \odot \tau),(\left|(\gamma g)^\tau\right|^{-1}(\gamma h)^{-1}(Xf)(\delta h)\left|(\delta g)^\tau\right|,(\delta g) \odot \tau))\Phi_{S^*}$$

where Φ_{S^*} *is given by* (2.1), g *is a homomorphism of* $(H_\alpha,\odot)$ *into* $(H_\beta,\odot)$ *given by*

(2.10)
$$0g = 0,\ \omega^\rho g = 0 \text{ for } 0 \leqq \rho < \mu, \text{ and}$$
$$\omega^{\mu+\rho} g = \lambda_\rho \text{ for } 0 \leqq \rho < \alpha - \mu,$$

and h *is a function from* H_α *into* G^* *given by*

(2.11)
$$\begin{aligned}(\gamma h) = &\left\{\prod_{j=0}^{c_m-1}[(U_{\gamma_m}^{(j\omega^{\gamma_m}g)})\left|(j\omega^{\gamma_m}g)^{(\omega^{\gamma_m}g)}\right|]\right\}\\ &\left\{\prod_{j=0}^{c_{m-1}-1}[(U_{\gamma_{m-1}}^{(j\omega^{\gamma_{m-1}})})\left|(j\omega^{\gamma_{m-1}}g)^{(\omega^{\gamma_{m-1}}g)}\right|]\right\}^{(c_m\omega^{\gamma_m}g)}\\ &\left|(c_m\omega^{\gamma_m}g)^{(c_{m-1}\omega^{\gamma_{m-1}}g)}\right|\cdots\\ &\left\{\prod_{j=0}^{c_1-1}[(U_{\gamma_1}^{(j\omega^{\gamma_1}g)})\left|(j\omega^{\gamma_1}g)^{(\omega^{\gamma_1}g)}\right|]\right\}^{(c_m\omega^{\gamma_m}\odot\cdots\odot c_2\omega^{\gamma_2})g}\\ &\left|(c_m\omega^{\gamma_m}\odot\cdots\odot c_2\omega^{\gamma_2})g^{(c_1\omega^{\gamma_1}g)}\right|.\end{aligned}$$

Then F *is a homomorphism from* S *into* S^*. *Conversely, every homomorphism of* S *into* S^* *is obtained in this fashion.*

PROOF. Assume the conditions of the theorem. Utilizing [4, Theorem 2.5], we have

$$(X,\gamma)N = ((Xf)(\gamma h),\gamma g)$$

as a homomorphism of P into P^* where P and P^* are the right unit subsemigroups of S and S^* respectively, g is given by (2.10), and h is given by (2.11). By use of [4, Theorem 2.9], N is a semi-lattice homomorphism. Thus, utilizing [7, Theorem 1.1], if $(E^*,\tau) \in P^*$ where E^* is the identity of G^*, then

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N(E^*,\tau),(Y,\delta)N(E^*,\tau))$$

is a homomorphism of $P^{-1}\circ P$ into $P^{*-1}\circ P^*$. Hence, if $(X,(\gamma,\delta))\in S$, we have

$$\begin{aligned}(X,(\gamma,\delta))\Phi_S^{-1}M &= ((E,\gamma),(X,\delta))M = ((E,\gamma)N(E^*,\tau),(X,\delta)N(E^*,\tau))\\ &= ((\gamma h,\gamma g)(E^*,\tau),((Xf)(\delta h),\delta g)(E^*,\tau))\\ &= (((\gamma h)\left|(\gamma g)^\tau\right|,(\gamma g)\odot\tau),((Xf)(\delta h)\left|(\delta g)^\tau\right|,(\delta g)\odot\tau))\\ &= ((E^*,\gamma g\odot\tau),(\left|(\gamma g)^\tau\right|^{-1}(\gamma h)^{-1}(Xf)(\delta h)\left|(\delta g)^\tau\right|,\delta g\odot\tau)).\end{aligned}$$

Consequently, $F = \Phi_S^{-1}M\Phi_{S^*}$ is a homomorphism of S into S^* as desired.

For the converse, let F be any homomorphism of S into S^* and define M by $M = \Phi_S F\Phi_{S^*}^{-1}$. By use of [7, Theorem 1.1], there exists a semi-lattice homomorphism N of P into P^* and $(Z,\tau)\in P^*$ such that M is given by

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N(Z,\tau),(Y,\delta)N(Z,\tau)).$$

Define $N^*\colon P\to P^*$ by $(X,\gamma)N^* = (Z^{-1},0)((X,\gamma)N)(Z,0)$. Thus,

$$\begin{aligned}((X,\gamma),(Y,\delta))M &= ((Z,0)(X,\gamma)N^*(E^*,\tau),(Z,0)(Y,\delta)N^*(E^*,\tau))\\ &= ((X,\gamma)N^*(E^*,\tau),(Y,\delta)N^*(E^*,\tau)).\end{aligned}$$

Since N^* is given by [4, Theorem 2.5], we have the conditions of that theorem satisfied so the conditions here are satisfied. From $F = \Phi_S^{-1}M\Phi_{S^*}$, we get (2.9) as desired.

THEOREM 2.2. *Let α and β be ordinal numbers such that $0<\beta\leqq\alpha$. Let*

$$S = (G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$$

be an ω^α-bisimple semigroup and let

$$S^* = (G^*, W_\beta, \{\Psi_\sigma\}, \{t(\delta,\sigma)\})$$

be an ω^β-bisimple semigroup. Let μ be an ordinal number such that $\alpha = \mu+\beta$. Let $\{U_\delta\colon 0\leqq\delta<\alpha\}$ be elements of G^* and let f be a homomorphism of G onto G^* such that

(2.12) $$fC_{U_\sigma} = \theta_\sigma f \text{ for } 0\leqq\sigma<\mu,$$

(2.13) $$(f\Psi_{\sigma-\mu})C_{U_\sigma} = \theta_\sigma f \text{ for } 0\leqq\mu\leqq\sigma<\alpha,$$

(2.14) $$((U_\delta\Psi_{\sigma-\mu})t(\sigma-\mu,\delta-\mu))C_{U_\sigma} = z(\sigma,\delta)f \quad \text{for } 0\leqq\mu\leqq\delta<\sigma<\alpha,$$

(2.15) $$(U_\delta\Psi_{\sigma-\mu})C_{U_\sigma} = z(\sigma,\delta)f \text{ for } 0\leqq\delta<\mu\leqq\sigma<\alpha,$$

and

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(2.16) $$U_\delta C_{U_\sigma} = z(\sigma,\delta)f \text{ for } 0 \leqq \delta < \sigma < \mu \leqq \alpha.$$

For each

$$(X,(\gamma,\delta)) = (X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{k} \odot \cdots \odot d_1\omega^{\delta_1})) \in S,$$

let

(2.17) $$(X,(\gamma,\delta))F = ((\gamma h)^{-1}(Xf)(\delta h),(\gamma g,\delta g))$$

where g is a homomorphism of $(H_\alpha, \odot)$ onto $(H_\beta, \odot)$ given by

(2.18) $$\gamma g = \begin{cases} 0 \text{ if } \gamma = 0, \\ c_m\omega^{\gamma_m-\mu} \odot \cdots \odot c_1\omega^{\gamma_1-\mu} \quad \text{if } \gamma \neq 0 \text{ and } \mu \leqq \gamma_m, \\ c_s\omega^{\gamma_s-\mu} \odot \cdots \odot c_1\omega^{\gamma_1-\mu} \quad \text{if } \gamma \neq 0 \text{ and} \\ \qquad s \text{ is such that } 1 \leqq s \leqq m-1 \\ \qquad \text{and } \gamma_{s+1} < \mu \leqq \gamma_s, \\ 0 \text{ if } \gamma \neq 0 \text{ and } \mu > \gamma_1, \end{cases}$$

and h is a function from H_α into G^* given by

(2.19) $$\gamma h = \begin{cases} E^* \text{ if } \gamma = 0, \\ \left(\prod_{j=0}^{c_m-1} U_{\gamma_m}\Psi^j_{\gamma_m-\mu}\right)\left(\prod_{j=0}^{c_{m-1}-1} U_{\gamma_{m-1}}\Psi^j_{\gamma_{m-1}-\mu}\Psi^{c_m}_{\gamma_m-\mu}\right)\cdots \\ \left(\prod_{j=0}^{c_1-1} U_{\gamma_1}\Psi^j_{\gamma_1-\mu}\Psi^{c_2}_{\gamma_2-\mu}\cdots\Psi^{c_m}_{\gamma_m-\mu}\right) \text{ if } \gamma \neq 0 \text{ and } \mu \leqq \gamma_m, \\ \left(\prod_{i=0}^{m-s-1} U^{c_{m-i}}_{\gamma_{m-i}}\right)\left(\prod_{j=0}^{c_s-1} U_{\gamma_s}\Psi^j_{\gamma_s-\mu}\right)\cdots \\ \left(\prod_{j=0}^{c_1-1} U_{\gamma_1}\Psi^j_{\gamma_1-\mu}\Psi^{c_2}_{\gamma_2-\mu}\cdots\Psi^{c_s}_{\gamma_s-\mu}\right) \text{ if } \gamma \neq 0 \text{ and } s \text{ is such that} \\ \qquad 1 \leqq s \leqq m-1 \text{ and } \gamma_{s+1} < \mu \leqq \gamma_s, \\ \left(\prod_{j=0}^{m-1} U^{c_{m-j}}_{\gamma_{m-j}}\right) \text{ if } \gamma \neq 0 \text{ and } \mu > \gamma_1. \end{cases}$$

Then, F is a homomorphism of S onto S^*. Conversely, if F is any homomorphism of S onto S^*, then there exists an ordinal number μ such that $\alpha = \mu + \beta$, elements $\{U_\delta : 0 \leqq \delta < \alpha\}$ of G^*, and a homomorphism f of G into G^* such that (2.12), (2.13), (2.14), (2.15), *and* (2.16) *are valid and F is given by* (2.17).

PROOF. Assume the conditions as given. Thus, by [4, Theorem 2.6],

$$(X,\gamma)N = ((Xf)(\gamma h),\gamma g)$$

is a homomorphism of P onto P^* where g is given by (2.18) and h is given by (2.19). By [4, Theorem 2.9], N is a semi-lattice homomorphism. From [7, Theorem 1.1],

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N,(Y,\delta)N)$$

is a homomorphism. It is clear that M is onto. Thus, if we let $F = \Phi_S^{-1}M\Phi_{S^*}$, then

$$(X,(\gamma,\delta))F = ((E,\gamma),(X,\delta))M\Phi_{S^*} = ((\gamma h,\gamma g),((Xf)(\delta h),\delta g))\Phi_{S^*}$$
$$= ((E^*,\gamma g),((\gamma h)^{-1}(Xf)(\delta h),\delta g))\Phi_{S^*} = ((\gamma h)^{-1}(Xf)(\delta h),(\gamma g,\delta g))$$

as in (2.17).

Conversely, let F be any homomorphism of S onto S^* and define M by $M = \Phi_S F\Phi_{S^*}^{-1}$. By [7, Theorem 1.1], there exists a semi-lattice homomorphism N of P into P^* and $(Z,\tau)\in P^*$ such that M is given by

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N(Z,\tau),(Y,\delta)N(Z,\tau)).$$

Let

$$((A,\rho),(B,\sigma))M = ((E^*,0),(E^*,0)) = ((A,\rho)N(Z,\tau),(B,\sigma)N(Z,\tau)).$$

Thus, $(A,\rho)N(Z,\tau) = (W,0)$ for some $W\in G^*$, whence $(Z,\tau) = (V,0)$ for some $V\in G^*$. If we define $N^*: P\to P^*$ by

$$(X,\gamma)N^* = (V^{-1},0)((X,\gamma)N)(V,0),$$

then

$$((X,\gamma)(Y,\delta))M = ((X,\gamma)N(V,0),(Y,\delta)N(V,0))$$
$$= ((V,0)((X,\gamma)N^*),(V,0)((Y,\delta)N^*)) = ((X,\gamma)N^*,(Y,\delta)N^*).$$

We use the fact that N^* is given by [4, Theorem 2.6] to complete the proof.

REMARK. Theorem 2.10 of [4] describes the isomorphisms of an ω^α-bisimple semigroup S onto an ω^α-bisimple semigroup S^*.

THEOREM 2.3. *There exists no homomorphism of an ω^α-bisimple semigroup S onto an ω^β-bisimple semigroup S^* for $\alpha<\beta$.*

PROOF. Suppose F is a homomophism of S onto S^*. Then, $M = \Phi_S F\Phi_{S^*}^{-1}$ is given by [7, Theorem 1.1]. That is,

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N(Z,\tau),(Y,\delta)N(Z,\tau))$$

for some $(Z,\tau)\in P^*$ and some semi-lattice homomorphism N of P into P^*. As in the proof of Theorem 2.2, we get

$$((X,\gamma),(Y,\delta))M = ((X,\gamma)N^*,(Y,\delta)N^*),$$

where N^* is a homomorphism of P into P^*. We note that N^* is given by [4, Theorem 2.5]. Thus,

$$(X,\gamma)N^* = ((Xf)(\gamma h),\gamma g),$$

where f is a homomorphism of G into G^* and g is a homomorphism of $(H_\alpha,\odot)$ into $(H_\beta,\odot)$ given by [4, (2.3)], [4, (2.4)] and [4, (2.5)]. However, if $\rho \in H_\beta$, there exists $(A,\sigma)\in P$ and $(V,0)\in P^*$ such that

$$(A,\sigma)N^* = ((Af)(\sigma h),\sigma g) = (V,0)(E^*,\rho) = (V,\rho).$$

That is, g must be onto. By [4, Theorem 2.4], g cannot be onto when $\alpha < \beta$.

THEOREM 2.4. *Let $S = \{G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\}\}$ be an ω^α-bisimple semigroup and let S_0 be a group. Let $\{U_\delta : 0 \leqq \delta < \alpha\}$ be elements of S_0 and let f be a homomorphism of G into S_0 such that*

$$fC_{U_\sigma} = \theta_\sigma f \text{ for } 0 \leqq \sigma < \alpha \tag{2.20}$$

and

$$U_\delta C_{U_\sigma} = z(\sigma,\delta)f \text{ for } 0 \leqq \delta < \sigma < \alpha. \tag{2.21}$$

Then

$$\begin{aligned}(X,(\gamma,\delta))F &= (X,(c_m\omega^{\gamma} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1})F \\ &= \left(\prod_{j=0}^{m-1} U_{\gamma_{m-j}}^{c_{m-j}}\right)^{-1}(Xf)\left(\prod_{j=0}^{k-1} U_{\delta_{k-j}}^{d_{k-j}}\right)\end{aligned} \tag{2.22}$$

is a homomorphism of S into S_0.

Conversely, every homomorphism of S into S_0 is obtained in this fashion.

PROOF. The proof is essentially contained in the proof of Theorem 2.1.

3. The Congruence Relations

In this section we study the congruence relations on an ω^α-bisimple semigroup. As mentioned previously, we show that any congruence relation on such a semigroup is a group congruence, an ω^β-bisimple congruence for some ordinal number β such that $0 < \beta < \alpha$, an ω^α-bisimple congruence which is an idempotent separating congruence, or an ω^α-bisimple congruence which is not an idempotent separating congruence. A description of the idempotent separating congruences and of the ω^β-bisimple congruences for $0 \leqq \beta < \alpha$ is presented.

THEOREM 3.1. *Let $S = (G, W_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$ be an ω^α-bisimple semigroup. Let E denote the identity of G and let R be a congruence relation on S. Then,*

R is an ω^β-bisimple congruence for $0 < \beta \leqq \alpha$ if and only if there exists an ordinal number μ such that $\alpha = \mu + \beta$ and the following condition is satisfied:

$$(E,(\gamma,\gamma))R(E,(\delta,\delta)) \text{ if and only if } \eta = \tau \text{ whenever}$$
$$\gamma = \eta\omega^\mu + \nu,\ 0 \leqq \nu < \omega^\mu, \text{ and } \delta = \tau\omega^\mu + \sigma,$$
$$0 \leqq \sigma < \omega^\mu.$$

Moreover, if $\mu > 0$, then R is not idempotent separating.

PROOF. Suppose R is an ω^β-bisimple congruence on S with $0 < \beta \leqq \alpha$. Let $F: S \to S/R$ be the natural mapping. Note that F is described by Theorem 2.2. Thus, there exists an ordinal number μ such that $\alpha = \mu + \beta$ and F is given by (2.17). Next, suppose $(E,(\gamma,\gamma))R(E,(\delta,\delta))$ with $\gamma = \eta\omega^\mu + \nu$, $0 \leqq \nu < \omega^\mu$, and $\delta = \tau\omega^\mu + \sigma$, $0 \leqq \sigma < \omega^\mu$. Hence,

$$(E,(\gamma,\gamma))F = (E,(\delta,\delta))F, (E^*,(\gamma g,\gamma g)) = (E^*,(\delta g,\delta g)),$$

and $\gamma g = \delta g$. It follows that $\eta = \tau$ using (2.18). If $\eta = \tau$ with $\gamma = \eta\omega^\mu + \nu$, $0 \leqq \nu < \omega^\mu$, and $\delta = \tau\omega^\mu + \sigma$, $0 \leqq \sigma < \omega^\mu$, then $\gamma g = \delta g$. Hence, by reversing the steps above, we get $(E,(\gamma,\gamma))R(E,(\delta,\delta))$.

Now suppose R is a congruence relation on S, suppose there exists an ordinal number μ such that $\alpha = \mu + \beta$, and suppose the condition is satisfied. The semigroup S/R is a bisimple inverse semigroup with identity. The set of idempotents of S/R is a kernel normal system of S [2, Theorem 7.48]. These facts, along with the information that for any ordinal number τ where $0 \leqq \tau < \omega^\beta$ we have

$$(E,(\tau\omega^\mu,\tau\omega^\mu))R(E,(\tau\omega^\mu + \sigma,\tau\omega^\mu + \sigma))$$

when $0 \leqq \sigma < \omega^\mu$, produce the result that S/R is an ω^β-bisimple semigroup. It is clear that R is not idempotent separating when $\mu > 0$.

THEOREM 3.2. *Let S be an ω^α-bisimple semigroup and let R be a congruence relation on S. Then, R is a group congruence, an ω^β-bisimple congruence for some ordinal number β such that $0 < \beta < \alpha$, an ω^α-bisimple congruence which is not idempotent separating, or an ω^α-bisimple congruence which is idempotent separating.*

PROOF. Let R be a congruence on S which is not a group congruence. Let $(E,(\delta,\delta))$ be the largest idempotent of S such that $(E,(\delta,\delta))R(E,(0,0))$. We wish to show that there exists an ordinal number μ such that $0 \leqq \mu < \alpha$ and $\delta = \omega^\mu$. To that end, let $\gamma < \delta$. Assume $\delta - \gamma < \delta$. Hence,

$$(E,(0,0))R(E,(\delta-\gamma,\delta-\gamma)),$$
$$(E,(\delta,\delta))R(E,(\delta,0))(E,(\delta-\gamma,\delta-\gamma))(E,(0,\delta)), \text{ and}$$
$$(E,(\delta,\delta))R(E,(\delta+(\delta-\gamma),\delta+(\delta-\gamma)).$$

We let $x = (E,(\gamma,\delta))$ and utilize [1, Lemma 1.31] to show that $\langle x, x^{-1}\rangle$ is the bicyclic semigroup W_1. By [1, Corollary 1.32], it follows that $R \mid \langle x, x^{-1}\rangle$ is idempotent separating. However,

$$x^{-1}x = (E,(\delta,\delta))R(E,(\delta+(\delta-\gamma),\delta+(\delta-\gamma)) = x^{-1}x^{-1}xx,$$

is contradictory. We conclude that $\delta - \gamma = \delta$ and $\gamma + \delta = \delta$. From [5, Theorem 1, p. 282] and [5, Theorem 1, p. 323], we conclude that $\delta = \omega^\mu$ for some μ where $0 \leqq \mu < \alpha$. Next, let $\beta = \alpha - \mu$ or $\alpha = \mu + \beta$. Let $\gamma = \eta\omega^\mu + \nu$ where $0 \leqq \nu < \omega^\mu$ and $\delta = \eta\omega^\mu + \sigma$ where $0 \leqq \sigma < \omega^\mu$. We wish to show $(E,(\gamma,\gamma))R(E,(\delta,\delta))$. To this end, suppose $\gamma < \delta$. Then, $\gamma + (\sigma - \nu) = \delta$ and $\delta - \gamma = \sigma - \nu < \omega^\mu$. Hence,

$$(E,(0,0))R(E,(\delta-\gamma,\delta-\gamma)),$$

$$(E,(\gamma,\gamma))R(E,(\gamma,0))(E,(\delta-\gamma,\delta-\gamma))(E,(0,\gamma)), \text{ and}$$

$$(E,(\gamma,\gamma))R(E,(\delta,\delta)).$$

Finally, suppose $(E,(\gamma,\gamma))R(E,(\delta,\delta))$ where $\gamma = \eta\omega^\mu + \nu, 0 \leqq \nu < \omega^\mu, \delta = \tau\omega^\mu + \sigma$, and $0 \leqq \sigma < \omega^\mu$. We wish to show that $\eta = \tau$. To this end, assume $\eta < \tau$. Then, $\delta > \gamma$ and $\delta - \gamma = (\tau - \eta)\omega^\mu + \sigma$. Hence, by previous considerations, we have

$$(E,(\delta-\gamma,\delta-\gamma))R(E,((\tau-\eta)\omega^\mu,(\tau-\eta)\omega^\mu)).$$

Now, $(E,(\gamma,\gamma))R(E,(\delta,\delta))$ implies

$$(E,(\gamma,0))(E,(0,0))(E,(0,\gamma))R(E,(\gamma,0))(E,(\delta-\gamma,\delta-\gamma))(E,(0,\gamma))$$

and we get $(E,(0,0))R(E,(\delta-\gamma,\delta-\gamma))$. Thus,

$$(E,(0,0))R(E,((\tau-\eta)\omega^\mu,(\tau-\eta)\omega^\mu)), \text{ and}$$
$$(E,(\omega^\mu,\omega^\mu))R(E,(\omega^\mu,\omega^\mu))(E,((\tau-\eta)\omega^\mu,(\tau-\eta)\omega^\mu)), \text{ or}$$
$$(E,(\omega^\mu,\omega^\mu)R(E,((\tau-\eta)\omega^\mu,(\tau-\eta)\omega^\mu)).$$

But, here we have $(E,(0,0))R(E,(\omega^\mu,\omega^\mu))$ contrary to the choice of ω^μ. We conclude that $\eta = \tau$. At this stage, we utilize Theorem 3.1 to conclude that R is an ω^β-bisimple congruence for $0 < \beta \leqq \alpha$ which is not idempotent separating in case $\mu > 0$. The conclusion of the theorem is now immediate.

Remark. If α is not a limit ordinal, it is impossible for a congruence relation on an ω^α-bisimple semigroup to be an ω^α-bisimple congruence which is not idempotent separating.

The following two theorems are straightforward generalizations of results o Warne [11].

Theorem 3.3. *Let $P = (G, H_\alpha, \{\theta_\sigma\}, \{z(\delta,\sigma)\})$ be an ω^α-right cancellative semigroup. The right normal divisors of P are the $\{\theta_\sigma\}$-invariant subgroups of G.*

PROOF. Let V be a right normal divisor of P and let $X \in V$. Then, if $0 \leqq \sigma < \alpha$ and E denotes the identity of G, we have

$$(E,\omega^{\sigma})(X,0) = ((X\theta_{\sigma}),\omega^{\sigma}) = (Y,0)(E,\omega^{\sigma}) = (Y,\omega^{\sigma})$$

where $Y \in V$. Hence, $X\theta_{\sigma} \in V$ and V is $\{\theta_{\sigma}\}$-invariant.

Conversely, let V be an $\{\theta_{\sigma}\}$-invariant subgroup of G. Let $Y \in V$ and let $(X,\gamma) \in P$ where

$$\gamma = c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}.$$

Then,

$$(X,\gamma)(Y,0) = \left(X\left(Y\prod_{i=1}^{m}\theta_{\gamma_i}^{c_i}\right),\gamma\right) = \left(X\left(Y\prod_{i=1}^{m}\theta_{\gamma_i}^{c_i}\right)X^{-1},0\right)(X,\gamma).$$

Consequently, V is a right normal divisor of P.

THEOREM 3.4. *Let $S = (G, W_{\alpha}, \{\theta_{\sigma}\}, \{z(\delta,\sigma)\})$ be an ω^{α}-bisimple semigroup. There is a one-to-one correspondence between the idempotent separating congruences of S and the $\{\theta_{\sigma}\}$-invariant subgroups of G. If R^V is the congruence relation corresponding to the $\{\theta_{\sigma}\}$-invariant subgroup V, then the congruence class containing $(X,(\gamma,\delta))$ is the set $\{(YX,(\gamma,\delta)): Y \in V\}$. If V and W are $\{\theta_{\sigma}\}$-invariant subgroups of G, then $V \subseteq W$ if and only if $R^V \subseteq R^W$. $\mathscr{H}$ is the maximal idempotent separating congruence on S. The idempotent separating congruences on S are those congruences such that the congruence class containing the identity is a group, and they are uniquely determined by this class.*

PROOF. This result follows from Theorem 1.6, Theorem 1.8, Theorem 3.3, [8, Lemma 1.2], [9, Theorem 2], and (2.1).

We now describe the congruence relations on an ω^{α}-bisimple semigroup which are ω^{β}-bisimple congruences where $0 < \beta \leqq \alpha$.

THEOREM 3.5. *Let $S = (G, W_{\alpha}, \{\theta_{\sigma}\}, \{z(\delta,\sigma)\})$ be an ω^{α}-bisimple semigroup and let R be an ω^{β}-bisimple congruence on S with $0 < \beta \leqq \alpha$. Then, there exists an ordinal number μ such that $\alpha = \mu + \beta$ and an $\{\theta_{\sigma}\}$-invariant subgroup V of G with the following satisfied:*

(3.1) *If for each ordinal number ν such that $0 \leqq \nu < \omega^{\beta}$ we let*

$$A_{\nu} = \{(X,(\gamma,\delta)): (X,(\gamma,\delta))R(E,(\nu\omega^{\mu},\nu\omega^{\mu}))\},$$

then the collection $A = \{A_{\nu}: 0 \leqq \nu < \omega^{\beta}\}$ is a kernel normal system and

$$R = \{(a,b) \in S \times S: aa^{-1}, bb^{-1}, ab^{-1} \in A_{\nu} \text{ for some } \nu\}.$$

(3.2) *If $\chi = \nu\omega^{\mu} + \gamma$, $\xi = \nu\omega^{\mu} + \rho$, $\tau = \kappa\omega^{\mu} + \delta$, and $\lambda = \kappa\omega^{\mu} + \sigma$*

where $0 \leqq \gamma < \omega^{\mu}$, $0 \leqq \rho < \omega^{\mu}$, $0 \leqq \delta < \omega^{\mu}$, and $0 \leqq \sigma < \omega^{\mu}$, then

$(X,(\chi,\tau))R(Y,(\xi,\lambda))$ *if and only if* $(X,(\gamma,\delta))R(Y,(\rho,\sigma))$.

(3.3) *Suppose* $\mu = 0$. *Then,* $(X,(\gamma,\delta)) \in A_0$ *if and only if* $\gamma = \delta = 0$ *and* $X \in V$. *Also,*

$$(X,(\gamma,\delta))R(Y,(\rho,\sigma)) \text{ if and only if } \gamma = \rho,\ \delta = \sigma,$$

and $XY^{-1} \in V$.

(3.4) *Suppose* $\mu > 0$ *is an ordinal number of the second kind. Then,*

$$(X,(\gamma,\delta)) = (X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1})) \in A_0$$

if and only if $0 \leqq \max\{\gamma_1,\delta_1\} < \mu$ *and*

$$X\theta_{r(\gamma,\delta)} \in V\left(\prod_{j=0}^{m-1} z^{c_{m-j}}(r(\gamma,\delta),\gamma_{m-j})\right)\left(\prod_{j=0}^{k-1} z^{d_{k-j}}(r(\gamma,\delta),\delta_{k-j})\right)^{-1}$$

where $r(\gamma,\delta) = \max\{\gamma_1,\delta_1\}+1$. *Also, if* $0 \leqq \gamma_1 < \mu$, $0 \leqq \delta_1 < \mu$, $0 \leqq \rho_1 < \mu$ *and* $0 \leqq \sigma_1 < \mu$, *then*

$$(X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1}))R$$
$$(Y,(r_q\omega^{\rho_q} \odot \cdots \odot r_1\omega^{\rho_1}, s_n\omega^{\sigma_n} \odot \cdots \odot s_1\omega^{\sigma_1}))$$

if and only if

$$\left(\prod_{j=0}^{m-1} z^{c_{m-j}}(s(\gamma,\delta,\rho,\sigma),\gamma_{m-j})\right)^{-1}(X\theta_{s(\gamma,\delta,\rho,\sigma)})\Bigg)$$

$$\left(\prod_{j=0}^{k-1} z^{d_{k-j}}(s(\gamma,\delta,\rho,\sigma),\delta_{k-j})\right)\left(\prod_{j=0}^{n-1} z^{s_{n-j}}(s(\gamma,\delta,\rho,\sigma),\sigma_{n-j})\right)^{-1}$$

$$(Y^{-1}\theta_{s(\gamma,\delta,\rho,\sigma)})\left(\prod_{j=0}^{q-1} z^{r_{q-j}}(s(\gamma,\delta,\rho,\sigma),\rho_{q-j})\right) \in V$$

where $s(\gamma,\delta,\rho,\sigma) = \max\{\gamma_1,\delta_1,\rho_1,\sigma_1\}+1$.

(3.5) *Suppose* μ *is an ordinal number of the first kind. Let* η *be defined by* $\eta + 1 = \mu$. *Let a non-negative integer p and an element B of G be specified as follows*: (a) *If* $(Y,(q\omega^\eta,0)) \notin A_0$ *for every* $Y \in G$ *and every positive integer q, then* $p = 0$ *and* $B = E$; (b) *If there exists* $Y \in G$ *and a positive integer q such that* $(Y,(q\omega^\eta,0)) \in A_0$, *then p is the smallest positive integer with this property and B is such that* $(B,(p\omega^\eta,0)) \in A_0$. *Then,*

$$(X,(\gamma,\delta)) = (X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1})) \in A_0$$

if and only if $0 \leqq \max\{\gamma_1,\delta_1\} < \mu$ *and*

$$X\theta_\eta \in V\left(\prod_{j=0}^{M(\omega^\eta,\gamma)} z^{c_{m-j}}(\eta,\gamma_{m-j})\right)B^r\left(\prod_{j=0}^{M(\omega^\eta,\delta)} z^{d_{k-j}}(\eta,\delta_{k-j})\right)^{-1}$$

where $rp = c_1N(\omega^\eta, \gamma) - d_1N(\omega^\eta, \delta)$. *Also, if* $0 \leqq \gamma_1 < \mu$, $0 \leqq \delta_1 < \mu$, $0 \leqq \rho_1 < \mu$, *and* $0 \leqq \sigma_1 < \mu$, *then*

$$(X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1}))R$$
$$(Y,(r_q\omega^{\rho} \odot \cdots \odot r_1\omega^{\rho_1}, s_n\omega^{\sigma_n} \odot \cdots \odot s_1\omega^{\sigma_1}))$$

if and only if

$$\left\{\left[\left(\prod_{j=0}^{M(\omega^\eta,\gamma)} z^{c_{m-j}}(\eta,\gamma_{m-j})\right)^{-1} X \left(\prod_{j=0}^{M(\omega^\eta,\delta)} z^{d_{k-j}}(\eta,\delta_{k-j})\right)\right]\theta_\eta^{r_1N(\omega^\eta,\rho)}\right\}$$
$$\left\{\left[\left(\prod_{j=0}^{M(\omega^\eta,\sigma)} z^{s_{n-j}}(\eta,\sigma_{n-j})\right)^{-1} Y^{-1} \left(\prod_{j=0}^{M(\omega^\eta,\rho)} z^{r_{q-j}}(\eta,\rho_{q-j})\right)\right]\theta_\eta^{c_1N(\omega^\eta,\gamma)}\right\} \in VB^t \text{ where}$$
$$pt = c_1N(\omega^\eta,\gamma) - d_1N(\omega^\eta,\delta) + s_1N(\omega^\eta,\sigma) - r_1N(\omega^\eta,\rho).$$

PROOF. Let $F: S \to S/R$ be the natural mapping where $S/R = S^* = (G^*, W_\beta, \{\Psi_\sigma\}, \{t(\delta,\sigma)\})$. Note that F is described by the converse in Theorem 2.2. Thus, there exists an ordinal number μ such that $\alpha = \mu + \beta$, elements $\{U_\delta: 0 \leqq \delta < \alpha\}$ of G^*, and a homomorphism f of G into G^* such that (2.12), (2.13), (2.14), (2.15), and (2.16) are valid and F is given by (2.17). If V is the kernel of f, then V is $\{\theta_\sigma\}$-invariant by use of (2.12) and (2.13).

Let $A = \{A_i: i \in \Delta\}$ be the set of idempotents of S/R. From [2, Theorem 7.48], A is a kernel normal system and $R = \{(a,b) \in S \times S: aa^{-1}, bb^{-1}, ab^{-1} \in A_i$ for some $i \in \Delta\}$. Next, let $(X,(\gamma,\delta)) \in A_i$ where

$$\gamma = c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1} \text{ and } \delta = d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1}.$$

Then, $((X,(\gamma,\delta))\,(X,(\gamma,\delta)))\,F = (X,(\gamma,\delta))\,F$. Hence,

$$(Z,(\gamma,\delta)(\gamma,\delta))F = (X,(\gamma,\delta))F$$

where $Z = X^2$ if $\delta = \gamma$, and

$$Z = X\left\{\left[\left(\prod_{j=0}^{M(\sigma,\gamma)} z^{c_{m-j}}(\sigma_1,\gamma_{m-j})\right)^{-1}(X\theta_{\sigma_1})\left(\prod_{j=0}^{M(\sigma,\delta)} z^{d_{k-j}}(\sigma_1,\delta_{k-j})\right)\right]\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}\right\}$$

if $\delta > \gamma$ and $\sigma = \delta - \gamma$ has the normal form $\sigma = s_p\omega^{\sigma_p} \odot \cdots \odot s_1\omega^{\sigma_1}$.

Since F is given by (2.17), we have

$$(X,(\gamma,\delta))R(E,(\gamma,\gamma)) \text{ if } \delta = \gamma,$$

$$(\gamma h)^{-1}(Zf)\,(\delta+(\delta-\gamma))h) = (\gamma h)^{-1}(Xf)(\delta h) \text{ and } (\delta+(\delta-\gamma))g = \delta g \text{ if } \delta > \gamma,$$

and

$$((\gamma+(\gamma-\delta))h)^{-1}(Zf)(\delta h) = (\gamma h)^{-1}(Xf)(\delta h) \text{ and } (\gamma+(\gamma-\delta))g = \gamma g \text{ if } \delta < \gamma.$$

Note that $\gamma g = \delta g$ in any case. If $\delta > \gamma$, we utilize [4, (2.11)] to produce

$$((\left|(\delta-\gamma)^\delta\right|)f)((\delta+(\delta-\gamma))h) = ((\delta-\gamma)h)(\delta h)$$

since $(\delta-\gamma)g = 0$. Hence, $(Zf)((\left|(\delta-\gamma)^\delta\right|)f)^{-1}((\delta-\gamma)h) = Xf$ and

$$\left\{\left[\left(\prod_{j=0}^{M(\sigma,\gamma)} z^{c_{m-j}}(\sigma_1,\gamma_{m-j}))^{-1}(X\theta_{\sigma_1}\right)\right.\right.$$
$$\left.\left.\left(\prod_{j=0}^{M(\sigma,\delta)} z^{d_{k-j}}(\sigma_1,\delta_{k-j})\right)\right]\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}\right\}f$$
$$((\left|(\delta-\gamma)^\delta\right|)f)^{-1}((\delta-\gamma)h) = E^*.$$

We utilize (1.9) to get

$$((\left|(\delta-\gamma)^\delta\right|)f = (\left|\sigma^\delta\right|)f = \left\{\left(\prod_{j=0}^{M(\sigma,\delta)} z^{\delta_{k-j}}(\sigma_1,\delta_{k-j})\right)\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}\right\}f.$$

Using these results and (1.10), it follows that

$$\left[\left(\prod_{j=0}^{M(\sigma,\gamma)} z^{c_{m-j}}(\sigma_1,\gamma_{m-j})\right)^{-1}\theta_{\sigma_1}^{s_1-1}\prod_{i=2}^{p}\theta_{\sigma_i}^{s_i}f\right](X^\sigma f)\,((\delta-\gamma)h) = E^*.$$

That is, $(\left|\sigma^\gamma\right|f)^{-1}(X^\sigma f)(\sigma h) = E^*$. From [4, 6, 2.18)] and the fact that $\sigma g = 0$, this becomes

$$(\left|\sigma^\gamma\right|f)^{-1}(\sigma h)(Xf) = E^*.$$

Then, from [4, (2.17)], we get

$$(\left|\sigma^\gamma\right|f)(\delta h) = (\sigma h)(\gamma h) \text{ so } (Xf) = (\gamma h)(\delta h)^{-1}.$$

This means that

$$(X,(\gamma,\delta))F = ((\gamma h)^{-1}(Xf)(\delta h),(\gamma g,\delta g)) = (E^*,(\gamma g,\gamma g)) = (E,(\gamma,\gamma))F,$$

or that $(X,(\gamma,\delta))R(E,(\gamma,\gamma))$. If $\delta < \gamma$, we begin with

$$((\gamma+(\gamma-\delta))h)^{-1}(Zf) = (\gamma h)^{-1}(Xf)$$

or $(X^{-1}f)(\gamma h) = (Z^{-1}f)((\gamma+(\gamma-\delta))h)$. As before, we have

$$(Z^{-1}f)((|\sigma^{\gamma}|)f)^{-1}((\gamma-\delta)h) = (X^{-1}f)$$

and $((|\sigma^{\delta}|)f)^{-1}(X^{\sigma}f)^{-1}(\sigma h) = E^*$. Hence,

$$((|\sigma^{\delta}|f)^{-1}(\sigma h)(X^{-1}f) = E^*$$

and $(Xf) = (\gamma h)(\delta h)^{-1}$. Again we get $(X,(\gamma,\delta))R(E,(\gamma,\gamma))$. Thus, in all cases we have $(X,(\gamma,\delta))R(E,(\gamma,\gamma))$. From Theorem 3.1, we conclude that A is given by (3.1).

Next, we consider (3.2). Then $(X,(\chi,\tau))R(Y,(\xi,\lambda))$ if and only if

$$(\chi h)^{-1}(Xf)(\tau h) = (\xi h)^{-1}(Yf)(\lambda h).$$

By utilizing [4, (2.17)], (1.9), and (1.1), we see that $(\tau h) = (\delta h)(\kappa\omega^{\mu}h)$, $(\lambda h) = (\sigma h)(\kappa\omega^{\mu}h)$, $(\chi h) = (\gamma h)(\nu\omega^{\mu}h)$, and $(\xi h) = (\rho h)(\nu\omega^{\mu}h)$. It is now clear that (3.2) is valid.

Now suppose $\mu = 0$. Let $(X,(\gamma,\delta))\in A_0$. Since F is given by (2.17), it follows that $\gamma = \delta = 0$ and $X\in V$. Conversely, if $\gamma = \delta = 0$ and $X \in V$, then $(X,(\gamma,\delta))\in A_0$. Since g is the identity mapping when $\mu = 0$, it is clear that $(X,(\gamma,\delta))R(Y,(\rho,\sigma))$ if and only if $\gamma = \rho$, $\delta = \sigma$, and $XY^{-1}\in V$. This completes the proof of (3.3).

Next, a brief outline of the proof of (3.4) is presented. We use the notation of (3.4). Thus, $(X,(\gamma,\delta))\in A_0$ if and only if

$$Xf = \left(\prod_{j=0}^{m-1} U_{\gamma_{m-j}}^{c_{m-j}}\right)\left(\prod_{j=0}^{k-1} U_{\delta_{k-j}}^{d_{k-j}}\right)^{-1}$$

and $0 \leqq \max\{\gamma_1,\delta_1\} < \mu$. We use (2.12) and (2.16) to show that the condition on X is equivalent to

$$X\theta_{r(\gamma,\delta)} \in V\left(\prod_{j=0}^{m-1} z^{c_{m-j}}(r(\gamma,\delta),\gamma_{m-j})\right)\left(\prod_{j=0}^{k-1} z^{d_{k-j}}(r(\gamma,\delta),\delta_{k-j})\right)^{-1}.$$

If $0 \leqq \gamma_1 < \mu$, $0 \leqq \delta_1 < \mu$, $0 \leqq \rho_1 < \mu$, and $0 \leqq \sigma_1 < \mu$, then

$$(X,(c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1}))R$$
$$(Y,(r_q\omega^{\rho} \odot \cdots \odot r_1\omega^{\rho_1}, s_n\omega^{\sigma_n} \odot \cdots \odot s_1\omega^{\sigma_1}))$$

if and only if

$$\left(\prod_{j=0}^{m-1} U_{\gamma_{m-j}}^{c_{m-j}}\right)^{-1}(Xf)\left(\prod_{j=0}^{k-1} U_{\delta_{k-j}}^{d_{k-j}}\right) = \left(\prod_{j=0}^{q-1} U_{\rho_{n-j}}^{r_{q-j}}\right)^{-1}(Yf)\left(\prod_{j=0}^{n-1} U_{\sigma_{n-j}}^{s_{n-j}}\right).$$

By (2.12) and (2.16), this condition is equivalent to the condition stated in (3.4).

Finally, we present an outline of the proof of (3.5). Let η, p, and B be as specified in (3.5). We use the notation of that result. Thus $(X,(\gamma,\delta))\in A_0$ if and only if $0 \leqq \max\{\gamma_1,\delta_1\} < \mu$ and

$$Xf = \left(\prod_{j=0}^{m-1} U_{\gamma_{m-j}}^{c_{m-j}}\right)\left(\prod_{j=0}^{k-1} U_{\delta_{k-j}}^{d_{k-j}}\right)^{-1} \quad \text{or}$$

$$Xf = \left(\prod_{j=0}^{M(\omega^\eta,\gamma)} U_{\gamma_{m-j}}^{c_{m-j}}\right) U_\eta^{c_1 N(\omega^\eta,\gamma) - d_1 N(\omega^\eta,\delta)} \left(\prod_{j=0}^{M(\omega^\eta,\delta)} U_{\delta_{k-j}}^{d_{k-j}}\right)^{-1}.$$

Following the procedure of prior work in utilizing (2.12) and (2.16) we show that this condition on X is equivalent to the condition

$$X\theta_\eta \in V\left(\prod_{j=0}^{M(\omega^\eta,\gamma)} z^{c_{m-j}}(\eta, \gamma_{m-j})\ B^r \left(\prod_{j=0}^{M(\omega^\eta,\delta)} z^{d_{k-j}}(\eta, \delta_{k-j})\right)^{-1}\right)$$

where $rp = c_1 N(\omega^\eta, \gamma) - d_1 N(\omega^\eta, \delta)$. If $0 \leqq \gamma_1 < \mu$, $0 \leqq \delta_1 < \mu$, $0 \leqq \rho_1 < \mu$, and $0 \leqq \sigma_1 < \mu$, then

$$(X, (c_m\omega^{\gamma_m} \odot \cdots \odot c_1\omega^{\gamma_1}, d_k\omega^{\delta_k} \odot \cdots \odot d_1\omega^{\delta_1}))R$$
$$(Y, (r_q\omega^{\rho_q} \odot \cdots \odot r_1\omega^{\rho_1}, s_n\omega^{\sigma_n} \odot \cdots \odot s_1\omega^{\sigma_1}))$$

if and only if

$$\left(\prod_{j=0}^{m-1} U_{\gamma_{m-j}}^{c_{m-j}}\right)^{-1}(Xf)\left(\prod_{j=0}^{k-1} U_{\delta_{k-j}}^{d_{k-j}}\right) = \left(\prod_{j=0}^{q-1} U_{\rho_{q-j}}^{r_{q-j}}\right)^{-1}(Yf)\left(\prod_{j=0}^{n-1} U_{\sigma_{n-j}}^{s_{n-j}}\right).$$

We use (2.16) to show this condition to be equivalent to

$$U_\eta^{r_1 N(\omega^\eta,\rho)}\left\{\left[\left(\prod_{j=0}^{M(\omega^\eta,\gamma)} z^{c_{m-j}}(\eta,\gamma_{m-j})\right)^{-1} X \left(\prod_{j=0}^{M(\omega^\eta,\delta)} z^{d_{k-j}}(\eta,\delta_{k-j})\right)\right] f\right\} U_\eta^{d_1 N(\omega,\delta)}$$
$$= U_\eta^{c_1 N(\omega^\eta,\gamma)}\left\{\left[\left(\prod_{j=0}^{M(\omega^\eta,\rho)} z^{r_{q-j}}(\eta,\rho_{q-j})\right)^{-1} Y \left(\prod_{j=0}^{M(\omega^\eta,\sigma)} z^{s_{n-j}}(\eta,\sigma_{n-j})\right)\right] f\right\} U_\eta^{s_1 N(\omega^\eta,\sigma)}.$$

Then, (2.12) is used to complete the proof of (3.5).

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. I (Math. Surveys No. 7, Amer. Math. Soc., Providence, 1961).

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. II. (Math. Surveys No. 7, Amer. Math. Soc., Providence, 1967).

[3] A. H. Clifford, ' A Class of *d*-Simple Semigroups', *Amer. J. Math.* 75 (1953), 547–556.
[4] J. W. Hogan, 'Bisimple Semigroups with Idempotents Well-ordered', (To appear).
[5] Waclaw Sierpinski, *Cardinal and Ordinal Numbers* (Second Ed. Revised, Panstwowe Wydawnictwo Naukowe (PWN — Polish Scientific Publishers), Warszawa (Poland), 1965).
[6] R. J. Warne, 'Matrix Representation of *d*-Simple Semigroups', *Trans. Amer. Math. Soc.* 106 (1963), 427–435.
[7] R. J. Warne, 'Homomorphisms of *d*-Simple Semigroups with Identity,' *Pacific J. Math.* 14 (1964), 1111–1122.
[8] R. J. Warne, 'A Class of Bisimple Inverse Semigroups,' *Pacific J. Math.* 18 (1966), 563–577.
[9] R. J. Warne, 'The Idempotent Separating Congruences of a Bisimple Inverse Semigroup', *Publicationes Mathematicae*, 13 (1966), 203–206.
[10] R. J. Warne, 'Bisimple Inverse Semigroups Mod Groups,' *Duke Math. J.* 34 (1967), 787–811.
[11] R. J. Warne, 'Congruences on ω^n -Bisimple Semigroups,' *J. Australian Math. Soc.* 9 (1969), 257–274.

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