OVALS, DUALITIES, AND DESARGUES'S THEOREM

T. G. OSTROM

Introduction. Consider a projective plane with $n + 1$ points on each line, An *oval* $\mathfrak{C}$ is a set of $n + 1$ points, no three of which are collinear.

DEFINITION. (1) A line which contains two points of $\mathfrak{C}$ will be called a *secant* of $\mathfrak{C}$.

(2) A line which contains exactly one point of $\mathfrak{C}$ will be called a *tangent* of $\mathfrak{C}$ or an *absolute line.*

(3) A line which contains no points of $\mathfrak{C}$ will be called an *exterior line* of $\mathfrak{C}$.

Each point of $\mathfrak{C}$ lies on exactly one tangent of $\mathfrak{C}$. Qvist **(10)** has shown that, for n odd, no three tangents of $\mathfrak{C}$ are concurrent. For n odd, we can make the following definitions.

DEFINITION. (4) A point which lies on two tangents of $\mathfrak{C}$ will be called an *exterior point* of $\mathfrak{C}$.

(5) A point which lies on no tangents of $\mathfrak{C}$ will be be called an *interior point* of $\mathfrak{C}$.

(6) Points of $\mathfrak{C}$ will sometimes be referred to as *absolute points*.

There is a natural correspondence between secants and exterior points. Each secant contains two absolute points; by taking the point of intersection of their tangents a definite exterior point is determined corresponding to the secant. Let the point corresponding to a given secant be called the *pole* of the secant, the secant will be called the *polar* of the point. The number of exterior points and the number of secants are both equal to $\frac{1}{2}n(n + 1)$ (the number of pairs of absolute points). Since there are $n^2 + n + 1$ points in all, $n + 1$ of which are absolute, the number of interior points is $\frac{1}{2}n(n - 1)$. Now Baer **(1)** has shown that the absolute points of a polarity form an oval unless n is a square or is even. Segre has shown that, in a Desarguesian plane, every oval determines a conic. It is well known that a conic in a Desarguesian plane determines a polarity.

The following form of Desargues's theorem holds in any plane admitting a polarity: "If two self-polar triangles are perspective from a point, they are perspective from the polar of that point." While we do not restrict ourselves to Desarguesian planes, we find that Desargues's configuration comes up again and again for suitably restricted pairs of triangles.

In §1, we study certain kinds of collineations which carry an oval into itself. In §2, we study certain respects in which an oval resembles a conic in a Desarguesian plane, especially in its relations to a modified type of harmonic set. In § 3, we specialize the results of §2 to the case where the oval consists of the absolute points of a polarity. In §4, we consider ovals in cyclic and other

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transitive planes. In §5, we show the existence of Fano's configuration for certain planes with n even. Except in §5, we restrict ourselves to odd values of n. The author is greatly indebted to the referee for suggestions which led to the inclusion of §2.

1. Collineations carrying an oval into itself. In this part, we consider collineations σ satisfying the following conditions: (1) σ must carry some oval $\mathfrak{C}$ into itself. (2) Excepting the identity, no power of σ leaves fixed any points which are not left fixed by σ itself.

σ, then, may leave certain points fixed, but the other points are arranged in cycles whose length is equal to the order of σ.

We list some fairly obvious properties of σ:

1.1. Any collineation which carries $\mathfrak{C}$ into itself must carry exterior points of $\mathfrak{C}$ into exterior points and interior points into interior points.

1.2. Since the g.c.d. of $n + 1$, $\frac{1}{2}n(n + 1)$ and $\frac{1}{2}n(n - 1)$ is 1, σ must leave at least one point fixed.

1.3. If an exterior point U is fixed, its polar is fixed, and vice versa. This follows from the fact that the two tangents through U must either be fixed or mapped into each other. In turn, the corresponding points of tangency must either be fixed or mapped into each other.

1.4. If the fixed points of σ constitute a proper subplane with $n_1 + 1$ points on a line, and one of the fixed points belongs to $\mathfrak{C}$, then $n_1 + 1$ points of $\mathfrak{C}$ are fixed and the fixed points of $\mathfrak{C}$ constitute an oval of the subplane.

Proof. Let A be a fixed point of $\mathfrak{C}$. There are $n_1 + 1$ fixed lines through A, one of which will be the tangent at A. The other will be secants. But if we have a fixed secant through a fixed absolute point, the other absolute point on the secant must also be fixed.

1.5. If three points of $\mathfrak{C}$ are fixed, the fixed points of the plane constitute a proper subplane, since the poles of the corresponding secants must also be fixed.

1.6. If (a) no points (b) one point (c) two points or (d) $n_1 + 1$ points of $\mathfrak{C}$ are fixed, then the order of σ divides (a) $n + 1$, (b) n, (c) $n - 1$, or (d) $n - n_1$. This is essentially a generalization of some of the results in **(9)**. See **(2**, Theorem 3**)** for a further characterization of σ if the order of σ is a prime power.

It should perhaps be noted that properties 1.1, 1.3, 1.4, and 1.5 apply to any collineation carrying $\mathfrak{C}$ into itself.

THEOREM 1.1. *If σ leaves invariant a proper subplane with n_1+1 points on a line, and if all of the fixed points are interior, then the order of σ divides*

$$(\tfrac{1}{2}(n + 1),\ n_1 + 1).$$

Proof. If a secant were fixed, the exterior point which is its pole would also be fixed. Hence there are no fixed secants. Similarly, no tangents are fixed. Hence the fixed lines are all exterior lines. Since each exterior point lies on two of the $n + 1$ tangents, it follows that half of the points on an exterior line are exterior and the other half are interior points. (Similarly, a secant contains $\frac{1}{2}(n - 1)$ exterior points and $\frac{1}{2}(n - 1)$ interior points). Consider a fixed exterior line u. The order of σ must divide the number of non-fixed interior points on u, which is $\frac{1}{2}(n + 1) - (n_1 + 1)$. But the number of non-fixed exterior points on u is $\frac{1}{2}(n + 1)$, so that the order of σ must also divide $\frac{1}{2}(n + 1)$.

THEOREM 1.2. *If σ is of even order, and leaves invariant a proper subplane with $n_1 + 1$ points per line, at least one point of which belongs to $\mathfrak{C}$, then $n = n_1^2$ and none of the fixed points is interior.*

Proof. If σ is of even order, some power of σ is of order 2 and has the same fixed points. Without loss of generality, take σ to be of order 2. By property 4, $n_1 + 1$ points of $\mathfrak{C}$ are fixed. These $n_1 + 1$ points determine $\frac{1}{2}n_1(n_1 + 1)$ fixed secants. In addition, the remaining $n - n_1$ points of $\mathfrak{C}$ are interchanged in pairs, thus determining $\frac{1}{2}(n - n_1)$ fixed secants. Each fixed secant determines a fixed point. The number of fixed interior points is then

$$n_1^2 + n_1 + 1 - (n_1 + 1) - \tfrac{1}{2}n_1(n_1 + 1) - \tfrac{1}{2}(n - n_1) = \tfrac{1}{2}(n_1^2 - n).$$

But **(6)** $n \geqslant n_1^2$.

This theorem would seem to be closely related to Mann's theorem **(7)** that a cyclic plane possesses a multiplier of even order only if n is a square, since, as we shall see, there are ovals in cyclic planes which are left invariant by the multipliers. We have been unable to obtain any interesting results for the case where no point of $\mathfrak{C}$ is fixed except that $\frac{1}{2}(n + 1)$ exterior points remain fixed if the order of σ is even.

THEOREM 1.3. *If the fixed points of σ consist of the points on a line u and a point U not belonging to u or $\mathfrak{C}$, then* (a) *σ is of order 2;* (b) *every triangle and its image satisfy Desargues's theorem;* (c) *if U is exterior, u is its polar;* (d) *if U is exterior and AB, CD are two secants through U, where A, B, C, and D are absolute points, then the secants AC and BD intersect in a point of u, similarly, AD and BC intersect in a point of u.*

Proof. (a) Every line through U is fixed. Let AB be a fixed secant through U. Then $A \rightleftarrows B$, or A and B are fixed. But at most two points of $\mathfrak{C}$ lie on u and there are at least $\frac{1}{2}(n - 1)$ (fixed) secants through U. Hence at least one pair of absolute points on a secant must be interchanged. (b) Let V, W, X be three non-fixed collinear points and let V_1, W_1, and X_1 be their images. Then the lines VV_1, WW_1, and XX_1 are fixed lines and must intersect in U. The lines VW and V_1W_1 are each other's images and must intersect in a point on u. Similarly, the other pairs of corresponding sides intersect in a point on u. (c) If U is exterior, its polar is a fixed line not through U and hence must be u. (d) Since $A \rightleftarrows B$ and $C \rightleftarrows D$, it follows that $AC \rightleftarrows BD$ and $AD \rightleftarrows BC$ so that these pairs of lines must intersect in u.

THEOREM 1.4. *Suppose that, for every point U not on an oval $\mathfrak{C}$, there is a collineation taking $\mathfrak{C}$ into itself which leaves U and every point on some line u fixed (u depending on U) then $\mathfrak{C}$ determines a polarity.*

Proof. Since all secants through U are fixed, their poles are fixed and lie on the line u. If U is interior, define the polar of U to be u, U the pole of u. With this extended relation of pole and polar, it is still true that a line is fixed if and only if its pole is fixed. Now, if we consider all lines through *any* point U, their poles must lie all on a certain line u. Note that only one such collineation is possible for each point U, since if there were two such collineations σ_1 and σ_2 then $\sigma_1\,\sigma_2^{-1}$ would leave every point of $\mathfrak{C}$ fixed and $\sigma_1\sigma_2^{-1}$ is the identity.

2. Harmonic sets and harmonic ovals.

DEFINITION. An ordered set of collinear points $\{AB\colon CD\}$ will be said to be *harmonic with respect to the oval* $\mathfrak{C}$ if (1) A and B belong to $\mathfrak{C}$, (2) C is an exterior point, and (3) D is on the polar of C. When only one oval is under discussion, we shall merely speak of a harmonic set of points. The point D will sometimes be referred to as the fourth harmonic point or the harmonic conjugate of C.

DEFINITION. A perspectivity between the points of a secant and the points of a secant will be said to be a *perspectivity on* $\mathfrak{C}$ if either (1) the point of intersection of the secants is not absolute and absolute points correspond to absolute points, or (2) the secants intersect in an absolute point A, the center of perspectivity is on the tangent at A, and absolute points correspond to absolute points. The symbol $\overline{\wedge}$ between two sets of points will indicate that they are related by a perspectivity on $\mathfrak{C}$.

ASSUMPTION A1. *Let $\{AB\colon UE\}$ and $\{A_1B_1\colon UE_1\}$ be harmonic sets on two secants intersecting in the exterior point U. Then $ABUE \,\overline{\wedge}\, A_1B_1UE_1$ and $ABUE \,\overline{\wedge}\, B_1A_1UE_1$.*

Remark. Assumption A1 is a consequence of Theorem 1.3 (d). In a Desarguesian plane, sets harmonic with respect to $\mathfrak{C}$ are harmonic sets in the ordinary sense, and Assumption A1 is satisfied. Much of what we say about ovals applies as well to a curve $\mathfrak{C}$ in an infinite plane provided that $\mathfrak{C}$ has the following properties: (1) no three points of $\mathfrak{C}$ are collinear; (2) no three tangents to $\mathfrak{C}$ are concurrent. (At each point of $\mathfrak{C}$ there must be a tangent which contains no other points of $\mathfrak{C}$.) Now in the infinite case, we have a very natural example of a plane in which A1 is satisfied. We refer to the well known example **(8)** of a non-Desarguesian plane constructed by distorting the interior of a conic in a Desarguesian plane.

DEFINITION. If Assumption A1 is satisfied for every exterior point of an oval $\mathfrak{C}$, we shall call $\mathfrak{C}$ an *harmonic oval.*

THEOREM 2.1. *If U and V are exterior points of the harmonic oval $\mathfrak{C}$, and V is on the polar of U, then U is on the polar of V.*

Proof. Let A be a point on $\mathfrak{C}$, not on the polars of either U or V. Then the line UA is not a tangent and intersects $\mathfrak{C}$ in another point B. Similarly, let the line VA intersect $\mathfrak{C}$ in a point A_1. Now A_1 cannot be on the polar of U, else the line VA is the polar of U, contrary to the choice of A. Hence the line UA_1 intersects $\mathfrak{C}$ in another point B_1. Let $\{AB: UE\}$ and $\{A_1B_1: UE_1\}$ be harmonic sets. By Assumption A1, the lines AA_1, BB_1, and EE_1 are concurrent. But EE_1 is the polar of U. Therefore, AA_1 and EE_1 intersect in V, since V is on the polar of U by hypothesis. Hence BB_1 goes through V.

Now let $\{AA_1: VD\}$ and $\{BB_1: VF\}$ be harmonic sets. Again, by Assumption A1, AB, A_1B_1, and DF are concurrent. But DF is the polar of V, while AB and A_1B_1 intersect in U. Hence U is on the polar of V.

LEMMA 2.1. *Let U and V be conjugate exterior points of an harmonic oval $\mathfrak{C}$, (i.e., U and V are on each other's polars) then* (a) *the line UV is a secant if $n \equiv 1 \pmod 4$*, (b) *the line UV is an exterior line if $n \equiv -1 \pmod 4$.*

Proof. Let A be an absolute point not on the polars of U or V and not on the line UV. As in the proof of Theorem 2.1, a set of four absolute points AA_1BB_1 is determined such that U and V are diagonal points of the quadrangle AA_1BB_1. The $n+1$ points of $\mathfrak{C}$ occur in sets of four points, omitting (1) two points each on the polars of U and V and (2) the absolute points, if any, on the line UV. If UV is a secant, then 4 must divide $n-1=(n+1)-2$. If UV is an exterior line, then 4 must divide $n+1$.

THEOREM 2.2. *If $n \equiv 1 \pmod 4$, the fourth harmonic point is always exterior, while if $n \equiv -1 \pmod 4$, the fourth harmonic point is always interior.*

Proof. Consider the lines connecting an exterior point U to the exterior points on its polar u. If $n \equiv 1 \pmod 4$, every one of these lines is a secant. There are $\frac{1}{2}(n-1)$ exterior points on u and $\frac{1}{2}(n-1)$ secants through U. Thus all of the secants are accounted for and in each one of these cases the fourth harmonic point is exterior. If $n \equiv -1 \pmod 4$, every one of these lines is an exterior line. All of the exterior lines are accounted for and the remaining non-absolute lines through U must be secants which intersect u in interior points.

COROLLARY. *Given a collineation which* (a) *carries the oval $\mathfrak{C}$ into itself* (b) *is not of order* 2, (c) *leaves fixed a proper subplane with n_1+1 points on each line*, (d) *leaves at least one point of $\mathfrak{C}$ fixed, then $n \equiv n_1 \pmod 4$.*

Proof. The fixed points of $\mathfrak{C}$ form an oval in the subplane. If n is not equal to 2, a secant is fixed if and only if its absolute points are fixed. A fixed exterior point of $\mathfrak{C}$ is an exterior point of the oval and is the pole of a fixed secant which will also be a secant of the oval $\mathfrak{C}_1$. Sets harmonic with respect to $\mathfrak{C}_1$

are harmonic with respect to $\mathfrak{C}$, and a point of the subplane is respectively interior, exterior or an absolute point of $\mathfrak{C}_1$ if and only if it is an interior, exterior or absolute point of $\mathfrak{C}$, as the case may be. Thus if the fourth harmonic point is always exterior or always interior in one case, it must be so in the other case also.

DEFINITION. A triangle consisting of an exterior point U and two absolute points A and B such that U is the pole of the secant AB will be called a *fundamental triangle.*

THEOREM 2.3 *Let A, B, A_1, B_1 be four points on the harmonic oval $\mathfrak{C}$. Let C and C_1 respectively be the poles of the secants AB and A_1B_1. Then if AB and A_1B_1 intersect in an exterior point U, the fundamental triangles ABC and $A_1B_1C_1$ are perspective from a point. If AA_1 and BB_1 also intersect in an exterior point, the triangles are also perspective from a line.*

Proof. Since U is on the polar AB of the exterior point C, C is on the polar of U. Similarly, C_1 is on the polar of U. Let E and E_1 be the respective points of intersection of CC_1 with AB and A_1B_1. Then $\{AB: UE\}$ and $\{A_1B_1: UE_1\}$ are harmonic sets. Therefore, AA_1, BB_1 and $EE_1 = CC_1$ are concurrent in a point V and the triangles are perspective from V. Let D and D_1 be the respective poles of the secants AA_1 and BB_1. Then the fundamental triangles AA_1D and BB_1D_1 are perspective from the line CC_1. Similarly, if V is an exterior point, the triangles ABC and $A_1B_1C_1$ are perspective from the line DF.

THEOREM 2.4 *Let A, B, C, A_1, B_1, C_1 be six distinct absolute points of an harmonic oval $\mathfrak{C}$. If AA_1, BB_1, and CC_1 intersect in an exterior point V, and if P, Q, R are the respective points of intersection of the pairs of lines AB_1, A_1B; BC_1, B_1C; AC_1, A_1C; then P, Q, and R are collinear.*

Proof. Let $\{AA_1: VE\}$ and $\{BB_1: VD\}$ be harmonic sets. Then $AA_1VE \barwedge B_1BVD$, i.e., lines AB_1, A_1B, and ED are concurrent, where the line ED is the polar of V. Thus P is on the polar of V. Similarly, Q and R are on the polar of V.

COROLLARY. *If two harmonic ovals have five points A, B, C, A_1, and B_1 in common, and if AA_1 and BB_1 intersect in a point U exterior to both, then either the line UC is a common tangent or contains another point C_1 common to both.*

Remark. Qvist **(10)** has shown that if two ovals have half of their points in common, they are identical. Our Corollary suggests that five points may determine an harmonic oval if there are no proper subplanes.

THEOREM 2.5. *Let A, B, C be three points on a secant, and let A^2, B^2, C^2 be three points on another secant, where A, B, A^2, B^2 are absolute points of the harmonic oval $\mathfrak{C}$ and the two secants intersect in an exterior point. Then A, B, and C can be carried into A^2, B^2, C^2 by the product of at most two perspectivities on $\mathfrak{C}$.*

Proof. Suppose that CA^2 is not a tangent. Then the line CA^2 contains another absolute point B_1. Let BB_1 and AA^2 intersect in O_1. Then

$$ABC \underset{\wedge}{-} A^2B_1C \underset{\wedge}{-} A^2B^2C^2.$$

If CA^2 is a tangent, a similar argument can be made using one of the lines CB^2, C^2A, or C^2B unless all of these lines are also tangents. In this case, C is the pole of A^2B^2, while C^2 is the pole of AB. Then, by Theorem 2.3, the fundamental triangles A^2B^2C and ABC^2 are perspective from some point O, and $ABC \underset{\wedge}{-} A^2B^2C^2$.

DEFINITION. A point to point transformation from a secant onto a secant will be called *projective with respect* to $\mathfrak{C}$ (or merely, *projective*) if it carries sets harmonic with respect to the harmonic oval $\mathfrak{C}$ into sets harmonic with respect to $\mathfrak{C}$.

THEOREM 2.6. *A collineation which carries $\mathfrak{C}$ into itself induces a projective mapping between the points of a secant and its image.*

DEFINITION. If $n \equiv 1 \pmod 4$, a triangle UVW such that each of the exterior points U, V, W is the pole with respect to the oval $\mathfrak{C}$ of the opposite side of the triangle will be called a *self polar triangle with respect to* $\mathfrak{C}$. If V and W are on the line $\bar{l}$, the triangle will be said to be a *self polar triangle on* $\bar{l}$.

THEOREM 2.7. *Let $n \equiv 1 \pmod 4$. Let $\bar{l}$ and $\bar{l}_1$ be two secants of the oval $\mathfrak{C}$, intersecting in an exterior point. Then a perspectivity from $\bar{l}$ onto $\bar{l}_1$ will be projective with respect to $\mathfrak{C}$ if and only if self polar triangles on $\bar{l}$ are perspective to self polar triangles on $\bar{l}_1$. If the center of perspectivity is an exterior point V, the triangles are also perspective from the polar of V.*

Proof. A perspectivity on $\mathfrak{C}$ carries absolute points into absolute points. If C is the pole of $\bar{l}$ and C_1 is the pole of $\bar{l}_1$, it follows from Theorem 2.3 that the center of perspectivity is on the line CC_1. If CDE is a self polar triangle, then $\{AB\colon DE\}$ is a harmonic set, where A and B are the absolute points on $\bar{l}$. The image $\{A_1B_1\colon D_1E_1\}$ of $\{AB\colon DE\}$ will be a harmonic set if and only if D_1 and E_1 are on each other's polars, i.e., $C_1D_1E_1$ is a self polar triangle. The last part of the Theorem follows from Theorem 2.1.

Assumption A1 does not tell us much about interior points. If $n \equiv -1 \pmod 4$, it is natural to extend this assumption in the following manner:

ASSUMPTION A2. *If $\{AB\colon UE\}$ and $\{A_1B_1\colon U_1E\}$ are harmonic sets with respect to the oval $\mathfrak{C}$, where the point E is an interior point, then*

$$ABUE \underset{\wedge}{-} A_1B_1U_1E, \quad ABUE \underset{\wedge}{-} B_1A_1U_1E.$$

We shall also make use of another Assumption. (Note Theorem 2.3.)

ASSUMPTION B: *Given two fundamental triangles ABC and $A_1B_1C_1$, where A, B, A_1, B_1 are absolute points of the oval $\mathfrak{C}$, then the triangles are perspective from the intersection of the secants AA_1, and BB_1 and also from the point of intersection of the secants AB_1 and A_1B.*

LEMMA 2.2. *If $n \equiv -1 \pmod 4$, and Assumptions* A1, A2, B *are satisfied for the oval* $\mathfrak{C}$, *then the poles of the secants through each interior point O are collinear.*

Proof. Given any secant, corresponding to each exterior point on that secant there is an interior point which is its harmonic conjugate. Since the number of exterior points on a secant is equal to the number of interior points, it is likewise true that for every interior point on a secant there is an exterior point on the secant which is its harmonic conjugate. Let AB be a secant through O, and let U be an exterior point such that $\{AB: UO\}$ is a harmonic set. Let U_1 be the pole of the secant AB. We shall show that the pole of every secant through O lies on the line UU_1.

Let CD be any other secant through O and let $\{CD: VO\}$ be a harmonic set. Let V_1 be the pole of CD. Then, by Assumption A2, the lines AC, BD, and UV are concurrent. Thus the line UV connects the point of intersection of the lines AC and BD with the point of intersection of AD and BC. Now consider the fundamental triangles ABU_1 and CDV_1. The two sets of lines AC, BD, U_1V_1 and AD, BC, U_1V_1 are respectively concurrent. The lines UV and U_1V_1 must be the same, i.e., V_1 lies on the line UU_1.

THEOREM 2.8. *If $n \equiv -1 \pmod 4$, and Assumptions* A1, A2, *and* B *are satisfied for the oval* $\mathfrak{C}$, *then* $\mathfrak{C}$ *determines a polarity.*

Proof. The polar of an interior point O can be taken as the line containing the poles of the secants through O. We have completely determined a correspondence between points and lines. This correspondence will be a polarity if, for any two points U and V such that V is on the polar of U, then U is on the polar of V. Theorem 2.1 takes care of the case where both U and V are exterior. The case where U is exterior and V is interior is taken care of by the way in which we defined the polar of an interior point. U interior and V exterior can be handled in a similar manner. The case where both U and V are interior is readily handled by an argument similar to those used in Theorems 2.1 and 2.3.

3. Conics in projective planes. As remarked before, Baer **(1)** has shown that the absolute points of a polarity form an oval unless n is even or a square. In §2, we had a sort of quasi-polarity but were restricted by the fact that the relation of pole and polar was restricted to exterior points and secants.

DEFINITION. A set of $n + 1$ points will be called a *quasi-conic* if (1) no three of them are collinear, and (2) they are the absolute points of a polarity.

DEFINITION. A set of four collinear points $\{AB: CD\}$ will be said to be *harmonic with respect to the quasi-conic* $\mathfrak{C}$ if (1) A and B belong to $\mathfrak{C}$, (2) C and D are conjugate points of the polarity.

ASSUMPTION A. *Let* $\{AB: UE\}$ *and* $\{A_1B_1: UE_1\}$ *be harmonic sets with respect to the quasi-conic* $\mathfrak{C}$. *Then* $ABUE \barwedge A_1B_1UE_1$ *and* $ABUE \barwedge B_1A_1UE_1$.

DEFINITION. If Assumption A is satisfied for every pair of sets harmonic with respect to the quasi-conic $\mathfrak{C}$, then $\mathfrak{C}$ will be said to be a *conic*.

We will be able to put some of the theorems of §2 in a simpler form and to obtain some new ones. Many of the proofs are similar to those of §2, so we shall give the proofs only for those theorems which have no counterpart in §2. We shall number the theorems to facilitate cross-reference—e.g., Theorem 3.3 will be analogous to Theorem 2.3. Theorems which carry over unchanged will be omitted. This will cause some gaps in the numbering of theorems.

THEOREM 3.3. *For a quasi-conic* $\mathfrak{C}$, *the following propositions are equivalent:*

(a) *Assumption* A.

(b) *Given two fundamental triangles* ABC *and* $A_1B_1C_1$, *where* A, B, A_1, *and* B_1 *are absolute points, the triangles are perspective from the intersection of the secants* AA_1, *and* BB_1, *and also from the point of intersection of the secants* AB_1 *and* A_1B.

(c) *The triangles in* (b) *are perspective from the polars of the points which are the centers of perspectivity in* (b).

(d) *If the points of a complete quadrangle are all absolute points, then the diagonal points form a self polar triangle—i.e., each point is the pole of the opposite side.*

THEOREM 3.4. *Let* A, B, C, A_1, B_1, C_1 *be six distinct absolute points of a conic. If* AA_1, BB_1, *and* CC_1 *intersect in a point* V, *and if* P, Q, *and* R *are the respective points of intersection of the pairs of lines* AB_1, A_1B; BC_1, B_1C; *and* AC_1, A_1C; *then* P, Q *and* R *are collinear.*

COROLLARY. *If two conics have five points in common, then either they have a common tangent at one of these points or there is another point common to both.*

THEOREM 3.5. *Let* A, B, C *be three points on a secant, and let* A^2, B^2, C^2 *be three points on another secant, where* A, B, A^2, B^2 *are absolute points of a conic* $\mathfrak{C}$. *Then* A, B, C *can be carried into* A^2, B^2, C^2 *by the product of at most two perspectivities on* $\mathfrak{C}$.

THEOREM 3.5.1. *Let* $\{AB: UT\}$ *be a harmonic set with respect to the conic* $\mathfrak{C}$, *where* U *is an exterior point. Then there is a complete quadrangle such that* U *and* T *are diagonal points, while* A *and* B *lie on the other two sides of the quadrangle—i.e.,* $\{AB: UT\}$ *forms a harmonic set in the sense of the usual definition.*

Proof. Let U_1 be the pole of the secant AB, so that U_1A and U_1B are tangents at A and B respectively. Let UA_1 and UB_1 be tangents from U, tangent at the absolute points A_1 and B_1. Since U is on AB, U and U_1 are conjugate so that U_1 lies on A_1B_1.

Let UA_1 and U_1A intersect in X,
UA_1 and U_1B intersect in Y,
UB_1 and U_1A intersect in Z,
UB_1 and U_1B intersect in W.

Then U and U_1 are two of the diagonal points of the complete quadrangle $XYZW$. We shall show that T is the third diagonal point, i.e., XW and YZ intersect in T.

Consider the fundamental triangles UA_1B_1 and U_1AB. By Theorem 3.3, they are perspective from the intersection of AB_1 and A_1B, and also from the polar of this point. The corresponding pairs of sides are UA_1, U_1B; UB_1, U_1A; and AB, A_1B_1. The points of intersection Y, Z, and T are therefore collinear. Similarly, X, W, and T are collinear. Thus T is the third diagonal point. Since A and B lie on XZ and YW respectively, this completes our proof.

THEOREM 3.7. *A perspectivity on a conic $\mathfrak{C}$ from a secant $\bar{l}$ onto a secant $\bar{l}_1$ will be projective with respect to $\mathfrak{C}$ if and only if self polar triangles on $\bar{l}$ are perspective to self polar triangles on $\bar{l}_1$.*

THEOREM 3.7.1. *If perspectivities on $\mathfrak{C}$ are projective with respect to the conic $\mathfrak{C}$, then any set harmonic with respect to the conic $\mathfrak{C}$ can be carried into any set harmonic with respect to $\mathfrak{C}$ by a product of perspectivities on C.*

The proof follows from Theorem 3.5.

4. Ovals in transitive planes. A cyclic plane is a plane which possesses a cyclic group of collineations, transitive on the points of the plane. Zappa **(12)** has studied a more general class of transitive planes. The author is indebted to Professor B. Segre for suggesting that some of the results of this part might be applied to the more general class of transitive planes.

A cyclic plane can be represented in the following manner: the points can be taken as a complete set of residues mod $N = n^2 + n + 1$. Let $a_0, a_1, \ldots, a_n$ form a difference set mod N. That is, the set $\{a_i - a_j\}, i \neq j\ (i,j = 0, 1, \ldots, \mathrm{n})$ contains each residue mod N exactly once. For any fixed s, the set $\{a_i - s\}$ $(i = 0, 1, \ldots, n)$ will also be a difference set. The sets $\{a_i - s\}$ can be taken as the lines of a projective plane and will be denoted by $\bar{l}_s$. In discussing cyclic planes, equations will denote congruences mod N.

There are two known collineations in a cyclic plane: (1) the mapping $x \to x + s$, (2) for certain choices of m, the mapping $x \to mx$ is a collineation. Such values of m are called multipliers. Hall **(5)** has showed that all factors of n are multipliers. There is at least one line which is left invariant by all multipliers. Take $\bar{l}_0$ to be such an invariant line.

Now the mapping $x \to l_x$ forms a natural polarity **(5)**. Two points x and y are conjugate points of the polarity if, for some i, $x + y = a_i$, since then $x \in l_y$ and $y \in l_x$. The absolute points of the polarity will be the points x such that $2x = a_i\ (i = 0, 1, \ldots, n)$ and will be the points of a quasi-conic. (If n is even, 2 is a multiplier and the absolute points all lie on l_0. See **(1**, p. 83**)**).

More generally, $x \to l_{x+2s}$ forms a polarity in which x and y are conjugate if $x + y = a_i - 2s$. The absolute points are the points x such that $2x = a_i - 2s$ or $x = \frac{1}{2}a_i - s$. Let $\mathfrak{C}_s$ denote the set $\frac{1}{2}a_i - s$ $(i = 0, 1, \ldots, n)$. $\mathfrak{C}_s$ then is a quasi-conic for each s, $s = 0, 1, \ldots, N$. Now, the set $\mathfrak{C}_0$ is a difference set mod N. The sets $\mathfrak{C}_s$ form a projective plane π_1 in the same way that the sets l_x formed the plane π. Moreover, $\mathfrak{C}_0$ is invariant under the collineation $x \to mx$, m a multiplier.

THEOREM 4.1. *Given any three collinear points A, B, C, there exists a unique point D such that the set $\{AB: CD\}$ is harmonic with respect to one of the quasi-conics $\mathfrak{C}_s$.*

Proof. Since the conics $\mathfrak{C}_s$ form the lines of a finite projective plane π_1, there is exactly one quasi-conic $\mathfrak{C}_s$ such that A and B are points on $\mathfrak{C}_s$. With respect to $\mathfrak{C}_s$, the point C has exactly one conjugate point D on the line containing A, B, and C.

Next, consider the correlation $x \to l_{mx}$, where m is a multiplier. (From here on, to avoid continual use of subscripts, we shall use small letters near the beginning of the alphabet to denote elements of l_0.) A point x will be an absolute point of the correlation if it lies on its image line $\bar{l}_{mx}$ i.e., if $x = a - mx$ for some $a \in l_0$. The absolute points of the correlation are thus the solutions of the equations $(m + 1)x = a$, $a \in l_0$. Now if $m + 1$ is a divisor of zero, then m is congruent to -1 modulo some factor of N. By Mann's theorem **(7)** on multipliers of even order, n must then be a square. If n is not a square, we have exactly $n + 1$ absolute points $x = a(m + 1)^{-1}$, where a may take on any of the values $a_0, a_1, \ldots, a_n$.

If $a(m + 1)^{-1}$ is an absolute point, its image line $\bar{l}_{ma(m+1)^{-1}} = \bar{l}_{b(m+1)^{-1}}$ is an absolute line, where $b = ma \in l_0$.

We know that each absolute line contains at least one absolute point. Suppose that $\bar{l}_{a(m+1)^{-1}}$ contains the absolute point $b(m + 1)^{-1}$. Then, for some $c \in \bar{l}_0$,

$$b(m + 1)^{-1} = c - a(m + 1)^{-1}.$$

Multiplying through by $m + 1$ and transposing, we get

$$b - cm = c - a$$

where $a, b, c, cm \in \bar{l}_0$. Since $\bar{l}_0$ is a difference set, this implies either

$$\begin{cases} b = cm \\ c = a \end{cases} \text{or} \begin{cases} b = c \\ cm = a \end{cases}$$

Therefore, either $b = am$ or $b = am^{-1}$. Conversely, if $b = am$ or am^{-1}, then the absolute point $b(m + 1)^{-1}$ belongs to the absolute line $\bar{l}_{k(m+1)^{-1}}$.

If $m = 1$, we have the polarity $x \to l_x$ and each absolute line contains one absolute point. Even if $m \neq 1$, we may have $ma = a$ for certain $a \in \bar{l}_0$ if a is a fixed point under the collineation $x \to mx$. In all other cases, each absolute line $\bar{l}_{a(m+1)^{-1}}$ contains the two absolute points $am^{-1}(m + 1)^{-1}$ and $am(m + 1)^{-1}$ i.e., the image and inverse image of the absolute line. Hence

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LEMMA 4.1. *Each absoltue line of the correlation contains at most two absolute points.*

LEMMA 4.2. *If* (a) *m is a multiplier,* (b) *n is not a square,* (c) *$a \in \bar{l}_0$, $ma \neq a$ then $(m + 1)a$ does not belong to $\bar{l}_0$.*

Proof. An element of $\bar{l}_0$ multiplied by a multiplier gives an element of $\bar{l}_0$. If $(m + 1)a \in \bar{l}_0$, then $m(m + 1)a$ and $m^2(m + 1)a$ also belong to $\bar{l}_0$. If a, ma, m^2a are distinct, we will be led to a contradiction of Lemma 4.1. If $m^2a = a$, but $ma \neq a$, then $(m - 1)a \neq 0$, $(m - 1)(m + 1)a = 0$ so that $m + 1$ is a divisor of zero which implies that n is a square.

THEOREM 4.2. *If n is not a square and if m is a multiplier $\neq 1$, then $m + 1$ is not a multiplier.*

Proof. If $m + 1$ is a multiplier and $a \in \bar{l}_0$ then $(m + 1)a \in \bar{l}_0$. By Lemma 4.2, this can only happen if $ma = a$ for every $a \in \bar{l}_0$. Let $(m - 1, N) = N_1$. Let $N_2 = N/N_1$. If $(m - 1)a = 0$, then a must contain N_2 as a factor. If this be true for every $a \in \bar{l}_0$ then all differences between elements of $\bar{l}_0$ are multiples of N_2 and $\bar{l}_0$ cannot be a difference set.

COROLLARY. *If m_1 and m_2 are multipliers, then $m_1 + m_2$ is not a multiplier.*

Remark. These are strengthened versions of Mann's theorem 3 and Corollary 1 in reference **(7)**. The corresponding proposition for $m = 1$ is as follows: "2 is a multiplier if and only if n is even." This has already been noted by Hall **(5)**.

Of particular interest is the case where $m = n$. Now the set $\{a_i\}$ of points $\in \bar{l}_0$ is the same as the set $\{n^{-1}a_i\}$; the set $a_i(n + 1)^{-1}$ of absolute points is the same as the set $\{-a_i\}$ since $n(n + 1) \equiv -1 \pmod{N = n^2 + n + 1}$.

THEOREM 4.3. *The absolute points of the correlation $x \rightarrow \bar{l}_{nx}$ form an oval $\mathfrak{C}^*$.*

Proof. We have already proved that there are $n + 1$ absolute points. Now if a and $b \in \bar{l}_0$ then the line $\bar{l}_{a+b}$ contains $-a = b - (a + b)$ and $-b = a - (a + b)$. Suppose that $-a$, $-b$, $-c$ are collinear, where a, b, $c \in \bar{l}_0$. Then $\bar{l}_{a+b} = \bar{l}_{a+c} = \bar{l}_{b+c}$ so that $a + b = a + c = b + c$ and a, b, and c are not distinct.

THEOREM 4.4. *Consider the collineation $x \rightarrow x + a - b$, which carries $-a$ into $-b$, where a, $b \in \bar{l}_0$. The intersection of each line through $-a$ and its image line through $-b$ is a point of $\mathfrak{C}^*$.*

Proof. The lines through $-a$ can be written in the form $\bar{l}_{a+c}$ where $c \in \bar{l}_0$. (If $c = a$, we have the tangent line at $-a$.) The image line under the collineation is the line $\bar{l}_{a+c-a+b} = \bar{l}_{c+b}$ which intersects $\bar{l}_{a+c}$ in the point $-c$.

We have earlier remarked that in any plane with a polarity, self polar triangles which are perspective from a point are also perspective from a line. We shall proceed to demonstrate pairs of self polar triangles which are perspective from a point.

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LEMMA 4.3. *If a belongs to l_0 but is not equal to* 0, $N/3$ *or* $2N/3$, *then* $-a$, $-na$, $-n^2a$ *are the vertices of a triangle which is self polar under the polarity* $x \to l_x$.

Proof. Since $a + na = -n^2a$, $-n^2a$ is the pole of $\bar{l}_{a+na}$, the line through $-a$ and $-na$. Similarly, $-a$ and $-na$ are the poles of the other two sides. $-a$, $-na$, $-n^2a$ are distinct unless $a = 0$, $N/3$ or $2N/3$. We have already seen that they are not collinear (Theorem 4.3).

THEOREM 4.5. *If n is odd and not divisible by* 3, *then there exist pairs of self polar triangles which are perspective from* 0 *and therefore from* $\bar{l}_0$.

Proof. 0 does not belong to $\bar{l}_0$ unless n is divisible by 3 **(3)**. If $n \equiv 1 \pmod 3$, $N \equiv 0 \pmod 3$ and there may be three lines invariant under all multipliers. If so, two of these will contain 0, but we can still take $\bar{l}_0$ as a line which does not contain 0.) Likewise, 0 does not belong to the oval $\mathfrak{C}^*$. Consider a secant intersecting $\mathfrak{C}^*$ in the points $-a$, $-b$ where $a, b \in \bar{l}_0$. Then the triangles $-a$, $-na$, $-n^2a$ and $-b$, $-nb$, $-n^2b$ are the desired triangles provided $a \neq N/3$ or $2N/3$.

Suppose that a projective plane π admits a group Σ of collineations which (a) is transitive on the points and lines of π, and (b) the only element of Σ which leaves fixed a given point 0 is the identity. Zappa **(12)** calls such a plane a regular transitive plane. We shall need to require in addition that (c) Σ is abelian.

Associated with each point x of π will be the element σ_x of Σ which carries 0 into x. Let $\bar{l}_0$ denote one of the lines of π. Let $\bar{l}_x$ be the image of $\bar{l}_0$ under the collineation σ_x^{-1}. Then if $x \in \bar{l}_y$, $\sigma_x = \sigma_a\sigma_y^{-1}$ for some $a \in \bar{l}_0$. If Σ is abelian, $\sigma_y = \sigma_a\sigma_x^{-1}$ and $y \in \bar{l}_x$. Hence the mapping $x \to \bar{l}_x$ is a polarity. The absolute points are the points x such that $\sigma_x^2 = \sigma_a$ for some $a \in \bar{l}_0$.

Let $-x$ denote the image of 0 under σ_x^{-1}; let $x + y$ be the image of 0 under the collineation $\sigma_x\sigma_y$. If a and $b \in \bar{l}_0$ then $\bar{l}_{a+b}$ contains $-a$ and $-b$. As in Theorem 4.3, if $a_0, a_1, \ldots, a_n$ are the points of $\bar{l}_0$ it follows that $-a_0, -a_1, \ldots, -a_n$ form an oval $\mathfrak{C}^*$. The analogue of Theorem 4.4 goes through. Let μ be a collineation (not the identity) which leaves 0 and $\bar{l}_0$ fixed. Then μ will also carry $\mathfrak{C}^*$ into itself. Denote the image of x under μ by $\mu(x)$. Then the mapping $x \to \bar{l}_{\mu(x)}$ is a correlation. If x is an absolute point of this collineation, then $\sigma_x = \sigma_a\sigma_{\mu(x)}^{-1}$ for some $a \in \bar{l}_0$ and $\sigma_x\sigma_{\mu(x)} = \sigma_a$. As in Lemma 4.2, it follows that we cannot have points a and $b \in \bar{l}_0$ such that $\sigma_b\sigma_{\mu(b)} = \sigma_a$ unless $\mu(b) = b$ or $\mu^2(b) = b$.

If μ is such that, for every point x,

$$\sigma_x\sigma_{\mu(x)}\sigma_{\mu^2(x)} = \sigma_0$$

then the absolute points of the correlation $x \to \bar{l}_{\mu(x)}$ satisfy the equation

$$\sigma_a^{-1} = \sigma_{\mu^2(x)}$$

i.e., $-a = \mu^2(x)$, $\mu^{-2}(-a) = x$ where $a \in \bar{l}_0$, so that $-a$ and $\mu^{-2}(-a)$ belong

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to $\mathfrak{C}^*$. Thus $\mathfrak{C}^*$ consists of the absolute points of the correlation $x \to \bar{l}_{\mu(x)}$. For the analogue to Theorem 4.5 to go through, μ must be of order 3.

5. The existence of Fano's configuration. In this part, we are concerned with the case where n is even. Here we shall not be concerned with ovals, which do not have the same properties as when n is odd. However, Theorem 5.1 is, in a sense, the counterpart of Theorem 4.5. Recall that, if n is even, and not a square, the absolute points of a polarity are the points of a line.

THEOREM 5.1. *Let the absolute points of a polarity be the points on a line o, and let O be the pole of o. Let A, B, C be the vertices of a self polar triangle, where A, B, C are not absolute points and are all different from O. Then the diagonal points of the complete quadrangle with vertices O, A, B, C are all on o.*

Proof. First we show that the lines OA and BC must intersect in a point of o. Let D denote the intersection of the line BC with o. Since A is the pole of BC, A is conjugate to D. Therefore, the polar of D goes through O and A. But D is an absolute point, hence the polar of D goes through D, i.e., the line OA contains D. Thus D is a diagonal point of the quadrangle $OABC$. Similarly, the other two diagonal points lie on o.

Remark. Note that Theorem 5.1 does not require that the plane be finite.

THEOREM 5.2. *In a cyclic plane, if n is even and not a square, if $a \in \bar{l}_0$, then $0, -a, -2a, -4a$ are the vertices of a complete quadrangle for which the diagonal points are collinear.*

Proof. For a cyclic plane, n cannot be divisible **(5)** by both 2 and 3. Now $N \equiv 0 \pmod 3$ if and only if $n \equiv 1 \pmod 3$. If $N \equiv 0 \pmod 3$, the multiplier 2 is even order and n must be a square. Thus $n \equiv 1 \pmod 3$; $0 \in \bar{l}_0$ and 3 is not a divisor of zero. We conclude that $0, -a, -2a, -4a$ are distinct. Theorem 4.3 applies as well when n is even, so $-a, -2a, -4a$ are not collinear. It may be shown by similar methods that no three of the points $0, -a, -2a, -4a$ are collinear.

Now, since 2 is a multiplier, $\bar{l}_0$ contains $a, 2a, 4a, 8a$. ($8a$ does not necessarily differ from a). Hence $\bar{l}_{2a}$ contains $2a - 2a = 0, a - 2a = -a, 4a - 2a = 2a$. Similarly, $\bar{l}_{6a}$ contains $-4a$, $-2a$, and $2a$. Thus the lines $0, -a$ and $-4a, -2a$ intersect in $2a$, i.e., $2a$ is one of the diagonal points. Similarly, the lines $0, -2a$ and $-a, -4a$ intersect in the point $-3a$. Now $\bar{l}_{8a}$ is the line $0, -4a$ while $\bar{l}_{3a}$ is the line $-a, -2a$. If we denote by x the intersection of $\bar{l}_{8a}$ and $\bar{l}_{3a}$ then for some $c, d \in \bar{l}_0$,

$$c - 8a = d - 3a = x.$$

Then c and d are the elements of $\bar{l}_0$ such that

$$c - d = 8a - 3a = 5a.$$

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To show that the diagonal points $2a$, $-3a$, x are collinear, consider the line $\bar{l}_{d+3a}$. This line contains

$$d - (d + 3a) = -3a, \quad c - (d + 3a) = 5a + d - (d + 3a) = 2a \quad \text{and} \quad 2d - (d + 3a) = x.$$

THEOREM 5.3. *Suppose that a projective plane admits a collineation of order two such that every point on a line $\bar{l}$ is fixed and every line through a point $P \in l$ is fixed. If A and B are two non-fixed points not collinear with P, if A_1 and B_1 are the respective images of A and B, then the diagonal points of the quadrangle ABA_1B_1 all lie on l.*

Proof. A and A_1 are collinear with P; similarly, B and B_1 are collinear with P. Hence P is one of the diagonal points. The line A_1B_1 is the image of AB; if Q is the intersection of AB with $\bar{l}$, A_1B_1 must also go through Q. Similarly, AB_1 and A_1B must intersect in a point on $\bar{l}$, since $AB_1 \rightleftarrows A_1B$.

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Montana State University

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