10

Quantum chromodynamics: spin in the world of massless partons

Quantum chromodynamics (QCD) is the beautiful theoretical structure believed to control the strong interactions of elementary particles. On the one hand, being a theory of strong interactions it is surprising that one can attack certain problems by perturbative methods, and where this has been done the agreement between theory and experiment is generally impressive. On the other hand a number of non-perturbative problems, which used to seem intractable, are now being attacked by lattice methods, but it is too early to say how significant the results are *vis-à-vis* experiment.

Because the theory deals with partons (quarks and gluons), whereas experiments are performed with hadrons, there is always some uncalculable piece in any theoretical treatment of a reaction. Consequently there is, to date, no single crucial experiment, which, analogous to the Lamb shift in QED, could be said to prove or disprove the validity of QCD. It is thus important to test the theory in as many ways as possible.

Historically, spin-dependent experiments have played a seminal rôle in verifying or falsifying theories. QCD has a very simple and clear-cut spin structure, so that the study of spin-dependent reactions should provide an excellent way to probe and test the theory further. In fact, as we shall see in Section 14.3 there is apparently serious disagreement between theory and experiment in several reactions, but it is now believed that this is a result of the naivety of the calculations. The situation is nonetheless tantalizing and should be resolved when results from the giant *pp* collider RHIC at Brookhaven, with polarized proton beams, start to emerge in a year or two.

10.1 A brief introduction to QCD

QCD is a non-abelian gauge theory describing the interaction of massless spin-1/2 objects, the 'quarks', which possess an internal degree of freedom called colour, and a set of massless gauge bosons (vector mesons), the

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'gluons', which mediate the force between quarks in much the same way that photons do in QED. Loosely speaking, the quarks come in three colours and the gluons in eight. More precisely, if $q^a(x), a = 1, 2, 3$ and $A_\mu^b(x), b = 1, \ldots, 8$, are the quark and gluon fields respectively then, under an $SU(3)$ transformation acting on the colour indices, q and A are defined to transform as the fundamental ($\underset{\sim}{3}$) and the adjoint ($\underset{\sim}{8}$) representations of $SU(3)$ respectively. These $SU(3)$ transformations, acting solely on the colour indices, have nothing at all to do with the usual $SU(3)$ that acts on the quark flavour labels; in what follows it must be understood that these flavour labels play no rôle in QCD since the gluons are taken to be flavourless, i.e. to be singlets under $SU(3)_f$, and electrically neutral, so they will not be displayed unless specifically needed.

The theory is *known* to possess the remarkable property of 'asymptotic freedom' and is *supposed* to possess the property of 'colour confinement'. The former implies that for interactions between quarks at very short distances, i.e. for large momentum transfers, the theory looks more and more like a free-field theory, without interactions. This, ultimately, is the justification for the parton model and for the use of perturbative methods for large momentum reactions. The latter means that only 'colourless' objects, that are colour singlets, can be found existing as real physical particles. In other words the forces between two coloured objects grow stronger with distance, so that they can never be separated. This property of confinement is also referred to as 'infrared slavery'. The proof of confinement is still lacking and remains one of the most burning theoretical questions.

The $SU(3)$ non-abelian, gauge-invariant theory for an octet of massless vector gluons interacting with a triplet of massless spin-1/2 quarks involves:

(1) *generalized field tensors* (i.e. non-abelian generalizations of the electromagnetic $F_{\mu\nu}$)

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c, \tag{10.1.1}$$

where A_μ^a is the gluon vector potential, the label $a = 1, \ldots, 8$ being the octet colour label, and where f_{abc} are the structure constants for $SU(3)$; the group generators obey

$$[T_a, T_b] = i f_{abc} T_c. \tag{10.1.2}$$

Note that colour indices will, for convenience, sometimes be written as subscripts, sometimes as superscripts – there is no difference in meaning;

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(2) *quark spinor fields* ψ_j, where $j = 1, 2, 3$ labels the quark colour. There will be a set of ψ_j for each flavour, but we leave out the flavour label to simplify the notation;

(3) *a covariant derivative operator*: symbolically one has the operator

$$\hat{D}_\mu \equiv \partial_\mu - igT_a A^a_\mu. \tag{10.1.3}$$

When acting on some given field that transforms according to a particular representation of the group, one replaces the T_a by the relevant representation matrices. Thus when acting on quark fields $\hat{D}_\mu$ is represented by

$$\left(D_\mu\right)_{ij} = \delta_{ij}\partial_\mu - igt^a_{ij}A^a_\mu \tag{10.1.4}$$

where the $t^a, a = 1, \ldots, 8$ are 3×3 hermitian matrices that, for the triplet representation of $SU(3)$, are just one half the Gell–Mann matrices λ^a.

Acting on the gluon fields the T_a are represented by the structure constants $(T_a)_{bc} \to -if_{abc}$, so that $\hat{D}_\mu$ is represented by

$$\left(D_\mu\right)_{bc} = \delta_{bc}\partial_\mu - gA^a_\mu f_{abc}. \tag{10.1.5}$$

The gauge-invariant interactions are described by the lagrangian

$$\mathscr{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar{\psi}_i\gamma_\mu \left(D^\mu\right)_{ij}\psi_j, \tag{10.1.6}$$

where the last term is in fact a sum of identical terms, one for each flavour, and where we have assumed massless quarks.

It is usually assumed that there are no quark mass terms in the original QCD lagrangian, so that it is perfectly flavour symmetric and chirally symmetric. The flavour symmetry is presumably spontaneously broken, the quarks acquiring masses from the electroweak Higgs mechanism and/or from non-perturbative spontaneous chiral-symmetry-breaking effects caused by non-zero vacuum expectation values of $\langle 0|\bar{\psi}_f\psi_f|0\rangle$, where f is some fixed flavour. (For an introductory discussion, see Leader and Predazzi, 1996.)

Since quarks are supposed not to exist as free physical particles their masses are not masses in the usual sense. The quark mass should be thought of simply as a parameter in the lagrangian, to be determined in principle from experiment. However, in perturbation theory, a quark propagator has a pole at $p^2 = m^2$, whereas in the exact theory it presumably has no pole at all. So perturbative calculations are only considered reliable in kinematics regions where the momentum transfers, energies etc., are all large compared with m, which can then be neglected. Thus, determination of quark masses must come from non-perturbative studies such as current algebra or QCD sum rules. (A comprehensive review is given in Gasser and Leutwyler, 1982.) One finds that u and d have masses of a few MeV only ($m_u \sim 4$ MeV$/c^2, m_d \sim 7$ MeV$/c^2$) and that $m_s \sim 125$–150 MeV$/c^2$; these are referred to as 'current quark masses' and should not be confused with the 'constituent quark masses' that are used in the non-relativistic treatment of hadron spectroscopy.

The field theory, not surprisingly, is riddled with infinities and has to be renormalized. In the renormalization the bare coupling constant g in the lagrangian, hidden in the $(D^\mu)_{ij}$ of (10.1.6), becomes replaced by the renormalized coupling, which has to be measured by comparing theory and experiment.

It turns out that there is a certain freedom in carrying out the renormalization, but physical quantities must be invariant under changes of the renormalization scheme. This leads to the concept of the *renormalization group*, under whose transformations the physics is invariant. (See, for example, Chapter 20 of Leader and Predazzi, 1996.) The main consequence for our discussion is that one can 'renormalization-group-improve' a perturbative calculation by replacing the strong coupling $\alpha_s \equiv g^2/4\pi$ by an *effective* or *running* coupling $\alpha_s(Q^2)$, where Q is some characteristic energy or momentum scale of the process one is studying. The variation of $\alpha_s(Q^2)$ with Q^2 is determined by the QCD renormalization group, and to lowest order

$$\alpha_s(Q^2) = \frac{12\pi}{(33 - 2N_f)\ln(Q^2/\Lambda^2)} \tag{10.1.7}$$

where N_f is the number of quark flavours and Λ (often written Λ_{QCD}) has to be determined by experiment (strictly speaking it should be called $\Lambda^{(0)}$ because (10.1.7) is only a lowest-order result) and one has $\Lambda \approx 200$ MeV.

In higher orders $\alpha_s(Q^2)$ and therefore Λ become scheme dependent (see Section 11.7) and require a label to indicate the scheme. And N_f is, strictly, not the total number of flavours but the *effective* number that is relevant, i.e. the number playing a rôle at the scale Q.

The power of using $\alpha_s(Q^2)$ is that $\alpha_s(Q^2) \to 0$ as $Q^2 \to \infty$ (asymptotic freedom) so that for reactions at a large scale Q the effective coupling is small and one can justify a perturbative approach.

When a reaction contains several widely disparate scales $Q_1, Q_2, \ldots$ the above argument becomes ambiguous and there is no obvious rule about what value of Q^2 to use in $\alpha_s(Q^2)$. However, there are many important reactions where one large scale does exist, e.g. deep inelastic lepton–hadron scattering at large momentum transfer (Chapter 11), hadron–hadron scattering at large momentum transfer (Chapter 13), the Drell–Yan process

$$\text{hadron} + \text{hadron} \to \left[(l^+l^-), Z^0, W\right] + X$$

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at large transverse momentum (Chapter 12) and $e^+e^- \rightarrow$ hadrons at high energies (see Leader and Predazzi, 1996, Section 22.1), so there is a host of experimental data against which the theory can be tested.

In summary we can apply perturbative QCD to hard processes where there is one energy or momentum scale sufficiently large to make $\alpha_s(Q^2) \ll 1$. At these scales we can ignore m_u, m_d and m_s and it is adequate to utilize the massless lagrangian (10.1.6). For the 'heavy' quarks $t(!), b$ and perhaps c, one should modify $\mathscr{L}$ to include quark mass terms, but we shall not have space to discuss this.

10.2 Local gauge invariance in QCD

The QCD lagrangian is invariant under local $SU(3)$ transformations. However, in order to do a concrete calculation one has to choose a definite gauge in which to work. In QED one often uses the covariant Lorentz gauge $\partial_\mu A^\mu(x) = 0$. In QCD covariant gauges are more complicated and it is necessary to include a *ghost* propagator in diagrams involving closed loops. (The Feynman rules are given in Appendix 11.) The reason for this difference is linked to the question whether one may replace a polarization vector $\epsilon_\mu(k)$ by $\epsilon_\mu(k) + ck_\mu$, c arbitrary, in the expression for a Feynman diagram involving external photons or gluons.

In both QED and QCD the total amplitude for a reaction involving any number of external photons or gluons respectively has the structure

$$\begin{aligned} A = {} & \epsilon^*_{\mu_1}(k'_1)\ldots\epsilon^*_{\mu_n}(k'_n)M(k'_1\ldots k'_n;k_1\ldots k_m)^{\mu_1\ldots\mu_n;\nu_1\ldots\nu_m} \\ & \times \epsilon_{\nu_1}(k_1)\ldots\epsilon_{\nu_m}(k_m) \end{aligned} \tag{10.2.1}$$

where in *this* expression all 4-vectors k_i, k'_j are on the mass shell, i.e. $k_i^2 = (k'_j)^2 = 0$. (In QCD M would also have colour labels.)

In QED, either for the whole amplitude or for the amplitude arising from any local-gauge-invariant subset of Feynman diagrams, one has the remarkably powerful property that, for any of the momenta,

$$\begin{aligned} (k'_j)_{\mu_j}M(k'_1\ldots k'_n;k_1\ldots k_m)^{\mu_1\ldots\mu_j\ldots\mu_n;\nu_1\ldots\nu_m} &= 0 \\ M(k'_1\ldots k'_n;k_1\ldots k_m)^{\mu_1\ldots\mu_n;\nu_1\ldots\nu_i\ldots\nu_m}(k_i)_{\nu_i} &= 0 \end{aligned} \tag{10.2.2}$$

irrespective of whether the ks in M are on or off the mass shell.

Clearly, then, in QED one is free to replace any $\epsilon_\mu(k)$ by $\epsilon_\mu(k) + ck_\mu$ as long as one is working with either *all* the diagrams of a given order or some local-gauge-invariant subset of them. (Any single Feynman diagram is usually not invariant!) This, as will be seen, allows huge simplifications in the calculations.

In QCD there is nothing like (10.2.2) involving just M itself. Instead, one gets the following rule.

• *QCD local-gauge-invariance rule.* In (10.2.1) we get zero if we replace one or more $\epsilon_{\mu_j}(k_j)$ by $(k_j)_{\mu_j}$, provided that amongst these ks *at most one* is off shell, i.e. does not satisfy $k^2 = 0$. (All the other ks in (10.2.1) are on mass shell, as previously stated.)

Although much weaker than (10.2.2) this rule is sufficient to permit one to replace any $\epsilon_\mu(k)$ by $\epsilon_\mu(k) + ck_\mu$ in the expression for the amplitude arising from any set of local-gauge-invariant diagrams in QCD. A detailed derivation of these results is given in Section 21.3 of Leader and Predazzi (1996).

The identification of local-gauge-invariant subsets of Feynman diagrams is relatively simple in QED. For the photon of interest, for which one wishes to replace $\epsilon_\mu(k)$ by $\epsilon_\mu(k) + ck_\mu$, one takes the set of diagrams in which this photon is attached to a fermion line in all possible ways. For instance, in lowest-order Compton scattering (see Fig. 10.7) neither diagram is local gauge invariant, but their sum is.

In QCD the identification is much more subtle and was solved in a classic paper by Cvitanović, Lauwers and Scharbach (1981). In any reaction involving gluons and quarks the amplitude will be labelled by a colour label for each external parton, $A(a, b, \ldots; i, j, \ldots)$. In colour space there are invariant tensors $F_r(a, b, \ldots; i, j, \ldots), r = 1, 2, \ldots$, for example t^a_{ij}, $t^a_{ij}t^b_{lm}$, f_{abc} etc., which will emerge from any calculation of any individual Feynman diagram. These tensors are generally not independent and may be related through the fundamental structure relations of the Lie group, for example,

$$\left[t^a, t^b\right] = if_{abc}t^c \tag{10.2.3}$$

or the so-called Jacobi identity

$$f_{abe}f_{ecd} + f_{cbe}f_{aed} + f_{dbc}f_{ace} = 0. \tag{10.2.4}$$

By repeated use of these, one can eliminate various F_r until one is left with a *linearly independent* set of tensors T_r. Note that several different Feynman diagrams could give contributions proportional to some given T_r. This set of tensors is called a *colour basis* for the given reaction.

After grouping together all terms proportional to a given T_r the amplitude will take the form

$$A(a, b, \ldots; i, j, \ldots) = \sum_r T_r(a, b, \ldots; i, j, \ldots)\mathscr{A}_r \tag{10.2.5}$$

where the $\mathscr{A}_r$ are functions of the momenta and helicities of the external partons. Since the QCD local-gauge-invariance rule applies to A, and since the terms in (10.2.5) are linearly independent, the rule must apply separately to each $\mathscr{A}_r$. Examples will be given in Section 10.11.

10.3 Feynman rules for massless particles

Since perturbation methods can only be applied to 'hard' processes, in which energies and momenta are large compared with the scale of a typical nucleon mass, the quark-partons of QCD may in many cases be taken as massless. It then turns out that one can reformulate the rules so that calculating the helicity amplitudes from a Feynman diagram becomes much simpler than in the traditional approach. In fact for low-order diagrams these methods remain efficient even when generalized to allow for non-zero-mass quarks. In addition the methods are especially suitable for numerical computation.

The existence of such methods is important because in a high energy collision of hadrons, final states with many jets or hadrons occur and these arise from partonic reactions involving a large number of partons. The number of Feynman diagrams for this kind of process, even in lowest order (known as Born or tree-level), is horrendous. For example for $GG \to 6G$ there are 34 300 diagrams!

Although it is not easy to imagine studying such reactions in order to test QCD, it often happens that one is trying to look for 'new physics' reactions, involving, for example, a sequential decay of some new heavy particle and giving rise to a multijet, multiparticle final state. The identification of a new reaction is impossible without any accurate knowledge of the standard QCD background.

The pioneering steps in this field were taken by De Causmaecker, Gastmans, Troos and Wu (De Causmaecker *et al.*, 1981), and Farrar and Neri (1983), and there followed many calculations in QED by what became known as the CALKUL collaboration. Berends and Giele (1987) approached the massless spinor problem using the dotted and undotted spinors of Weyl and van der Warden and calculated the cross-section for $2G \to 4G$. A further advance was due to Xu, Zhang and Chang (Xu *et al.*, 1987), who simplified the form of the polarization vectors for gluons. Interesting applications have been made by Kleiss and Stirling (1985) to $\bar{p}p \to W/Z +$ jets, by Mangano, Parke and Xu (1987) to multigluon scattering and by Kleiss (1986) to $e^+e^- \to e^+e^-\gamma$ and $e^+e^- \to f\bar{f}\gamma$ (where f is a fermion). For a review of the subject see Mangano and Parke (1991) and for access to the latest literature see Mahlon and Parke (1997) and Benn *et al.* (1997). The reader is warned that in some of these papers the phase conventions do not correspond to the helicity convention utilized in this book and used widely in the literature. Also, in the CALKUL papers what is labelled as helicity ± 1 corresponds to what is normally called helicity ∓ 1 respectively. However, since these papers calculate only cross-sections, i.e. sum over helicities, this does not affect their results. But there could be confusion regarding signs of polarizations etc.

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In the following we reformulate the approach with due care for the phase conventions and in such a fashion that it generalizes to the case of massive particles.

10.3.1 The calculus of massless spinors

The properties of massless spinors are derived in Appendix 12. Here we recall the most important results and introduce a new notation that takes advantage of these properties. As discussed in subsection 4.6.3, in the limit $m \to 0$ the helicity states become states of definitive chirality (R or L) which we shall henceforth designate by $+$ or $-$. We have then

$$\gamma_5 u_\pm = \pm u_\pm \qquad \gamma_5 v_\pm = \mp v_\pm \tag{10.3.1}$$

and eqns (A12.8) and (A12.29) become, for $p^2 = 0$,

$$\bar{u}_\pm(\mathbf{p})u_\pm(\mathbf{p}) = \bar{v}_\pm(\mathbf{p})v_\pm(\mathbf{p}) = 0 \tag{10.3.2}$$

$$\left.\begin{aligned} u_+(\mathbf{p})\bar{u}_+(\mathbf{p}) + u_-(\mathbf{p})\bar{u}_-(\mathbf{p}) &= \not{p} \\ v_+(\mathbf{p})\bar{v}_+(\mathbf{p}) + v_-(\mathbf{p})\bar{v}_-(\mathbf{p}) &= \not{p} \end{aligned}\right\} \tag{10.3.3}$$

Also, from (A12.53) we have

$$\tfrac{1}{2}(1 \pm \gamma_5)\not{p} = u_\pm(\mathbf{p})\bar{u}_\pm(\mathbf{p}) = v_\mp(\mathbf{p})\bar{v}_\mp(\mathbf{p}). \tag{10.3.4}$$

In the Weyl representation (A12.43) we have a simple form for the spinors

$$\begin{aligned} u_+(\mathbf{p}) = v_-(\mathbf{p}) &= \sqrt{2p^0}\begin{pmatrix} \chi_+(\mathbf{p}) \\ 0 \end{pmatrix} \\ u_-(\mathbf{p}) = v_+(\mathbf{p}) &= \sqrt{2p^0}\begin{pmatrix} 0 \\ \chi_-(\mathbf{p}) \end{pmatrix} \end{aligned} \tag{10.3.5}$$

where the two-component spinors $\chi_\pm(\mathbf{p})$ are given in eqn (4.6.28).

We take advantage of the above by introducing the following notation (*only* when $p^2 = 0$)

$$|\mathbf{p}_\pm\rangle \equiv u_\pm(\mathbf{p}) \qquad \langle\mathbf{p}_\pm| \equiv \bar{u}_\pm(\mathbf{p}) \tag{10.3.6}$$

so that if the 4-vectors p, q are such that $p^2 = q^2 = 0$, $p_0 > 0$, $q_0 > 0$, we have the *spinor product*

$$\langle q_{\lambda'}|p_\lambda\rangle = \bar{u}_{\lambda'}(\mathbf{q})u_\lambda(\mathbf{p}), \tag{10.3.7}$$

where throughout this chapter $\lambda = \pm 1$ is the chirality.

From (A12.47) we have for $\lambda' = \lambda$

$$\langle q_\lambda|p_\lambda\rangle = 0. \tag{10.3.8}$$

The symmetry properties of the spinor product are very simple, and all spinor products can be expressed in terms of a basic one, say $\langle q_-|p_+\rangle$.

If the vectors $\mathbf{q}$, $\mathbf{p}$ have polar angles θ, ϕ and θ', ϕ' respectively, then one finds explicitly

$$\langle q_-|p_+\rangle = 2\sqrt{p_0q_0}\left[\cos\left(\frac{\phi'-\phi}{2}\right)\sin\left(\frac{\theta'-\theta}{2}\right) + i\sin\left(\frac{\phi'-\phi}{2}\right)\sin\left(\frac{\theta'+\theta}{2}\right)\right] \tag{10.3.9}$$

$$= \sqrt{2p\cdot q}\,e^{i\Phi_{qp}}. \tag{10.3.10}$$

The phase Φ_{qp} is given by

$$\tan\Phi_{qp} = \tan\left(\frac{\phi'-\phi}{2}\right)\sin\left(\frac{\theta'+\theta}{2}\right)\Big/\sin\left(\frac{\theta'-\theta}{2}\right) \tag{10.3.11}$$

and its quadrant is fixed by demanding that

$$\mathrm{sign}\left[\sin\Phi_{qp}\right] = \mathrm{sign}\left[\sin\left(\frac{\phi'-\phi}{2}\right)\sin\left(\frac{\theta'+\theta}{2}\right)\right]. \tag{10.3.12}$$

It is important to remember that spinors are multivalued functions of the components of a vector, so care must be taken to specify polar angles in a consistent fashion.

It is easy to demonstrate the following elegant properties of the spinor product.

(1) Reversal of chiralities:

$$\langle q_+|p_-\rangle = -\langle q_-|p_+\rangle^*. \tag{10.3.13}$$

(2) Interchange of vectors:

$$\langle p_-|q_+\rangle = -\langle q_-|p_+\rangle. \tag{10.3.14}$$

(3) Interchange of initial and final state:

$$\langle p_+|q_-\rangle = \langle q_-|p_+\rangle^*. \tag{10.3.15}$$

Most importantly one finds that

$$|\langle q_-|p_+\rangle|^2 = 2p\cdot q. \tag{10.3.16}$$

It follows that if q is a multiple of p, $q = Cp$, then $\langle Cp_-|p_+\rangle = 0$. Of particular importance is the case $C = 1$:

$$\langle p_-|p_+\rangle = 0. \tag{10.3.17}$$

Let $p^\mu = (p, \mathbf{p})$ be a null vector with polar coordinates $\mathbf{p} = (p, \theta, \phi)$. We define the *conjugate* four vector $\tilde{p}^\mu$ by

$$\tilde{p}^\mu \equiv (p, -\mathbf{p}), \tag{10.3.18}$$

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where, in accordance with subsection 1.2.2, the polar coordinates of $-\mathbf{p}$ are given by

$$(-\mathbf{p}) \equiv (p, \pi - \theta, \phi + \pi). \tag{10.3.19}$$

Then from (10.3.9) we find

$$\langle \tilde{p}_- | p_+ \rangle = -2ip = -i\sqrt{2p \cdot \tilde{p}}. \tag{10.3.20}$$

Also if p^μ, q^μ are any two null vectors then one finds that

$$\langle \tilde{q}_- | \tilde{p}_+ \rangle = -\langle q_- | p_+ \rangle^* = \langle q_+ | p_- \rangle . \tag{10.3.21}$$

Furthermore one may interchange the conjugacy:

$$\langle q_- | \tilde{p}_+ \rangle = \langle \tilde{q}_- | p_+ \rangle^* = -\langle \tilde{q}_+ | p_- \rangle . \tag{10.3.22}$$

One should beware of the fact that although $\tilde{\tilde{p}}^\mu = p^\mu$ the polar coordinates of $\tilde{\tilde{p}}$ are $p, \theta, \phi + 2\pi$, so that

$$|\tilde{\tilde{p}}_\pm\rangle = -|p_\pm\rangle \tag{10.3.23}$$

When dealing with Feynman diagrams it will turn out that the vertices give rise to matrix elements of the form $\langle q_\lambda | \gamma^\mu | p_\lambda \rangle$. It is easy to demonstrate the following useful properties.

(1) Reversal of chiralities:

$$\langle q_- | \gamma^\mu | p_- \rangle = \langle q_+ | \gamma^\mu | p_+ \rangle^*. \tag{10.3.24}$$

(2) Interchange of initial and final states:

$$\langle p_+ | \gamma^\mu | q_+ \rangle = \langle q_+ | \gamma^\mu | p_+ \rangle^*. \tag{10.3.25}$$

Combining these we have

$$\langle q_- | \gamma^\mu | p_- \rangle = \langle p_+ | \gamma^\mu | q_+ \rangle. \tag{10.3.26}$$

In the expression for the amplitude of a Feynman diagram the γ^μ in a vertex either will be contracted with the polarization vector of an external vector meson or will be linked via a vector meson propagator to some other vertex. We can choose from the outset to work in the Feynman gauge (see Appendix 11), so that the propagator contains only the term $g_{\mu\nu}$ and we end up with contractions of the form

$$\langle a_+ | \gamma^\mu | b_+ \rangle \langle c_- | \gamma_\mu | d_- \rangle \qquad \text{or} \qquad \langle a_+ | \gamma^\mu | b_+ \rangle \langle c_+ | \gamma_\mu | d_+ \rangle .$$

To evaluate these we use (A12.56):

$$2|b_+\rangle\langle a_+| = \langle a_+ | \gamma_\mu | b_+ \rangle \gamma^\mu \tfrac{1}{2}(1 - \gamma_5). \tag{10.3.27}$$

Multiplying on the left by $\langle c_-|$ and on the right by $|d_-\rangle$ yields

$$\langle a_+ | \gamma_\mu | b_+ \rangle \langle c_- | \gamma^\mu | d_- \rangle = 2 \langle a_+ | d_- \rangle \langle c_- | b_+ \rangle . \tag{10.3.28}$$

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For the other possibility we use (10.3.26):

$$\langle a_+|\gamma_\mu|b_+\rangle \langle c_+|\gamma^\mu|d_+\rangle = \langle a_+|\gamma_\mu|b_+\rangle \langle d_-|\gamma^\mu|c_-\rangle \\ = 2\langle a_+|c_-\rangle \langle d_-|b_+\rangle \,. \tag{10.3.29}$$

These are really just special cases of the Fierz rearrangement theorem (Appendix 12).

There are some further useful rearrangement-type results.

Let $|b_+\rangle, |c_+\rangle$ be independent in the sense that the scalar product $b{\cdot}c \neq 0$. Then since the massless spinors are in essence two-component objects it must be possible to expand any $|a_+\rangle$ in terms of $|b_+\rangle$ and $|c_+\rangle$:

$$|a_+\rangle = B|b_+\rangle + C|c_+\rangle.$$

Taking the spinor product with $\langle b_-|, \langle c_-|$ yields

$$\langle c_-|a_+\rangle = B\langle c_-|b_+\rangle \qquad \langle b_-|a_+\rangle = C\langle b_-|c_+\rangle \,..$$

Therefore

$$|a_+\rangle = \frac{\langle c_-|a_+\rangle}{\langle c_-|b_+\rangle}|b_+\rangle + \frac{\langle b_-|a_+\rangle}{\langle b_-|c_+\rangle}|c_+\rangle. \tag{10.3.30}$$

An analogous expansion holds for $|a_-\rangle$ with all chiralities reversed.

Multiplying on the left by some $\langle d_-|$, using (10.3.14) and relabelling into alphabetical order we get

$$\langle a_-|b_+\rangle \langle c_-|d_+\rangle = \langle a_-|d_+\rangle \langle c_-|b_+\rangle + \langle a_-|c_+\rangle \langle b_-|d_+\rangle \,. \tag{10.3.31}$$

Finally, for *any* 4-vector P^μ we introduce the notation

$$\not{P}_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_5)\not{P}. \tag{10.3.32}$$

If $p^2 = 0$, $p^0 > 0$ then from (10.3.4) we have that

$$\not{p} = \not{p}_+ + \not{p}_- \qquad \not{p}_\pm = |p_\pm\rangle\langle p_\pm|. \tag{10.3.33}$$

10.4 The helicity theorem for massless fermions

Because our primary interest is in QCD, QED and the V–A electroweak theory we consider massless fermions coupled to vector bosons $(\gamma, Z^0, W^\pm, G)$ via γ^μ or $\gamma^\mu\gamma_5$ vertices only. There follows a remarkable and powerful result. Consider *any* Feynman diagram, no matter how complicated, in which a fermion line enters in the initial state, continues through the diagram and emerges in the final state, as shown in Fig. 10.1.

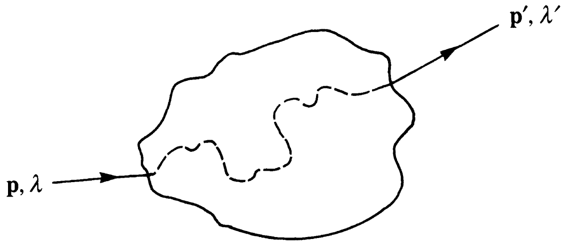

Fig. 10.1 Arbitrary Feynman diagram with fermion line connecting initial and final states.

Label the chiralities of the initial and final fermion f by λ and λ' and call the amplitude $A_{\lambda'\lambda}$. We shall prove that

$$A_{\lambda'\lambda} = 0 \qquad \text{if } \lambda' \neq \lambda. \tag{10.4.1}$$

Focus on the vertices attached to the fermion line under consideration, as shown in Fig. 10.2, where Γ^μ is either γ^μ or $\gamma^\mu\gamma_5$.

Ignoring the denominators of the propagators, the fermion line has associated with it the expression

$$L_{\lambda'\lambda} \equiv \bar{u}_{\lambda'}(\mathbf{p}')\Gamma^{\mu_{n+1}} \not{p}_n \Gamma^{\mu_n} \cdots \not{p}_2 \Gamma^{\mu_2} \not{p}_1 \Gamma^{\mu_1} u_\lambda(\mathbf{p}). \tag{10.4.2}$$

Now replace u and $\bar{u}$ using the fact that for chirality, see (4.6.52),

$$\begin{aligned} u_\lambda(\mathbf{p}) &= \eta_\lambda \gamma_5 u_\lambda(p) \\ \bar{u}_{\lambda'}(\mathbf{p}') &= -\eta_{\lambda'} \bar{u}_{\lambda'}(\mathbf{p}')\gamma_5 \end{aligned} \qquad \eta_\pm = \pm 1 \tag{10.4.3}$$

and commute the rightmost γ_5 through all the Γ^{μ_j} and $\not{p}_j$ until it hits the leftmost γ_5, yielding $\gamma_5^2 = 1$. Now, γ_5 anticommutes with both $\not{p}_j$ and γ^μ or $\gamma^\mu\gamma_5$ so we end up with

$$L_{\lambda'\lambda} = (-1)^N(-\eta_{\lambda'}\eta_\lambda)L_{\lambda'\lambda} \tag{10.4.4}$$

where N is the number of commutations involved. It is easy to see that

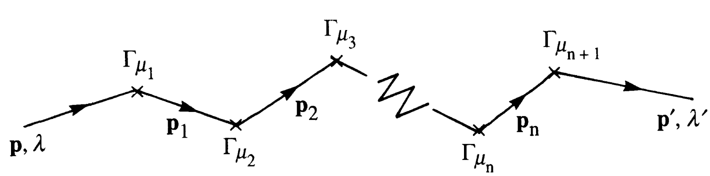

Fig. 10.2. Vertex structure along the fermion line in Fig. 10.1.

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N is always an odd number, so that

$$L_{\lambda'\lambda} = \eta_{\lambda'}\eta_\lambda L_{\lambda'\lambda} \tag{10.4.5}$$

implying $\eta_{\lambda'}\eta_\lambda = +1$ if $L_{\lambda'\lambda} \neq 0$.

Thus we can only have

$$\lambda' = \lambda. \tag{10.4.6}$$

The same result holds for an antifermion passing through the diagram.

For a fermion line that begins and ends in the initial state (i.e. $f\bar{f}$ annihilation) or in the final state (i.e. $f\bar{f}$ production), one finds that the amplitude is zero unless

$$\lambda' = -\lambda. \tag{10.4.7}$$

The conditions for non-zero amplitude are summarized in the diagram in Fig. 10.3.

10.5 Spin structure from a fermion line

Consider the massless fermion line discussed in the previous section (Fig. 10.2) but with all vertices Γ^μ representing γ^μ only. We define the *spin string* associated with it as the ordered product of spinors, propagator factors $\not{p}_j$ and vertices, leaving out all denominators and factors of i. We indicate such a string by the initial and final spinor involved, with a long dash between them. Thus for $\lambda = +1$

$$\begin{aligned}\bar{u}_+(\mathbf{p}')\text{—}u_+(\mathbf{p}) &= \bar{u}_+(\mathbf{p}')\gamma^{\mu_{n+1}}\not{p}_n \cdots \not{p}_1\gamma^{\mu_1}u_+(\mathbf{p}) \\ &= \bar{u}_+(\mathbf{p}')\gamma^{\mu_{n+1}}\left(\frac{1+\gamma_5}{2}\not{p}_n\right)\cdots\left(\frac{1+\gamma_5}{2}\not{p}_1\right)\gamma^{\mu_1}u_+(\mathbf{p})\end{aligned} \tag{10.5.1}$$

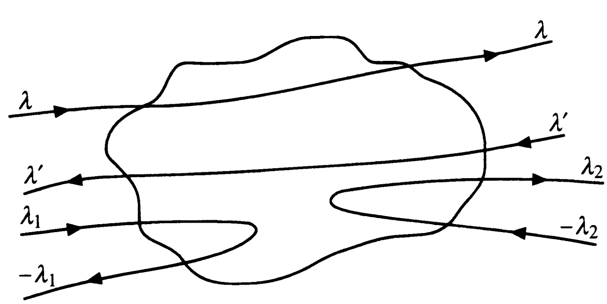

Fig. 10.3. Helicity rules for fermion lines in an arbitrary Feynman diagram.

where we have used the fact that

$$u_+(\mathbf{p}) = \tfrac{1}{2}(1+\gamma_5)u_+(\mathbf{p}) \qquad \text{and} \qquad \left[\tfrac{1}{2}(1+\gamma_5)\right]^2 = \tfrac{1}{2}(1+\gamma_5). \tag{10.5.2}$$

Thus

$$\bar{u}_+(\mathbf{p}')\text{—}u_+(\mathbf{p}) = \langle p'_+|\gamma^{\mu_{n+1}}\not{p}^n_+\cdots\not{p}^1_+\gamma^{\mu_1}|p_+\rangle. \tag{10.5.3}$$

We note that each internal p_j^μ will generally not be a null vector. However, it will always be expressible as a sum of null vectors (in a trivial way in a tree diagram). So each $\not{p}_j$ will give rise, via (10.3.33), to terms of the form $|q_+\rangle\langle q_+|$ with $q^2 = 0$ and the spin string (10.5.3) will be made up of a sum of factors all of the form

$$\langle p'_+|\gamma^{\mu_{n+1}}|q_+\rangle\langle q_+|\gamma^{\mu_n}|r_+\rangle\cdots\langle t_+|\gamma^{\mu_1}|p_+\rangle \tag{10.5.4}$$

with, of course, $q^2 = r^2 = \cdots = t^2 = 0$.

For a string with $\lambda = -1$ we have an analogous expression, except that every factor $\langle r_+|\gamma^\mu|s_+\rangle$ is replaced by $\langle r_-|\gamma^\mu|s_-\rangle$.

For an antifermion line, if the internal momentum labels refer to the flow of physical momentum and are thus directed opposite to the flow of fermion number, the spin string in Fig. 10.4 will be

$$\bar{v}_\lambda(\mathbf{p})\text{—}v_{\lambda'}(\mathbf{p}') = \bar{v}_\lambda(\mathbf{p})\gamma^{\mu_1}(-\not{p}_1)\gamma^{\mu_2}(-\not{p}_2)\cdots(-\not{p}_n)\gamma^{\mu_{n+1}}v_{\lambda'}(\mathbf{p}'). \tag{10.5.5}$$

Using (10.3.5) and (10.3.25) one finds

$$\begin{aligned}\bar{v}_\lambda(\mathbf{p})\text{—}v_{\lambda'}(\mathbf{p}') &= (-i)^n\left[\bar{u}_{-\lambda'}(\mathbf{p}')\text{—}u_{-\lambda}(\mathbf{p})\right]^* \\ &= (-i)^n\left[\bar{u}_{\lambda'}(\mathbf{p}')\text{—}u_\lambda(\mathbf{p})\right],\end{aligned} \tag{10.5.6}$$

by (10.3.21).

If some of the vertices, say m of them, are axial-vector, i.e. $\gamma^\mu\gamma_5$, then in (10.5.6) there is an additional factor $(-1)^m$.

The above is sufficient to deal with all processes not involving external vector mesons. We shall illustrate the simplicity of the approach by an example.

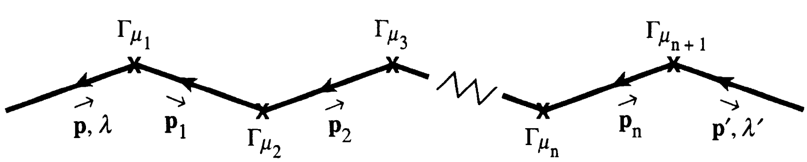

Fig. 10.4. Antifermion line giving rise to the spin string of (10.5.5).

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10.6 Example: high energy $e^- + \mu^- \to e^- + \mu^-$

We work in the CM and ignore the lepton masses since it is assumed that $E \gg m$ for all of them. The momentum vectors must be specified with care (see Section 4.1). For the initial particles we take $p^\mu_{\text{electron}} \equiv p^\mu = (p, \mathbf{p})$ with polar coordinates $\mathbf{p} = (p, 0, 0)$, and $p^\mu_{\text{muon}} = (p, -\mathbf{p}) = \tilde{p}^\mu$, as defined in (10.3.18) and (10.3.19). For the final particles, $p'^\mu_{\text{electron}} \equiv p'^\mu = (p, \mathbf{p}')$ with $\mathbf{p} = (p', \theta, \phi)$ and $p'^\mu_{\text{muon}} = (p, -\mathbf{p}') = \tilde{p}'^\mu$.

The Feynman diagram is shown in Fig. 10.5.

Firstly we take out a factor F coming from couplings, propagator denominators etc., using the standard Feynman rules for QED:

$$F = (-ie)^2 \left(-\frac{i}{k^2}\right) = \frac{ie^2}{k^2}. \tag{10.6.1}$$

The Feynman amplitudes are then as follows:

$$\begin{aligned} M_{++;++} &= F\langle p'_+|\gamma^\mu|p_+\rangle\langle\tilde{p}'_+|\gamma_\mu|\tilde{p}_+\rangle \\ &= 2F\,\langle p'_+|\tilde{p}'_-\rangle\,\langle\tilde{p}_-|p_+\rangle \qquad \text{by (10.3.29)} \\ &= 2F(-2ip)^*(-2ip) \qquad \text{by (10.3.15) and (10.3.20)} \\ &= \frac{8ie^2p^2}{k^2} = 2ie^2\frac{s}{t} \end{aligned} \tag{10.6.2}$$

where, as usual,

$$s = (p + \tilde{p})^2 = 4p^2 \qquad t = (p - p')^2 = k^2; \tag{10.6.3}$$

$$\begin{aligned} M_{+-;+-} &= F\langle p'_+|\gamma^\mu|p_+\rangle\langle\tilde{p}'_-|\gamma_\mu|\tilde{p}_-\rangle \\ &= 2F\,\langle p'_+|\tilde{p}_-\rangle\,\langle\tilde{p}'_-|p_+\rangle \qquad \text{by (10.3.28)} \\ &= -\,2F\,\langle\tilde{p}'_-|p_+\rangle^2 \qquad \text{by (10.3.22) and (10.3.13)} \\ &= -\,2F(2ip\cos\theta/2)^2 \qquad \text{by (10.3.9)} \\ &= \frac{8ie^2p^2\cos^2\theta/2}{k^2} = -2ie^2\frac{u}{t} \end{aligned} \tag{10.6.4}$$

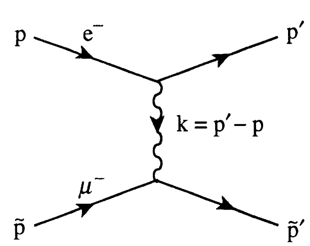

Fig. 10.5. Feynman diagram for $e^- + \mu^- \to e^- + \mu^-$.

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where

$$u = (p - \tilde{p}')^2 = -2p^2(1 + \cos\theta). \tag{10.6.5}$$

The remaining non-zero amplitudes are, by parity invariance (see subsection 4.2.1),

$$M_{--;--} = M_{++;++} \quad \text{and} \quad M_{-+;-+} = M_{+-;+-}. \tag{10.6.6}$$

It is then a trivial matter to compute cross-sections, polarizations, spin correlations etc. using (5.6.3) and (5.6.4). Note that if we are only interested in the cross-sections we can immediately use results such as $|\langle\tilde{p}'_-|p_+\rangle|^2 = 2p\cdot\tilde{p}'$, etc.

Let us now consider reactions with external photons or gluons. To begin with, we return to massive spinors and relate them to massless ones.

10.7 Massive spinors

Let $P^\mu = (E, \mathbf{p})$ be a time-like 4-vector with $P^2 = m^2 \neq 0$. With P^μ we associate two null vectors

$$p^\mu \equiv (p, \mathbf{p}) \tag{10.7.1}$$

and its conjugate

$$\tilde{p}^\mu \equiv (p, -\mathbf{p}), \tag{10.7.2}$$

with polar angles as in (10.3.19).

Then from (A12.44), in the Weyl representation,

$$u\left(P, \frac{\lambda}{2}\right) = \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix} E+m+p\lambda \\ E+m-p\lambda \end{pmatrix}\chi_{\lambda/2}(\hat{\mathbf{p}}); \tag{10.7.3}$$

here $\lambda = \pm 1$ corresponds to *helicity* $\pm 1/2$.

Now, for massless spinors

$$\begin{aligned} |p_+\rangle &= \sqrt{2p}\begin{pmatrix}1\\0\end{pmatrix}\chi_{1/2}(\hat{\mathbf{p}}) \\ |\tilde{p}_-\rangle &= \sqrt{2p}\begin{pmatrix}0\\1\end{pmatrix}\chi_{-1/2}(-\hat{\mathbf{p}}), \end{aligned} \tag{10.7.4}$$

but from (A12.41) one finds

$$\chi_{\lambda/2}(-\hat{\mathbf{p}}) = i\chi_{-\lambda/2}(\hat{\mathbf{p}}). \tag{10.7.5}$$

Thus we can write

$$u_{1/2}(P) = \alpha|p_+\rangle - i\beta|\tilde{p}_-\rangle \tag{10.7.6}$$

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where

$$\begin{aligned}\alpha(p) &\equiv \frac{E+m+p}{2\sqrt{p(E+m)}} \\ \beta(p) &\equiv \frac{E+m-p}{2\sqrt{p(E+m)}}.\end{aligned} \tag{10.7.7}$$

Similarly one finds

$$\begin{aligned}u_{-1/2}(P) &= \alpha|p_-\rangle - i\beta|\tilde{p}_+\rangle \\ v_{1/2}(P) &= \alpha|p_-\rangle + i\beta|\tilde{p}_+\rangle \\ v_{-1/2}(P) &= \alpha|p_+\rangle + i\beta|\tilde{p}_-\rangle.\end{aligned} \tag{10.7.8}$$

Note that for $E \gg m$ one has, to leading order in m/E,

$$\left.\begin{aligned}u_{\lambda/2}(P) &= |p_\lambda\rangle - \frac{im}{2p}|\tilde{p}_{-\lambda}\rangle \\ v_{\lambda/2}(P) &= |p_{-\lambda}\rangle + \frac{im}{2p}|\tilde{p}_\lambda\rangle\end{aligned}\right\} \quad (p \gg m). \tag{10.7.9}$$

Using these results it is clear that the amplitude for any Feynman diagram can be expressed as a combination of amplitudes with massless external fermions.

For present-day applications we are mainly interested in high energy collisions so that all external fermions can usually be taken to be massless. But care must be exercised in deciding whether the mass term in an internal fermion propagator is important. For example in the diagram in Fig. 10.6 for $e^-e^+ \to 2\gamma$, for *small momentum transfer* one should keep the full numerator $\not{p} - \not{k} + m$ even at high energies. The term m will induce a non-zero amplitude for annihilation from states of equal helicity or chirality.

10.8 Polarization vectors

Consider a massive vector meson with 4-momentum $K^\mu = (\omega, \mathbf{k})$, $K^2 = m^2$. As discussed in Section 3.4 the standard polarization vectors for helicity

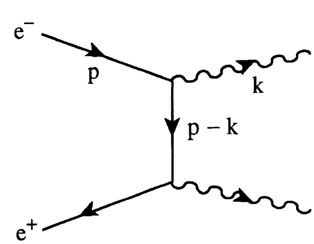

Fig. 10.6. Feynman diagram for $e^-e^+ \to 2\gamma$.

$\lambda = \pm 1, 0$ are

$$\epsilon^{\mu}_{\pm}(K) = \tfrac{1}{\sqrt{2}}(0, \mp\cos\theta\cos\phi + i\sin\phi,$$
$$\mp\cos\theta\sin\phi - i\cos\phi, \pm\sin\theta) \qquad (10.8.1)$$

$$\epsilon^{\mu}_{0}(K) = \frac{1}{m}(k, \omega\hat{\mathbf{k}}) \qquad (10.8.2)$$

when $\mathbf{k}$ has polar angles θ, ϕ. These are the polarization vectors associated with an *incoming* vector meson. For *outgoing* mesons one uses ϵ^{*}_{λ}. By going to the rest frame, where $K^{\mu} = \overset{\circ}{K}{}^{\mu} = (m, 0, 0, 0)$, one can see how to write the matrices $\not{\epsilon}(\overset{\circ}{K})$ in terms of products of the rest spinors $u(\overset{\circ}{K}), v(\overset{\circ}{K})$ etc. Then applying the helicity boost $D[h(\mathbf{k})]$, see (A12.24), one eventually obtains an expression for $\not{\epsilon}(K)$ in terms of massive spinors $u(K), v(K)$ etc. One finds after some labour

$$\not{\epsilon}_{\lambda=\pm 1}(K) = \frac{1}{\sqrt{2}m}\left\{u_{\lambda/2}(K)\bar{v}_{\lambda/2}(K) - v_{-\lambda/2}(K)\bar{u}_{-\lambda/2}(K)\right\} \qquad (10.8.3)$$

$$\not{\epsilon}_{0}(K) = \frac{1}{2m}\Big\{u_{1/2}(K)\bar{v}_{-1/2}(K) + u_{-1/2}(K)\bar{v}_{1/2}(K)$$
$$+ v_{1/2}(K)\bar{u}_{-1/2}(K) + v_{-1/2}(K)\bar{u}_{1/2}(K)\Big\}. \qquad (10.8.4)$$

Introducing as before the null vectors

$$k^{\mu} = (k, \mathbf{k}) \qquad \tilde{k}^{\mu} = (k, -\mathbf{k}) \qquad (10.8.5)$$

and utilizing (10.7.6) and (10.7.8) one eventually finds the very simple result

$$\not{\epsilon}_{\lambda=\pm 1}(K) = \frac{-i}{\sqrt{2}k}\left\{|k_{\lambda}\rangle\langle\tilde{k}_{\lambda}| + |\tilde{k}_{-\lambda}\rangle\langle k_{-\lambda}|\right\}. \qquad (10.8.6)$$

For $\not{\epsilon}_{0}(K)$ it is simpler to write

$$\epsilon^{\mu}_{0}(K) = \frac{1}{2mk}\left\{(\omega + k)k^{\mu} - (\omega - k)\tilde{k}^{\mu}\right\} \qquad (10.8.7)$$

so that

$$\not{\epsilon}_{0}(K) = \frac{1}{2mk}\left\{(\omega + k)\not{k} - (\omega - k)\tilde{\not{k}}\right\}. \qquad (10.8.8)$$

It is important to note that the above forms for $\not{\epsilon}_{\lambda}$ correspond to the expressions (10.8.1) and (10.8.2) for the standard ϵ^{μ}_{λ}.

For the *massless* case, i.e. for a photon or gluon of 4-momentum $k^{\mu} = (k, \mathbf{k})$, there is of course no helicity-zero state but (10.8.6) continues to hold for helicities ± 1.

In this case, however, it is possible to make use of gauge invariance to choose other forms for ϵ^{μ} that simplify the calculations.

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Let q^μ and q'^μ be any two null vectors such that $q \cdot k \neq 0$, $q' \cdot k \neq 0$. Then from (10.3.30) we can write

$$|\tilde{k}_{-\lambda}\rangle = \frac{\left\langle q_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle q_\lambda|k_{-\lambda}\rangle}|k_{-\lambda}\rangle + \frac{\left\langle k_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle k_\lambda|q_{-\lambda}\rangle}|q_{-\lambda}\rangle \tag{10.8.9}$$

$$\langle\tilde{k}_\lambda| = \langle k_\lambda|\frac{\left\langle q'_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle q'_\lambda|k_{-\lambda}\rangle} + \langle q'_\lambda|\frac{\left\langle k_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle k_\lambda|q'_{-\lambda}\rangle}; \tag{10.8.10}$$

substituting in (10.8.6) and using

$$\left\langle k_\lambda|\tilde{k}_{-\lambda}\right\rangle = 2ik \tag{10.8.11}$$

we get, for $\lambda = \pm 1$,

$$\begin{aligned}\not{\epsilon}_\lambda(k) = &\sqrt{2}\left(\frac{|k_\lambda\rangle\langle q'_\lambda|}{\langle k_\lambda|q'_{-\lambda}\rangle} + \frac{|q_{-\lambda}\rangle\langle k_{-\lambda}|}{\langle k_\lambda|q_{-\lambda}\rangle}\right)\\ &- \frac{i}{\sqrt{2}k}\left(\frac{\left\langle q'_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle q'_\lambda|k_{-\lambda}\rangle}|k_\lambda\rangle\langle k_\lambda| + \frac{\left\langle q_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle q_\lambda|k_{-\lambda}\rangle}|k_{-\lambda}\rangle\langle k_{-\lambda}|\right).\end{aligned} \tag{10.8.12}$$

In (10.8.12) let us first choose $q'^\mu = q^\mu$; then the second group of terms becomes

$$-\frac{i}{\sqrt{2}k}\frac{\left\langle q_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\langle q_\lambda|k_{-\lambda}\rangle}\left(|k_\lambda\rangle\langle k_\lambda| + |k_{-\lambda}\rangle\langle k_{-\lambda}|\right) = -\frac{i\left\langle q_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\sqrt{2}k\,\langle q_\lambda|k_{-\lambda}\rangle}\not{k} \tag{10.8.13}$$

by (10.3.33) and (10.4.1). It thus corresponds to a term proportional to k^μ and may be discarded if gauge invariance allows the substitution $\epsilon^\mu \to \epsilon^\mu + ck^\mu$. When this is so we may therefore simply use the expression

$$\not{\epsilon}_\lambda(k;q) \equiv \frac{\sqrt{2}}{\langle k_\lambda|q_{-\lambda}\rangle}\left(|k_\lambda\rangle\langle q_\lambda| + |q_{-\lambda}\rangle\langle k_{-\lambda}|\right) \tag{10.8.14}$$

where q^μ is *any* null vector for which $q \cdot k \neq 0$. In this expression q should be thought of as a reference vector specifying a family of equivalent polarization vectors.

The polarization vector that corresponds to the expression (10.8.14) is clearly

$$\epsilon^\mu_\lambda(k;q) = \epsilon^\mu_\lambda(k) + \frac{i\left\langle q_\lambda|\tilde{k}_{-\lambda}\right\rangle}{\sqrt{2}k\,\langle q_\lambda|k_{-\lambda}\rangle}k^\mu. \tag{10.8.15}$$

There is a very useful form for the $\epsilon^\mu_\lambda(k;q)$, which can be obtained via the relation

$$\epsilon^\mu_\lambda(k;q) = \tfrac{1}{4}\ \mathrm{Tr}\ \left[\gamma^\mu\not{\epsilon}_\lambda(k;q)\right].$$

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Using (10.8.14) and (10.3.26) one finds

$$\epsilon_\lambda^\mu(k;q) = \frac{\langle q_\lambda|\gamma^\mu|k_\lambda\rangle}{\sqrt{2}\,\langle k_\lambda|q_{-\lambda}\rangle}. \qquad (10.8.16).$$

This is particularly helpful when evaluating scalar products of ϵ_λ^μ with some other 4-vector.

It follows that polarization vectors specified by different reference vectors q differ only by a term proportional to the 4-vector k.

It can be checked via (10.3.9) that

$$q_\mu \epsilon_\lambda^\mu(k;q) = 0. \qquad (10.8.17)$$

This implies that using the form (10.8.14) for $\not\epsilon_\lambda(k;q)$ is similar to working in an axial gauge $A_\mu^a q^\mu = 0$, which is convenient for ladder-type diagrams. Other useful properties are

$$\epsilon_\lambda(k_1;q)\cdot\epsilon_\lambda(k_2;q) = 0 \qquad (10.8.18)$$

$$\epsilon_\lambda^*(k_1;k_2)\cdot\epsilon_\lambda(k_2;q) = 0 \qquad \text{or} \qquad \epsilon_{-\lambda}(k_1;k_2)\cdot\epsilon_\lambda(k_2;q) = 0. \qquad (10.8.19)$$

It is crucial, in a non-abelian gauge theory, where the gauge mesons couple to themselves, to remember that (10.8.15) or (10.8.16) *must* be used in conjunction with (10.8.14), if one wishes to work with the vector ϵ_λ^μ itself.

It is easily checked that, as usual,[1]

$$\epsilon_\lambda^{\mu *}(k;q) = -\epsilon_{-\lambda}^\mu(k;q). \qquad (10.8.20)$$

We see now that the standard polarization vectors given in (10.8.6) just correspond to the 4-vector choice $q = \tilde{k}$ in (10.8.14).

The standard form (10.8.6) is adequate for all $2 \to 2$ reactions. For multiparticle production a judicious choice of the reference vector q may simplify the calculation.

Let us return now to the more general expression (10.8.12) in the case where $q'^\mu \neq q^\mu$. We cannot, in general, discard the second group of terms, since, via (10.3.33) and (10.3.32), it contains both $\not k$ and $\gamma_5\not k$ and thus does not correspond to adding a vector ck^μ to ϵ^μ.

However, in massless QED, in any gauge-invariant subset of diagrams a given photon is attached to one single fermion line. In that case ϵ_λ^μ enters only in the form $\not\epsilon_\lambda$ and the γ_5 is innocuous since it will act on a massless

[1] Note that contrary to all other textbooks on field theory, Mangano and Parke (1991) uses $\epsilon_\lambda(k)$ for *outgoing* photons and gluons. Moreover, in their phase convention (10.8.20) holds with a plus sign on the right-hand side.

spinor and convert itself into ± 1. Thus in massless QED one can use a two-parameter family of polarization vectors

$$\not{\epsilon}_\lambda(k;q,q') = \sqrt{2}\left(\frac{|k_\lambda\rangle\langle q'_\lambda|}{\langle k_\lambda|q'_{-\lambda}\rangle} + \frac{|q_{-\lambda}\rangle\langle k_{-\lambda}|}{\langle k_\lambda|q_{-\lambda}\rangle}\right). \tag{10.8.21}$$

Spectacular simplifications ensue from a judicious choice of the 4-vectors q, q', usually from choosing q, q' equal to the initial and final momenta of the fermion line to which the photon is attached.

Before looking at some examples we shall introduce a shorthand notation for the spinor products.

10.9 Shorthand notation for spinor products

To simplify the expressions that occur in the calculation of the amplitudes we introduce, for positive-energy null vectors $a, b, c, \ldots$,

$$\langle ab\rangle \equiv \langle a_-|b_+\rangle \tag{10.9.1}$$

$$[ab] \equiv \langle a_+|b_-\rangle\,. \tag{10.9.2}$$

Equations (10.3.13) to (10.3.22) and (10.3.31) then become

$$\langle ab\rangle = -\langle ba\rangle \qquad [ab] = -[ba] \tag{10.9.3}$$

$$[ab] = \langle ba\rangle^* = -\langle ab\rangle^* \tag{10.9.4}$$

$$\langle ab\rangle\,[ba] = 2a\cdot b \tag{10.9.5}$$

$$\langle \tilde{a}a\rangle = -2ia_0 \tag{10.9.6}$$

$$\left\langle \tilde{a}\tilde{b}\right\rangle = [ab] \tag{10.9.7}$$

$$\left\langle a\tilde{b}\right\rangle = -[\tilde{a}b] \tag{10.9.8}$$

$$\langle ab\rangle\,\langle cd\rangle = \langle ad\rangle\,\langle cb\rangle + \langle ac\rangle\,\langle bd\rangle\,. \tag{10.9.9}$$

For the polarization vectors (10.8.16) one has

$$\epsilon_+^\mu(a;b) = \frac{\langle b_+|\gamma^\mu|a_+\rangle}{\sqrt{2}[ab]} \tag{10.9.10}$$

$$\epsilon_-^\mu(a;b) = \frac{\langle b_-|\gamma^\mu|a_-\rangle}{\sqrt{2}\,\langle ab\rangle} \tag{10.9.11}$$

and, for scalar products involving polarization vectors,

$$\epsilon_+(a;b)\cdot\epsilon_+(c;d) = \frac{\langle ac\rangle\,[db]}{[cd]\,[ab]} \tag{10.9.12}$$

$$\epsilon_+(a;b)\cdot\epsilon_-(c;d) = \frac{\langle ad\rangle\,[cb]}{\langle cd\rangle\,[ab]} \tag{10.9.13}$$

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$$\epsilon_+(a;b)\cdot k = \frac{\langle ak\rangle\,[kb]}{\sqrt{2}[ab]} \tag{10.9.14}$$

$$\epsilon_-(a;b)\cdot k = \frac{[ak]\,\langle kb\rangle}{\sqrt{2}\,\langle ab\rangle}. \tag{10.9.15}$$

We shall illustrate these techniques by two examples.

10.10 QED: high energy Compton scattering

The lowest-order diagrams are shown in Fig. 10.7. We work in the CM, hence $|\mathbf{k}| = |\mathbf{p}| \equiv k$ and we have the 4-vector association $k = \tilde{p}$, $k' = \tilde{p}'$. We take

$$p^\mu = (p, 0, 0, p) \qquad p'^\mu = (p, \mathbf{p}) \quad \text{with} \quad \mathbf{p} = (p, \theta, 0).$$

While neither diagram is gauge invariant, their sum is, so we may make use of our freedom in choosing the form of the polarization vectors. We may utilize the very general form (10.8.21), in which, with an eye to the structure of the diagrams, we take $q'^\mu = p^\mu$, $q^\mu = p'^\mu$.

For the incoming photon we then have

$$\begin{aligned}\not\epsilon_\lambda(k) &\equiv \not\epsilon_\lambda(k;p',p)\\ &= \sqrt{2}\left(\frac{|k_\lambda\rangle\langle p_\lambda|}{\langle k_\lambda|p_{-\lambda}\rangle} + \frac{|p'_{-\lambda}\rangle\langle k_{-\lambda}|}{\langle k_\lambda|p'_{-\lambda}\rangle}\right).\end{aligned} \tag{10.10.1}$$

For the outgoing photon we must use $\epsilon^{\mu*}_\lambda(k')$. We shall denote $\gamma_\mu\epsilon^{\mu*}_\lambda(k')$ by $\not\epsilon^*_\lambda(k')$. Then via (10.8.20)

$$\begin{aligned}\not\epsilon^*_\lambda(k') &= -\not\epsilon_{-\lambda}(k')\\ &= -\sqrt{2}\left(\frac{|k'_{-\lambda}\rangle\langle p_{-\lambda}|}{\langle k'_{-\lambda}|p_\lambda\rangle} + \frac{|p'_\lambda\rangle\langle k'_\lambda|}{\langle k'_{-\lambda}|p'_\lambda\rangle}\right).\end{aligned} \tag{10.10.2}$$

We may start with helicity $+1/2$ for the initial fermion, and we know from (10.4.6) that the final helicity must then be $+1/2$ also.

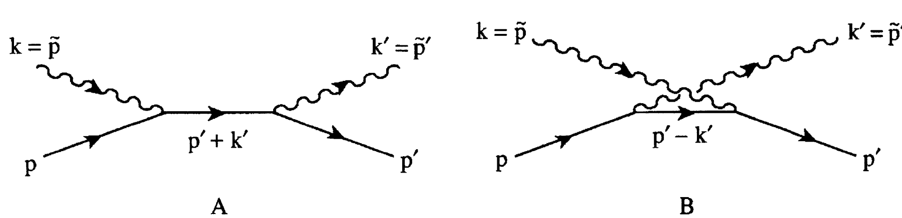

Fig. 10.7. Feynman diagrams for Compton scattering in QED.

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From diagram A we have

$$M^{(\mathrm{A})}_{\lambda'+;\lambda+} = (-ie)^2 \frac{i}{2p\cdot k} \langle p'_+|\not{\epsilon}^*_{\lambda'}(k')(\not{p}' + \not{k}')\not{\epsilon}_\lambda(k)|p_+\rangle. \tag{10.10.3}$$

By (10.3.8) and (10.3.17)

$$\not{\epsilon}_-(k)|p_+\rangle = 0 \qquad \langle p'_+|\not{\epsilon}^*_-(k') = 0. \tag{10.10.4}$$

Thus the only independent non-zero amplitude from diagram A is

$$\begin{aligned} M^{(\mathrm{A})}_{1+;1+} &= -\frac{(-ie)^2 2i}{2p\cdot k} \frac{[p'k']\,\langle pk'\rangle\,[k'p']\,\langle kp\rangle}{\langle k'p\rangle\,[kp']} \\ &= (-ie)^2 \frac{i}{p\cdot k} \frac{[p'k'][k'p']\,\langle kp\rangle}{[kp']} \qquad \text{by (10.9.3)} \\ &= -\frac{ie^2}{2p^2} \frac{[p'\tilde{p}'][\tilde{p}'p']\,\langle \tilde{p}p\rangle}{[kp']} \\ &= -\frac{ie^2}{2p^2} \frac{(2ip)(-2ip)(-2ip)}{(-2ip\cos\theta/2)} \end{aligned}$$

by (10.9.3), (10.9.6), (10.9.8) and (10.3.9). Thus

$$M^{(\mathrm{A})}_{1+;1+} = \frac{-2ie^2}{\cos\theta/2} = -2ie^2\sqrt{\frac{s}{-u}}. \tag{10.10.5}$$

(The singularity at $\theta = \pi$ is, of course, an artifact of our having neglected the fermion mass in the Feynman denominators.)

From diagram B we have

$$M^{(\mathrm{B})}_{\lambda'+;\lambda+} = (-ie)^2 \frac{i}{-2p\cdot k'} \langle p'_+|\epsilon_\lambda(k)(\not{p} - k')\not{\epsilon}^*_{\lambda'}(k')|p_+\rangle \tag{10.10.6}$$

and now we see that

$$\not{\epsilon}^*_+(k')|p_+\rangle = 0 \qquad \langle p_+|\not{\epsilon}_+(k) = 0. \tag{10.10.7}$$

So diagram B does not contribute to $M_{1+;1+}$; for B the only independent non-zero amplitude is

$$\begin{aligned} M^{(\mathrm{B})}_{-1+;-1+} &= \frac{2ie^2}{2p\cdot k'} \frac{[p'k]\,\langle pk'\rangle\,[k'p']\,\langle k'p\rangle}{\langle kp\rangle\,[k'p']} \\ &= \frac{ie^2}{p\cdot k'} \frac{[p'\tilde{p}]\,\langle p\tilde{p}'\rangle\,\langle \tilde{p}'p\rangle}{(-2ip)} \\ &= \frac{-e^2}{2pp\cdot k'} \langle p\tilde{p}'\rangle^3 = 2ie^2\cos\frac{\theta}{2} \\ &= 2ie^2\sqrt{\frac{-u}{s}}. \end{aligned} \tag{10.10.8}$$

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The amplitudes for negative helicity fermions can be obtained by the parity rules in subsection 4.2.1.

The above approach is much simpler and shorter than the conventional one, both for the cross-section and for spin-dependent observables. It is typical of the method that some helicity amplitudes receive contributions from one diagram only.

10.11 QCD: gluon Compton scattering

An important process is

$$G + q \to G + q,$$

which is the QCD analogue of electromagnetic Compton scattering. There is now an extra diagram in lowest order arising from the triple gluon coupling, as shown in Fig. 10.8, where i, j and a, b are colour labels.

The kinematic structure of Fig. 10.8, parts A and B, is exactly the same as in the QED diagrams A and B in Fig. 10.7. From Appendix 11 we see that

$$M_{\rm QCD}^{\rm (A)} = (t^b t^a)_{ji} \tilde{M}^{\rm (A)} \tag{10.11.1}$$

$$M_{\rm QCD}^{\rm (B)} = (t^a t^b)_{ji} \tilde{M}^{\rm (B)} \tag{10.11.2}$$

where

$$\tilde{M}^{\rm (A)} = M_{\rm QED}^{\rm (A)}(e^2 \to g^2) \qquad \tilde{M}^{\rm (B)} = M_{\rm QED}^{\rm (B)}(e^2 \to g^2). \tag{10.11.3}$$

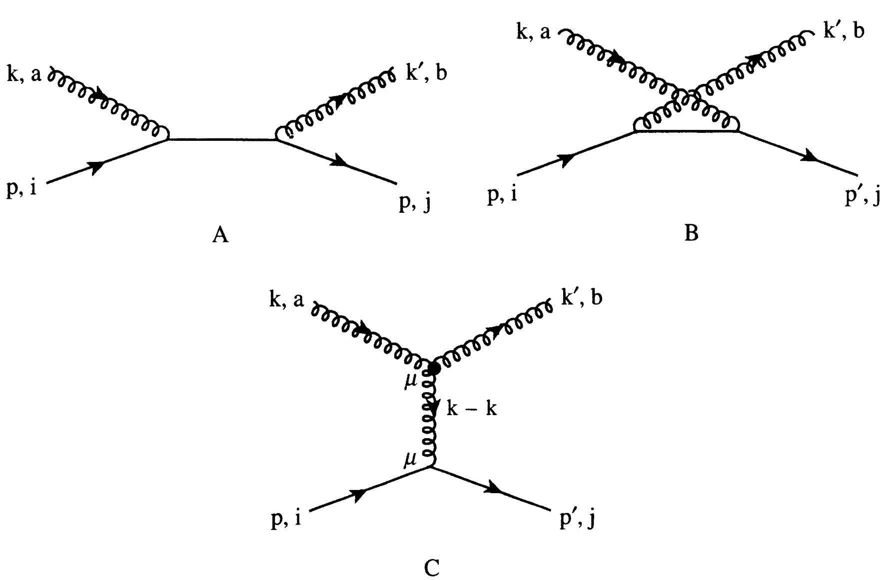

Fig. 10.8. Feynman diagrams for gluon Compton scattering.

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In QED, when $\epsilon^\mu(k)$ is replaced by k^μ the gauge invariance is achieved by a cancellation between $M^{(\mathrm{A})}_{\mathrm{QED}}$ and $M^{(\mathrm{B})}_{\mathrm{QED}}$. Clearly, in the QCD case, because of the different colour structure this can no longer happen and gauge invariance is reinstated only when diagram C in Fig. 10.8 is included.

The colour dependent factor in diagram C is $f_{cba}(t^c)_{ji}$. But the fundamental Lie-algebra commutation relation is

$$[t^a, t^b] = if_{abc}t^c \tag{10.11.4}$$

so that we can make the replacement

$$f_{cba}(t^c)_{ji} = i(t^a t^b)_{ji} - i(t^b t^a)_{ji}. \tag{10.11.5}$$

The linearly independent invariant tensors in colour space may thus be taken as $(t^a t^b)_{ji}$ and $(t^b t^a)_{ji}$; diagram C contributes to both. Writing

$$M^{(\mathrm{C})} = f_{cba}(t^c)_{ji}\tilde{M}^{(\mathrm{C})} \tag{10.11.6}$$

we have, for the total amplitude,

$$M = (t^b t^a)_{ji}\left[\tilde{M}^{(\mathrm{A})} - i\tilde{M}^{(\mathrm{C})}\right] + (t^a t^b)_{ji}\left[\tilde{M}^{(\mathrm{B})} + i\tilde{M}^{(\mathrm{C})}\right] \tag{10.11.7}$$

and we may use different choices of polarization vectors in evaluating the combinations in the first and second pairs of square brackets.

For the first term in (10.11.7) we use (10.8.14), with $q^\mu = p^\mu$ for the incoming gluon and $q^\mu = p'^\mu$ for the outgoing gluon. Let us call this gauge 1. Taking the quark helicity to be $+1/2$, by methods similar to the above one finds that diagram A only contributes to $M_{1+;1+}$, and $\tilde{M}^{(\mathrm{A})}_{1+;1+} = -2ig^2\cos\theta/2$.

For the contribution of the three-gluon vertex in diagram C one has, from Appendix 11, aside from the factor gf_{cba},

$$\begin{aligned}V^\mu_{\lambda'\lambda} &= 2k'\cdot\epsilon_\lambda(k)\epsilon^{\mu\,*}_{\lambda'}(k') - (k+k')^\mu\epsilon^*_{\lambda'}(k')\cdot\epsilon_\lambda(k)\\ &\quad + 2k\cdot\epsilon^*_{\lambda'}(k')\epsilon^\mu_\lambda(k)\end{aligned} \tag{10.11.8}$$

leading to

$$\tilde{M}^{(\mathrm{C})}_{\lambda'+;\lambda+} = -\frac{g^2}{2k\cdot k'}\langle p'_+|\not{V}_{\lambda'\lambda}|p_+\rangle, \tag{10.11.9}$$

which, after some straightforward algebra, yields

$$\begin{aligned}\tilde{M}^{(\mathrm{C})}_{\lambda'+;\lambda+}\Big|_{\mathrm{gauge1}} &= \frac{4g^2p}{k\cdot k'}\left\{\frac{\sin\theta/2}{\sqrt{2}}\left[\delta_{\lambda'1}k'\cdot\epsilon_\lambda(k;p) - \delta_{\lambda1}k\cdot\epsilon^*_{\lambda'}(k';p')\right] + p(\cos\theta/2)\epsilon^*_{\lambda'}\cdot\epsilon_\lambda\right\}\\ &= -2g^2(\cos\theta/2)\left[\lambda\delta_{\lambda'1} + \lambda'\delta_{\lambda1} + \delta_{\lambda'-\lambda} + (\cot^2\theta/2)\delta_{\lambda\lambda'}\right]\\ &= -2g^2(\cos\theta/2)\left[2\delta_{\lambda1} + \cot^2\theta/2\right]\delta_{\lambda\lambda'};\end{aligned} \tag{10.11.10}$$

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to obtain the penultimate expression we have used (10.8.1) for the polarization vectors.

Thus there is no gluon helicity-flip. For the $1+ \to 1+$ amplitude from diagram C we have

$$\tilde{M}^{(C)}_{1+;1+}\Big|_{\text{gauge1}} = -2g^2(\cos\theta/2)\left(2+\cot^2\theta/2\right). \tag{10.11.11}$$

and the first term in (10.11.7) becomes

$$(t^b t^a)_{ji} 2ig^2 \frac{\cot\theta/2}{\sin\theta/2}. \tag{10.11.12}$$

For the second term in (10.11.7) we choose polarization vectors with reference vectors $q = p'$ in $\epsilon_\lambda(k;q)$ and $q = p$ in $\epsilon_{\lambda'}(k';q)$. Call this gauge 2. Then diagram B does not contribute to $1+ \to 1+$. For diagram C we now find

$$\begin{aligned}\tilde{M}^{(C)}_{\lambda'+;\lambda+}\Big|_{\text{gauge2}} = \frac{4pg^2}{k\cdot k'}\Bigg\{&\frac{\sin\theta/2}{\sqrt{2}}\left[\delta_{\lambda',-1}k'\cdot\epsilon_\lambda(k;p') - \delta_{\lambda,-1}k\cdot\epsilon^*_{\lambda'}(k';p)\right]\\ &+p(\cos\theta/2)\epsilon^*_{\lambda'}\cdot\epsilon_\lambda\Bigg\}\end{aligned} \tag{10.11.13}$$

where, via (10.8.15) we find that the polarization vectors in (10.11.13) are related to those in (10.11.10) as follows:

$$\begin{aligned}\epsilon^\mu_\lambda(k;p') &= \epsilon^\mu_\lambda(k;p) - \frac{\lambda}{\sqrt{2}k}(\tan\theta/2)k^\mu\\ \epsilon^\mu_{\lambda'}(k';p) &= \epsilon^\mu_{\lambda'}(k';p') + \frac{\lambda'}{\sqrt{2}k}(\tan\theta/2)k'^\mu.\end{aligned} \tag{10.11.14}$$

Hence

$$\begin{aligned}\tilde{M}^{(C)}_{1+;1+}\Big|_{\text{gauge2}} &= \frac{2g^2\cos\theta/2}{\sin^2\theta/2}\epsilon^*_+(k';p)\cdot\epsilon_+(k;p')\\ &= -\frac{2g^2}{\cos\theta/2\sin^2\theta/2}.\end{aligned} \tag{10.11.15}$$

Thus, for the contribution to the amplitude $1+ \to 1+$, the second term in (10.11.7) yields

$$-(t^a t^b)_{ji}\frac{2ig^2}{\cos\theta/2\sin^2\theta/2}. \tag{10.11.16}$$

The sum of (10.11.12) and (10.11.16) then gives the complete amplitude for $1+ \to 1+$.

It is an interesting exercise to calculate $\tilde{M}^{(A)}$ in gauge 2. One finds

$$\tilde{M}^{(A)}_{1+;1+}\Big|_{\text{gauge2}} = -\frac{2ig^2}{\cos\theta/2}. \tag{10.11.17}$$

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Using (10.11.16) we can now calculate the first term of (10.11.7) in gauge 2, finding for $1+ \to 1+$

$$(t^b t^a)_{ji} \left[\frac{-2ig^2}{\cos\theta/2} - i \frac{-2g^2}{\cos\theta/2 \sin^2\theta/2} \right] = (t^b t^a)_{ji} 2ig^2 \frac{\cot\theta/2}{\sin\theta/2} \qquad (10.11.18)$$

exactly as in (10.11.12).

It is a straightforward matter to calculate the other independent amplitude $M_{-1+;-1+}$ in a similar fashion. Note that there is no gluon helicity-flip in any of these amplitudes.

Of course, choosing two different gauges for the above is a somewhat sledgehammer approach in such a simple problem. But in more complicated, higher-order, diagrams great simplification can be achieved.

As emphasized by Cvitanović, Lauwers and Scharbach (1981), certain properties of the gauge-invariant subsets of diagrams become clearer if linear combinations of the invariant colour tensors are used that transform simply under permutations of the symmetric group. For example, in gluon Compton scattering we could utilize

$$\begin{aligned} \left(T_+^{ba}\right)_{ji} &\equiv \tfrac{1}{2}(t^b t^a + t^a t^b)_{ji} \\ \left(T_-^{ba}\right)_{ji} &\equiv \tfrac{1}{2}(t^b t^a - t^a t^b)_{ji} \end{aligned} \qquad (10.11.19)$$

in which case (10.11.7) becomes

$$M = \left(T_+^{ba}\right)_{ji} \left[\tilde{M}^{(\mathrm{A})} + \tilde{M}^{(\mathrm{B})}\right] + \left(T_-^{ba}\right)_{ji} \left[\tilde{M}^{(\mathrm{A})} - \tilde{M}^{(\mathrm{B})} - 2i\tilde{M}^{(\mathrm{C})}\right]. \qquad (10.11.20)$$

We see that the first term contains only the abelian QED amplitudes. This is a general result. For any number of partons the totally symmetric colour tensor singles out the QED-like contributions to the amplitude and the non-abelian effects are contained in the other-gauge invariant subsets.

10.12 QCD: Multigluon amplitudes

In dealing with purely gluonic reactions it is simpler to deal with the symmetric situations where all the gluons are *incoming*. Let the n gluons labelled $1, 2, \ldots, n$ have colours $a_1, \ldots, a_n$, helicities $\lambda_1, \ldots, \lambda_n$ and momenta $k_1, \ldots, k_n$, respectively. We shall abbreviate the amplitude by

$$M(1, 2, \ldots, n) \equiv M(k_1, \lambda_1, a_1; \ldots; k_n, \lambda_n, a_n). \qquad (10.12.1)$$

The contribution to M from each Feynman diagram will be of the form

$$F(a_1, \ldots, a_n)\epsilon_{\mu_1}(k_1; \lambda_1) \ldots \epsilon_{\mu_n}(k_n; \lambda_n) M^{\mu_1 \ldots \mu_n}(k_1, \ldots, k_n) \qquad (10.12.2)$$

where $F(a_1, \ldots, a_n)$ is a colour factor.

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For the purpose of understanding the helicity structure we may ignore energy and momentum conservation and pretend that all incoming gluons have positive energy in (10.12.2).

The amplitude for a reaction with some outgoing gluons is obtained as follows. Let the jth gluon be outgoing with momentum $\bar{k}_j$ and helicity $\bar{\lambda}_j$. Then the amplitude for

$$G_1 + G_2 + \cdots + G_{j-1} + G_{j+1} + \cdots + G_n \to G_j$$

is given by (see (10.8.20)):

(1) replacing $\epsilon_{\mu_j}(k_j, \lambda_j)$ by

$$\epsilon^*_{\mu_j}(\bar{k}_j, \bar{\lambda}_j) = -\epsilon_{\mu_j}(\bar{k}_j, -\bar{\lambda}_j); \tag{10.12.3}$$

(2) putting $k_j = -\bar{k}_j$ in

$$M^{\mu_1 \dots \mu_n}(k_1, \dots, k_n). \tag{10.12.4}$$

Thus as far as helicity structure is concerned:

an ingoing helicity λ is equivalent to an outgoing helicity $-\lambda$. (10.12.5)

10.12.1 The colour structure

Now the colour factors, whether due to three-gluon or four-gluon vertices, always contain typical products like $f_{abe}f_{ecd}$. From (10.11.4) and the fact that

$$\mathrm{Tr}\ (t^a t^b) = \tfrac{1}{2}\delta_{ab} \tag{10.12.6}$$

one has that

$$\mathrm{Tr}\ (t^e[t^c, t^d]) = \frac{i}{2} f_{ecd} \tag{10.12.7}$$

and therefore

$$\begin{aligned} f_{abe}f_{ecd} &= -2i\ \mathrm{Tr}\ (f_{abe}t^e[t^c, t^d]) \\ &= -2i\ \mathrm{Tr}\ ([t^a, t^b][t^c, t^d]). \end{aligned} \tag{10.12.8}$$

Ultimately one ends up with traces of products, in all possible permutations of all the t^{a_j}. Since the trace is invariant under cyclic permutations the set of independent colour tensors for tree diagrams is just the set of *non-cyclic* permutations of the trace $\mathrm{Tr}\ (t^{a_1}t^{a_2}\cdots t^{a_n})$. The total Feynman amplitude, as will be seen, then has the structure

$$\begin{aligned} M = \frac{2}{i^n} \sum_{\text{perm(23...n)}} &\mathrm{Tr}\ (t^{a_1}t^{a_2}\cdots t^{a_n}) \\ &\times \tilde{M}(k_1, \lambda_1; \dots; k_n, \lambda_n) \end{aligned} \tag{10.12.9}$$

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where the momentum- and helicity-dependent amplitudes $\tilde{M}$ are gauge invariant. Each $\tilde{M}$, which is defined by the order of the labels in it, will contain contributions from several Feynman diagrams, as was the case for the $\tilde{M}$ in Section 10.10. The $\tilde{M}$ are calculated from Feynman diagrams in which the colour factor f_{abc} is simply left out at each trilinear gluon vertex. For the quadrilinear gluon vertices, one starts from the modified form

$$\frac{2}{i^2}(-ig^2) \sum_{\text{perm}(234)} \operatorname{Tr}\,(t^{a_1}t^{a_2}t^{a_3}t^{a_4}) \\ \times \left(2g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right), \quad (10.12.10)$$

which, as can be checked, coincides with the usual expression given in Appendix 11. So, in calculating contributions to $\tilde{M}$ from a Feynman diagram involving a quadrilinear vertex one must use, for the cyclic order (1234),

$$-ig^2\left[2g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3} - g_{\mu_1\mu_2}g_{\mu_3\mu_4}\right]. \quad (10.12.11)$$

Since the $\tilde{M}(1,2,\ldots,n)$ are gauge invariant, the reference vectors q_j used in specifying the polarization vectors $\epsilon(k_j;q_j)$ can be chosen differently for the calculation of each $\tilde{M}$.

Each q_j will always be equal to one of the k_i, say $k_{f(j)}$, i.e. the polarization vectors will have the form $\epsilon(k_j;k_{f(j)})$.

Let the mapping $i \to P_i$ be a permutation of $i = 1,2,\ldots,n$. If, when we calculate $\tilde{M}(P_1,P_2,\ldots,P_n)$, we utilize the set of polarization vectors $\epsilon(k_{P_j};k_{P_{f(j)}})$ then it is clear that we can evaluate $\tilde{M}(P_1,P_2,\ldots,P_n)$ from the result for $\tilde{M}(1,2,\ldots,n)$ by simply carrying out the permutation $i \to P_i$ in the result.

In subsection 10.12.3 we shall illustrate these rather abstract arguments with a concrete example, the four-gluon amplitude, and in subsection 10.12.6 we give some general properties of n-gluon amplitudes. First, however, it will be helpful to deduce two very powerful rules for the helicity structure of gluon amplitudes.

10.12.2 Helicity structure of the n-gluon amplitude

Consider the amplitude for the n-gluon reaction. For a tree diagram consisting solely of trilinear couplings it is easy to see that the number of trilinear vertices, N_V, is related to n by

$$N_V = n - 2. \quad (10.12.12)$$

If there are quadrilinear vertices present in the diagram then for the trilinear vertices one will have

$$N_V < n - 2. \quad (10.12.13)$$

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Now, as can be seen from (10.11.8) each trilinear vertex has mass dimension 1, i.e. $[m]$, whereas the quadrilinear vertices, (10.12.11), are dimensionless. Hence the mass dimension of the *numerator* of any Feynman tree diagram for the n-gluon reaction is d_n, where

$$d_n \leq n-2. \qquad (10.12.14)$$

However, the numerator of the amplitude is linear in the n polarization vectors ϵ of the gluons, which can only occur in combinations of the type $\epsilon \cdot p$ of dimension $[m]$, where p is some momentum, or $\epsilon_i \cdot \epsilon_j$, which is dimensionless. It follows from (10.12.14) that the number of factors of the type $\epsilon \cdot p$ must be $\leq n-2$ and therefore *at least one factor of the type $\epsilon_i \cdot \epsilon_j$ must occur*. This simple result has powerful consequences, as follows:

(1) Consider the amplitude where *all gluon helicities are equal*. Now choose the same reference vector q in (10.8.15) to define the polarization vectors for all the gluons. Then by (10.8.18) every scalar product $\epsilon_\lambda(k_i;q) \cdot \epsilon_\lambda(k_j;q) = 0$ and thus the entire amplitude vanishes.

For a physical scattering reaction, for example for $2G \to nG$, this implies, via (10.12.5), that the amplitude for

$$G_1(-\lambda) + G_2(-\lambda) \to G_3(\lambda) + G_4(\lambda) + \ldots + G_{n+2}(\lambda),$$

i.e. for a maximum change of helicity, is zero.

(2) Suppose now that one gluon, say the lth, has helicity opposite to all the rest, i.e. $\lambda_j = \lambda$ for all $j \neq l$, $\lambda_l = -\lambda$. Now choose the reference vector $q = k_l$ for all gluons except the lth. Then the only possibility for a non-vanishing amplitude must come from scalar products involving $\epsilon_{-\lambda}(k_l;q')$. But these will be of the form $\epsilon_\lambda(k_j;k_l) \cdot \epsilon_{-\lambda}(k_l;q')$, which vanishes by (10.8.19).

For the physical process $2G \to nG$ this implies, via (10.12.5), that

$$A[G_1(-\lambda) + G_2(-\lambda) \to G_3(\lambda) + \cdots + G_l(-\lambda) + \ldots + G_{n+2}(\lambda)] = 0 \qquad (10.12.15)$$

$$A[G_1(\lambda) + G_2(-\lambda) \to G_3(\lambda) + \cdots + G_l(\lambda) + \ldots + G_{n+2}(\lambda)] = 0 \qquad (10.12.16)$$

etc.

For the important reaction

$$G + G \to G + G$$

we immediately see that there are two non-zero independent amplitudes, for example $M_{11;11}$ and $M_{1-1;1-1}$. The other non-zero amplitudes are obtained via parity invariance or symmetry arguments.

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10.12.3 *The amplitude for* $G + G \to G + G$

To make full use of the symmetry, let us suppose that all the gluons are incoming. The Feynman diagrams are shown in Fig. 10.9.

Using the result (10.12.8), diagrams A, B and C have the form

$$\begin{aligned} &-2\ \mathrm{Tr}\ (1234 + 4321 - 1243 - 3421)\ \tilde{m}^{(\mathrm{A})} \\ &-2\ \mathrm{Tr}\ (1324 + 4231 - 1243 - 3421)\ \tilde{m}^{(\mathrm{B})} \\ &-2\ \mathrm{Tr}\ (1234 + 4321 - 1423 - 3241)\ \tilde{m}^{(\mathrm{C})} \end{aligned} \tag{10.12.17}$$

where the shorthand notation

$$\mathrm{Tr}\ (t^{a_1} t^{a_2} t^{a_3} t^{a_4}) = \mathrm{Tr}\ (1234) \tag{10.12.18}$$

is used, and the $\tilde{m}^{(\mathrm{A})}$ are the momentum- and helicity-dependent amplitudes calculated without the colour factors f_{abc}.

Now, if we write

$$\tilde{m}^{(\mathrm{A})} \equiv \tilde{m}^{(\mathrm{A})}(1, 2, 3, 4) \tag{10.12.19}$$

etc. then it is clear that

$$\tilde{m}^{(\mathrm{B})} = \tilde{m}^{(\mathrm{A})}(1, 3, 2, 4), \qquad \tilde{m}^{(\mathrm{C})} = \tilde{m}^{(\mathrm{A})}(4, 1, 2, 3). \tag{10.12.20}$$

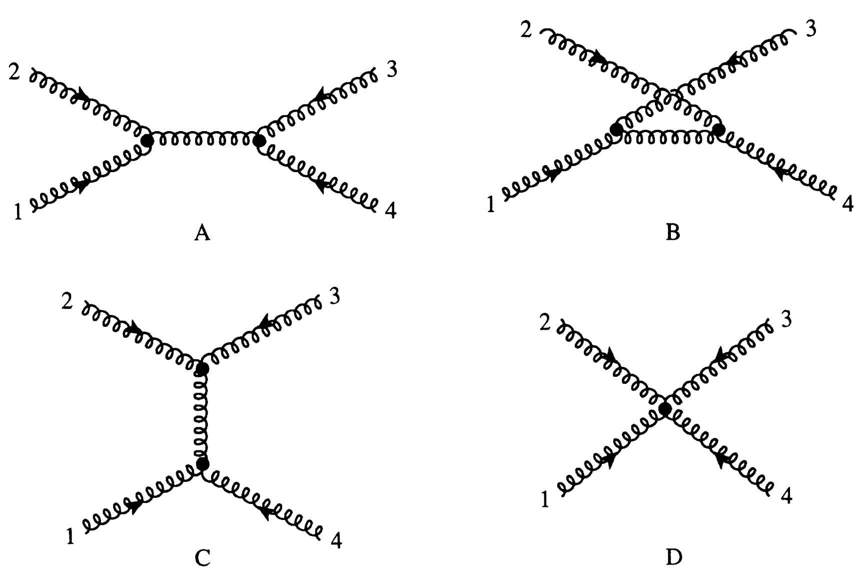

Fig. 10.9. Lowest-order Feynman diagrams for $G + G \to G + G$.

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For diagram D, using (10.12.10) we have

$$-2 \sum_{\text{perm }(234)} \text{Tr }(1234)\ \tilde{m}^{(\text{D})}(1,2,3,4) \tag{10.12.21}$$

where $\tilde{m}^{(\text{D})}$ is calculated using (10.12.11).

Now (10.12.17) can be re-arranged in the form

$$\begin{aligned}
&-2\ \text{Tr }(1234+4321)\left[\tilde{m}^{(\text{A})}(1,2,3,4)+\tilde{m}^{(\text{A})}(4,1,2,3)\right]\\
&-2\ \text{Tr }(1324+4231)\left[\tilde{m}^{(\text{A})}(1,3,2,4)-\tilde{m}^{(\text{A})}(4,1,2,3)\right]\\
&+2\ \text{Tr }(1243+3421)\left[\tilde{m}^{(\text{A})}(1,2,3,4)+\tilde{m}^{(\text{A})}(1,3,2,4)\right].
\end{aligned} \tag{10.12.22}$$

We shall soon see explicitly that

$$\tilde{m}^{(\text{A})}(1,2,3,4)=\tilde{m}^{(\text{A})}(2,1,4,3)=\tilde{m}^{(\text{A})}(4,3,2,1) \tag{10.12.23}$$

$$=-\tilde{m}^{(\text{A})}(2,1,3,4)=-\tilde{m}^{(\text{A})}(1,2,4,3) \tag{10.12.24}$$

so that (10.12.22) becomes

$$\begin{aligned}
&-2\ \text{Tr }(1234+4321)\left[\tilde{m}^{(\text{A})}(1,2,3,4)+\tilde{m}^{(\text{A})}(4,1,2,3)\right]\\
&-2\ \text{Tr }(1324+4231)\left[\tilde{m}^{(\text{A})}(1,3,2,4)+\tilde{m}^{(\text{A})}(4,1,3,2)\right]\\
&-2\ \text{Tr }(1243+3421)\left[\tilde{m}^{(\text{A})}(1,2,4,3)+\tilde{m}^{(\text{A})}(3,1,2,4)\right]
\end{aligned} \tag{10.12.25}$$

Using the second of eqns (10.12.23) we have finally for diagrams A + B + C + D

$$M=-2 \sum_{\text{perm }(234)} \text{Tr }(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\ \tilde{M}(1,2,3,4) \tag{10.12.26}$$

where

$$\tilde{M}(1,2,3,4)=\tilde{m}^{(\text{A})}(1,2,3,4)+\tilde{m}^{(\text{A})}(4,1,2,3)+\tilde{m}^{(\text{D})}(1,2,3,4). \tag{10.12.27}$$

Equation (10.12.26) is precisely in the form (10.12.9). In Section 10.12.4 we shall explain how the amplitudes $\tilde{M}$ are constructed in the general n-gluon case. Here we look in more detail into the four-gluon amplitude.

Using the form of the three-gluon vertex given in Appendix 11, without the colour factor f_{abc}, one finds after some algebra that

$$\begin{aligned}\tilde{m}^{(\mathrm{A})}(1,2,3,4) = &-\frac{ig^2}{(k_1+k_2)^2}\Big((\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot\epsilon_4)(k_1-k_2)\cdot(k_3-k_4)\\ &+4\left\{(\epsilon_1\cdot\epsilon_2)\left[(\epsilon_3\cdot k_1)(\epsilon_4\cdot k_2)-(\epsilon_3\cdot k_2)(\epsilon_4\cdot k_1)\right]\right.\\ &\quad+(\epsilon_3\cdot\epsilon_4)\left[(\epsilon_1\cdot k_3)(\epsilon_2\cdot k_4)-(\epsilon_1\cdot k_4)(\epsilon_2\cdot k_3)\right]\\ &\quad+(\epsilon_1\cdot\epsilon_3)(\epsilon_2\cdot k_1)(\epsilon_4\cdot k_3)+(\epsilon_2\cdot\epsilon_4)(\epsilon_1\cdot k_2)(\epsilon_3\cdot k_4)\\ &\quad-(\epsilon_1\cdot\epsilon_4)(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_4)\\ &\quad\left.-\ (\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_2)(\epsilon_4\cdot k_3)\right\}\Big) \end{aligned} \tag{10.12.28}$$

from which the properties (10.12.23) and (10.12.24) can be read off.

As we know from subsection 10.12.2 we can take as independent non-zero amplitudes just $M(++;--)$ and $M(+-;+-)$, corresponding to $\lambda_1 = \lambda_2 = +1$, $\lambda_3 = \lambda_4 = -1$ and $\lambda_1 = \lambda_3 = +1$, $\lambda_2 = \lambda_4 = -1$ respectively.

Consider first, in an obvious notation, the amplitude

$$\tilde{M}(1^+, 2^+, 3^-, 4^-).$$

Choose the reference q_j in such a way that the polarization vectors are

$$\epsilon_+(k_1;k_3) \quad \epsilon_+(k_2;k_3) \quad \epsilon_-(k_3;k_2) \quad \epsilon_-(k_4;k_2). \tag{10.12.29}$$

Then, via (10.8.18) and (10.8.19), the only non-zero scalar product of two polarization vectors is $\epsilon_+(k_1;k_3)\cdot\epsilon_-(k_4;k_2)$. Hence

$$\tilde{m}^{(\mathrm{D})}(1^+,2^+,3^-,4^-) = 0 \tag{10.12.30}$$

and

$$\tilde{m}^{(\mathrm{A})}(1^+,2^+,3^-,4^-) = \frac{4ig^2}{(k_1+k_2)^2}(\epsilon_1^+\cdot\epsilon_4^-)(\epsilon_2^+\cdot k_1)(\epsilon_3^-\cdot k_4). \tag{10.12.31}$$

Further, from (10.12.28),

$$\tilde{m}^{(\mathrm{A})}(4^-,1^+,2^+,3^-) \propto (\epsilon_4\cdot\epsilon_1)\left[(\epsilon_2\cdot k_4)(\epsilon_3\cdot k_1)-(\epsilon_2\cdot k_1)(\epsilon_3\cdot k_4)\right],$$

which, using (10.8.17), will vanish when energy–momentum conservation is enforced. Thus we end up with the remarkably simple result

$$\tilde{M}(k_1^+,k_2^+,k_3^-,k_4^-) = \frac{4ig^2}{(k_1+k_2)^2}(\epsilon_1^+\cdot\epsilon_4^-)(\epsilon_2^+\cdot k_1)(\epsilon_3^-\cdot k_4), \tag{10.12.32}$$

Consider now the amplitude $\tilde{M}(1^+,2^-,3^+,4^-)$. Choose reference momenta such that the polarization vectors are:

$$\epsilon_+(k_1;k_4) \quad \epsilon_+(k_3;k_4) \quad \epsilon_-(k_2;k_1) \quad \epsilon_-(k_4;k_1). \tag{10.12.33}$$

and the only non-zero scalar product is $\epsilon_+(k_3;k_4)\cdot\epsilon_-(k_2;k_1)$. Hence

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$$\tilde{m}^{(\mathrm{D})}(1^+, 2^-, 3^+, 4^-) = 0 \tag{10.12.34}$$

$$\tilde{m}^{(\mathrm{A})}(1^+, 2^-, 3^+, 4^-) = \frac{4ig^2}{(k_1+k_2)^2}(\epsilon_2^- \cdot \epsilon_3^+)(\epsilon_1^+ \cdot k_2)(\epsilon_4^- \cdot k_3) \tag{10.12.35}$$

and, as before,

$$\tilde{m}^{(\mathrm{A})}(4^-, 1^+, 2^-, 3^+) = 0.$$

Hence

$$\tilde{M}(k_1^+, k_2^-, k_3^+, k_4^-) = \frac{4ig^2}{(k_1+k_2)^2}(\epsilon_2^- \cdot \epsilon_3^+)(\epsilon_1^+ \cdot k_2)(\epsilon_4^- \cdot k_3). \tag{10.12.36}$$

The physical amplitudes $\tilde{M}_{\lambda_1'\lambda_2';\lambda_1\lambda_2}$ for the reaction

$$(k_1, \lambda_1) + (k_2, \lambda_2) \to (k_1', \lambda_1') + (k_2', \lambda_2')$$

are then, via (10.12.3), (10.12.4) and (10.12.17),

$$\begin{aligned} M_{11;11} = & -2 \sum_{\text{perm (234)}} \mathrm{Tr}\,(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,(-2ig^2) \\ & \times \left\{ \frac{1}{k_1 \cdot k_2}\,[\epsilon_1^+(k_1;-k_3)\cdot\epsilon_4^-(-k_4;k_2)]\,[\epsilon_2^+(k_2;-k_3)\cdot k_1] \right. \\ & \left. \times\,[\epsilon_3^+(-k_3;k_2)\cdot(-k_4)] \right\}, \end{aligned} \tag{10.12.37}$$

where

$$k_3 \equiv -k_1' \qquad k_4 \equiv -k_2', \tag{10.12.38}$$

and

$$\begin{aligned} M_{1-1;1-1} = & -2 \sum_{\text{perm (234)}} \mathrm{Tr}\,(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\,(-2ig^2) \\ & \times \left\{ \frac{1}{k_1 \cdot k_2}\,[\epsilon_2^-(k_2;k_1)\cdot\epsilon_3^+(-k_3;-k_4)]\,[\epsilon_1^+(k_1;-k_4)\cdot k_2] \right. \\ & \left. \times\,[\epsilon_4^-(-k_4;k_1)\cdot(-k_3)] \right\} \end{aligned} \tag{10.12.39}$$

Using eqns (10.9.1)–(10.9.11) we get for the parts of (10.12.37) and (10.12.39) within the braces

$$-\frac{\langle k_1 k_2\rangle^2\,[k_1' k_2']^2}{4(k_1\cdot k_2)(k_2\cdot k_3)} \qquad \text{and} \qquad -\frac{\langle k_1 k_1'\rangle^2\,[k_2' k_2]^2}{4(k_1\cdot k_2)(k_2\cdot k_3)} \tag{10.12.40}$$

respectively; we have used the fact that $k_1 \cdot k_4 = k_2 \cdot k_3$.

It is clear that the structure of the numerators just corresponds to a pairing of the momenta of the gluons with the same helicity label, $(\pm)$,

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after each ϵ_λ^* is replaced by $-\epsilon_{-\lambda}$, with the correspondence

$$(+) \to \langle \quad \rangle$$
$$(-) \to [\quad].$$

Thus, since permutation does not alter which gluon has which helicity, one has

$$M_{11;11} = -ig^2 \langle k_1 k_2 \rangle^2 [k_1' k_2']^2 \times \sum_{\text{perm (234)}} \frac{\text{Tr } (t^{a_1} t^{a_2} t^{a_3} t^{a_4})}{(k_1 \cdot k_2)(k_2 \cdot k_3)} \tag{10.12.41}$$

$$M_{1-1;1-1} = -ig^2 \langle k_1 k_1' \rangle^2 [k_2' k_2]^2 \times \sum_{\text{perm (234)}} \frac{\text{Tr } (t^{a_1} t^{a_2} t^{a_3} t^{a_4})}{(k_1 \cdot k_2)(k_2 \cdot k_3)}. \tag{10.12.42}$$

Note that since $k_1 + k_2 + k_3 + k_4 = 0$,

$$\sum_{\text{perm (234)}} \frac{1}{(k_1 \cdot k_2)(k_2 \cdot k_3)} = 0 \tag{10.12.43}$$

which is actually a reflection of a general property referred to as a *dual Ward identity* (see subsection 10.12.6 below).

We turn now to consider the colour structure.

10.12.4 Colour sums for gluon reactions

All physical observables are bilinear in the helicity amplitudes and one almost always wishes to carry out a sum over the colours of the gluons. One thus has to carry out colour sums of the type

$$S = \sum_{\text{all } a_j} [\text{ Tr } (t^{a_1} t^{a_2} \cdots t^{a_n})] \left[\text{ Tr } (t^{a_1} t^{b_2} \cdots t^{b_n}) \right]^* \tag{10.12.44}$$

where $(b_2 \dots b_n)$ is some permutation of $(a_2 \dots a_n)$. Because the t^a are hermitian one has

$$S = \sum_{\text{all } a_j} \text{Tr } (t^{b_n} \cdots t^{b_2} t^{a_1}) \text{ Tr } (t^{a_1} t^{a_2} \cdots t^{a_n}). \tag{10.12.45}$$

The sum can be carried out step by step using the relations, valid for $SU(N)$,

$$\sum_a t^a_{ij} t^a_{kl} = \frac{1}{2} \left[\delta_{il} \delta_{jk} - \frac{1}{N} \delta_{ij} \delta_{kl} \right], \tag{10.12.46}$$

$$\sum_a (t^a t^a)_{ij} = \frac{N^2 - 1}{2N} \delta_{ij}. \tag{10.12.47}$$

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We briefly indicate how this works. We have

$$S = \sum_{\text{all } a_j} (t^{b_n} \cdots t^{b_2} t^{a_1})_{ji} t^{a_1}_{ij} t^{a_1}_{kl} (t^{a_2} \cdots t^{a_n})_{lk}$$
$$= \frac{1}{2} \sum_{a_2 \dots a_n} \Big[\text{Tr } (t^{b_n} \cdots t^{b_2} t^{a_2} \cdots t^{a_n})$$
$$- \frac{1}{N} \text{Tr } (t^{b_n} \cdots t^{b_2}) \text{ Tr } (t^{a_2} \cdots t^{a_n}) \Big]. \quad (10.12.48)$$

To reduce further the second term one uses invariance under cyclic permutations to put it in the form

$$\text{Tr } (t^{b_n} \cdots t^{b_2}) \text{ Tr } (t^{b_2} t^{c_3} \cdots t^{c_n}),$$

where $(c_3 c_4 \dots c_n)$ is a permutation of $(b_3 b_4 \dots b_n)$, and then repeats the process used in (10.12.48).

For the first term there are two possibilities. If $b_2 = a_2$ we have

$$\sum_{a_2} \text{Tr } (t^{b_n} \cdots t^{b_3} t^{a_2} t^{a_2} t^{a_3} \cdots t^{a_n})$$
$$= \frac{N^2 - 1}{N} \text{Tr } (t^{b_n} \cdots t^{b_3} t^{a_3} \cdots t^{a_n}). \quad (10.12.49)$$

If $b_2 \neq a_2$ then by cyclic permutation the first term can be put in the form Tr $(\Lambda_1 t^{a_2} \Lambda_2 t^{a_2})$, where $\Lambda_{1,2}$ are products of t^{a_j}, $j = 3, \dots, n$. Then by (10.12.46)

$$\sum_{a_2} \text{Tr } (\Lambda_1 t^{a_2} \Lambda_2 t^{a_2}) = \frac{1}{2} \left[\text{Tr } (\Lambda_1) \text{ Tr } (\Lambda_2) - \frac{1}{N} \text{ Tr } (\Lambda_1 \Lambda_2) \right]. \quad (10.12.50)$$

Some useful results that hold when the colour group is $SU(3)$ are given below. For brevity we write

$$(a_1 a_2 a_3 a_4) \equiv \text{ Tr } (t^{a_1} t^{a_2} t^{a_3} t^{a_4}); \quad (10.12.51)$$

then

$$\sum_{a_i} (a_1 a_2) = 4 \quad (10.12.52)$$
$$\sum_{a_i} (a_1 a_2 a_1 a_2) = -2/3 \quad (10.12.53)$$
$$\sum_{a_i} (a_1 a_2 a_2 a_1) = 16/3 \quad (10.12.54)$$
$$\sum_{a_i} (a_1 a_2)(a_1 a_2) = 2. \quad (10.12.55)$$

There are two independent products of traces of three ts; the rest can be obtained by cyclic permutations.

$$\sum_{a_i}(a_3a_2a_1)(a_1a_2a_3) = 7/3 \tag{10.12.56}$$

$$\sum_{a_i}(a_2a_3a_1)(a_1a_2a_3) = -2/3. \tag{10.12.57}$$

There are six independent products of traces of four ts; the rest can be obtained by cyclic permutations.

$$\sum_{a_i}(a_4a_3a_2a_1)(a_1a_2a_3a_4) = 19/6 \tag{10.12.58}$$

$$\sum_{a_i}(a_1a_2a_3a_4)(a_1a_2a_3a_4) = 2/3 \tag{10.12.59}$$

$$\sum_{a_i}(a_3a_4a_2a_1)(a_1a_2a_3a_4) = -1/3 \tag{10.12.60}$$

$$\sum_{a_i}(a_1a_2a_4a_3)(a_1a_2a_3a_4) = -1/3 \tag{10.12.61}$$

$$\sum_{a_i}(a_4a_2a_3a_1)(a_1a_2a_3a_4) = -1/3 \tag{10.12.62}$$

$$\sum_{a_i}(a_1a_3a_2a_4)(a_1a_2a_3a_4) = -1/3. \tag{10.12.63}$$

10.12.5 Colour sum for GG → GG

Let us now apply these results to gluon–gluon scattering using the amplitudes (10.12.41), (10.12.42). Suppose we are interested in calculating the cross-section. In that case we need for example

$$\begin{aligned}\sum_{\text{colours}} |M_{11;11}|^2 = g^4 |\langle k_1k_2\rangle|^4 |[k_1'k_2']|^4 \\ \times \sum_{\text{colours}} \left[\sum_{\text{perm}(234)} \frac{(a_1a_2a_3a_4)}{(k_1\cdot k_2)(k_2\cdot k_3)}\right] \\ \times \left[\sum_{\text{perm}(234)} \frac{(a_1a_2a_3a_4)}{(k_1\cdot k_2)(k_2\cdot k_3)}\right]^* .\end{aligned} \tag{10.12.64}$$

It is clear from (10.12.25) that actually the trace $(a_1a_2a_3a_4)$ and the trace $(a_1a_4a_3a_2) = (a_4a_3a_2a_1)$, obtained by the reflection permutation $(1234) \rightarrow (4321)$, should be multiplied by the same kinematic amplitude. This is seen to be true since

$$(k_1\cdot k_2)(k_2\cdot k_3) = (k_1\cdot k_4)(k_4\cdot k_3).$$

Thus we can replace $(a_1a_2a_3a_4)$ by

$$[a_1a_2a_3a_4] \equiv (a_1a_2a_3a_4) + (a_4a_3a_2a_1) \tag{10.12.65}$$

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and sum only over permutations of (234) that are not reflections (NR) of each other, i.e. NR permutations of (234) are (234), (243), (324).

Furthermore the two separate sets of permutations can be rearranged so that

$$\sum_{\text{colours}} |M_{11;11}|^2 = g^4 |\langle k_1 k_2 \rangle|^4 |[k_1' k_2']|^4$$
$$\times \sum_{\text{colours}} \sum_{\substack{\text{NRperm}\\(234)}} \frac{[a_1 a_2 a_3 a_4]}{(k_1 \cdot k_2)(k_2 \cdot k_3)}$$
$$\times \left\{ \frac{[a_1 a_2 a_3 a_4]^*}{(k_1 \cdot k_2)(k_2 \cdot k_3)} + \frac{[a_1 a_2 a_4 a_3]^*}{(k_1 \cdot k_2)(k_2 \cdot k_4)} + \frac{[a_1 a_3 a_2 a_4]^*}{(k_1 \cdot k_3)(k_3 \cdot k_2)} \right\}. \tag{10.12.66}$$

Using eqns (10.12.58)–(10.12.63) one obtains

$$\sum_{\text{colours}} [a_1 a_2 a_3 a_4][a_1 a_2 a_3 a_4]^* = 23/3 \tag{10.12.67}$$

$$\sum_{\text{colours}} [a_1 a_2 a_3 a_4][a_1 a_2 a_4 a_3]^* = -4/3 \tag{10.12.68}$$

$$\sum_{\text{colours}} [a_1 a_2 a_3 a_4][a_1 a_3 a_2 a_4]^* = -4/3 \tag{10.12.69}$$

Writing $23/3 = 9 - 4/3$, the terms in multiplying $-4/3$ in (10.12.66), vanish by (10.12.43). So we are left with

$$\sum_{\text{colours}} |M_{11;11}|^2 = 9g^4 |\langle k_1 k_2 \rangle|^4 |[k_1' k_2']|^4$$
$$\times \sum_{\substack{\text{NRperm}\\(234)}} \left[\frac{1}{(k_1 \cdot k_2)(k_2 \cdot k_3)} \right]^2 . \tag{10.12.70}$$

Writing this in terms of the usual Mandelstam variables,

$$\sum_{\text{colours}} |M_{11;11}|^2 = 144 s^4 \left(\frac{1}{s^2 t^2} + \frac{1}{s^2 u^2} + \frac{1}{t^2 u^2} \right). \tag{10.12.71}$$

Summing over helicities and using symmetry arguments to evaluate $M_{1-1;1-1}$ and parity invariance for the other Ms,

$$\sum_{\substack{\text{helicities}\\ \text{colours}}} |M^2| = 288(s^4 + t^4 + u^4) \left(\frac{1}{s^2 t^2} + \frac{1}{s^2 u^2} + \frac{1}{t^2 u^2} \right), \tag{10.12.72}$$

which can be simplified using $s + t + u = 0$. Dividing by 4×64 to obtain

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an average over initial spins and colours gives

$$\overline{|M|^2} = \frac{9}{2}g^4\left(3 - \frac{ut}{s^2} - \frac{us}{t^2} - \frac{st}{u^2}\right) \tag{10.12.73}$$

which is a well-known result.

10.12.6 Some properties of *n*-gluon amplitudes

The set of all Feynman tree diagrams, for a given n, consists of certain 'basic' structures from which the rest can be generated by permuting the gluon labels in all possible ways, subject to the restriction that the diagrams thus generated are topologically independent. For example, for the Feynman diagrams for four gluons shown in Fig. 10.9, diagrams (B) and (C) were obtained from (A) by permuting gluon labels; see (10.12.20).

As a consequence it turns out that the kinematic amplitudes $\tilde{M}$ in (10.12.9) are invariant under *cyclic* permutations of the gluon momentum and helicity labels. Note, however, that the application of a permutation to the result for an amplitude $\tilde{M}$ can be rather subtle. For example, in (10.12.41) the factors $\langle k_1 k_2\rangle$ and $[k_1' k_2']$ refer to the momenta of gluons *with particular helicities* and this does not change under a permutation, i.e. if gluon k_1 has helicity $+1$ it still does so after permuting the arguments of the function.

For a given n the Feynman tree diagrams can be grouped into subsets, each subset J being represented by one characteristic diagram, D_J, from which the other members of the Jth subset can be generated by permutation of the gluons.

The coefficient of a particular trace, say $\mathrm{Tr}\,(t^{a_1}t^{a_3}\cdots t^{a_j})$, is

$$\tilde{M}(k_1,\lambda_1;k_3,\lambda_3;\dots;k_j,\lambda_j),$$

where the kinematic amplitude $\tilde{M}$ is a sum of contributions labelled by the characteristic diagram D_j, with the kinematic variables in the same order as the gluon labels inside the trace,

$$\tilde{M}(1,3,\dots,j) = \sum_{D_j} \tilde{M}_{D_j}(1,3,\dots,j) \tag{10.12.74}$$

and for each characteristic diagram D_j, $\tilde{M}_{D_j}$ is a sum of the amplitude $\tilde{m}_{D_j}$ arising from diagram D_j plus those cyclic permutations of it that correspond to topologically independent diagrams:

$$\tilde{M}_{D_j}(1,3,\dots,j) = \sum_{\substack{\text{cyclic}\\ \text{perms}}}{}' \tilde{m}_{D_j}(1,3,\dots,j). \tag{10.12.75}$$

In the review of Mangano and Parke (1991), it is shown that the kinematic amplitudes $\tilde{M}$ possess remarkable general properties, being

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essentially so-called *dual amplitudes* and related to string amplitudes. Two important properties, which were shown, in subsection 10.12.4, to be true for the four-gluon amplitude, are the *dual Ward identity*

$$\tilde{M}(1,2,3,\dots,n) + \tilde{M}(2,1,3,\dots,n) + \cdots + \tilde{M}(2,3,4,\dots,1,n) = 0 \tag{10.12.76}$$

and the symmetry under reversal of the order of the labels,

$$\tilde{M}(n,n-1,\dots,2,1) = (-1)^n \tilde{M}(1,2,\dots,n-1,n).$$

Moreover, it is shown how supersymmetry can be used to relate amplitudes for pure gluonic reactions to those where a pair of gluons is replaced by a quark–antiquark pair. For these general developments and many results for specific amplitudes the reader is referred to the above review, but care must be taken since the same symbols have been given differing normalizations in the review and in the earlier papers of the same authors.