Canad. J. Math. Vol. 72 (2), 2020 pp. 373–391
http://dx.doi.org/10.4153/S0008414X18000664

Marginals with Finite Repulsive Cost

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Abstract. We consider a multimarginal transport problem with repulsive cost, where the marginals are all equal to a fixed probability $\rho \in \mathcal{P}(\mathbb{R}^d)$. We prove that, if the concentration of ρ is less than $1/N$, then the problem has a solution of finite cost. The result is sharp, in the sense that there exists ρ with concentration $1/N$ for which the cost is infinite.

1 Introduction

Consider a system of N unitary-charged particles of negligible mass under the effect of the Coulomb force. We can describe the stationary states using a wave-function $\psi(x_1, \dots, x_N)$, where $x_j \in \mathbb{R}^3$; via the Born interpretation, $|\psi(x_1, \dots, x_N)|^2$ can be viewed as the density of the probability that the particles occupy the positions $x_1, \dots, x_N$, and it is symmetric, since the particles are indistinguishable.

When the semi-classical limit is considered, as already proved in [2, 7, 8, 16], the stationary states reach the minimum of potential energy, *i.e.*,

$$V_0 = \min_{\psi} V(\psi) = \min \int_{\mathbb{R}^{3N}} c(x_1, \dots, x_N) \, |\psi(x_1, \dots, x_N)|^2 \, dx_1 \cdots dx_N, \tag{1.1}$$

where c is the Coulomb (potential) cost function $c\colon (\mathbb{R}^3)^N \to \mathbb{R}$ defined as

$$c(x_1, \dots, x_N) = \sum_{1 \le i < j \le N} \frac{1}{|x_i - x_j|}.$$

This can also be viewed as the exchange correlation functional linking the Kohn–Sham to the Hohenberg–Kohn approach; see, for instance, [13].

Given any wave-function ψ, define its single-particle density as

$$\rho^{\psi}(x) = \int_{\mathbb{R}^{3(N-1)}} |\psi(x, x_2, \dots, x_N)|^2 \, dx_2 \cdots dx_N,$$

which is quite natural from the physical point of view, since the charge density is a fundamental quantum-mechanical observable.

It is a well-known result by Lieb [17] (see also Levy [15]) that the set of all possible marginal densities is

$$\mathcal{R} = \Big\{ \rho \in L^1(\mathbb{R}^d) \mid \rho \ge 0, \sqrt{\rho} \in H^1(\mathbb{R}^d), \int_{\mathbb{R}^d} \rho(x) dx = 1 \Big\}.$$

One can thus consider

$$C(\rho) = \min \Big\{ \int_{\mathbb{R}^{3N}} c(x_1, \dots, x_N) \, |\psi(x_1, \dots, x_N)|^2 \, dx_1 \cdots dx_N \,\Big|\, \rho^{\psi} = \rho \Big\},$$

Received by the editors October 2, 2018; revised October 15, 2018.
Published online on Cambridge Core May 7, 2019.
AMS subject classification: **49J10,** 49K10.
Keywords: multi-marginal, optimal transport, repulsive cost.

and factorize the original minimum problem (1.1) as

$$V_0 = \min_{\rho\in\mathcal{R}} \min_{\rho^\psi=\rho} V(\psi) = \min_{\rho} C(\rho).$$

This is a well-known approach, which dates back to Thomas and Fermi, and was later revised by Hohenberg and Kohn [14], Levy [15], and Lieb [17], whose questions are still sources of ideas for this field.

In this paper, we first generalize the physical dimension $d = 3$ to any $d \geq 1$. Moreover, we adopt a measure-theoretic approach: instead of considering wave-functions, we set the problem for every probability over $(\mathbb{R}^d)^N$ and formulate the corresponding relaxed minimum problem

$$\mathcal{C}(P) = \min \int_{(\mathbb{R}^d)^N} c(x_1,\dots,x_N)\,\mathrm{d}P(x_1,\dots,x_N),$$

where $P \in \mathcal{P}((\mathbb{R}^d)^N)$ is a probability measure. In this fashion, the single-particle density constraint gives rise to a multi-marginal optimal transport problem of the form

$$C(\rho) = \inf\Big\{ \int_{(\mathbb{R}^d)^N} c(x_1,\dots,x_N)\,\mathrm{d}P(x_1,\dots,x_N) \Big| \; P \in \mathcal{P}((\mathbb{R}^d)^N), \pi^i_\# P = \rho, i = 1,\dots,N \Big\}, \tag{1.2}$$

where ρ is a fixed probability measure over $\mathbb{R}^d$ and π^i is the projection over the i-th factor of $(\mathbb{R}^d)^N$. It is a simple and well-known observation that the infimum (1.2) is equal to

$$C(\rho) = \inf\Big\{ \int_{(\mathbb{R}^d)^N} c(x_1,\dots,x_N)\,\mathrm{d}P(x_1,\dots,x_N) \Big| \; P \in \mathcal{P}((\mathbb{R}^d)^N), P \text{ symmetric }, \pi^i_\# P = \rho, i = 1,\dots,N \Big\}. \tag{1.3}$$

In order to give an even stronger result, we take as a cost function a general repulsive potential, as in the following definition.

Definition 1.1 A function $c\colon (\mathbb{R}^d)^N \to \mathbb{R}$ is a *repulsive cost function* if it is of the form

$$c(x_1,\dots,x_N) = \sum_{1\leq i<j\leq N} \frac{1}{\omega(|x_i - x_j|)},$$

where $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, strictly increasing, and differentiable on $(0,+\infty)$, with $\omega(0) = 0$.

Although there are many works about this formulation, and the multi-marginal transport problem in general (see for instance [3, 5, 6, 9, 10]), none of them gives a condition on ρ that assures that the infimum in (1.3) is finite. We found that the correct quantity to consider is the one given by the following definition.

Definition 1.2 If $\rho \in \mathcal{P}(\mathbb{R}^d)$, the *concentration* of ρ is

$$\mu(\rho) = \sup_{x\in\mathbb{R}^d} \rho(\{x\}).$$

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This allows us to state the main result.

Theorem 1.3 *Let c be a repulsive cost function, and let $\rho \in \mathcal{P}(\mathbb{R}^d)$ with*

$$\mu(\rho) < \frac{1}{N}. \tag{1.4}$$

Then the infimum in (1.3) is finite.

Remark 1.4 After this paper was submitted, the author became aware of an independent work in preparation by F. Stra, S. Di Marino, and M. Colombo about the same problem. The techniques are different and the second result, although not yet available in preprint form, seems to be closer in the approach to some arguments in [3].

Structure of the paper In Section 2 we give some notation and collect some definitions, constructions, and results to be used later. In particular, we state and prove a simple but useful result about partitioning $\mathbb{R}^d$ into measurable sets with prescribed mass.

We then show in Section 3 that condition (1.4) is sharp; *i.e.*, given any repulsive cost function, there exists $\rho \in \mathcal{P}(\mathbb{R}^d)$ with $\mu(\rho) = 1/N$, and $C(\rho) = \infty$. The construction of this counterexample is explicit, but it is important to note that the marginal ρ depends on the given cost function.

Finally we devote Sections 4 to 6 to the proof of Theorem 1.3. The construction is universal, in the following sense: given $\rho \in \mathcal{P}(\mathbb{R}^d)$ such that (1.4) holds, we exhibit a symmetric transport plan P that has support outside the region

$$D_\alpha = \left\{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N \mid \exists i \neq j \text{ with } \left|x_i - x_j\right| < \alpha\right\}$$

for some $\alpha > 0$. This implies that $C(P)$ is finite for any repulsive cost function.

2 Notation and Preliminary Results

In the following, x and x_j denote elements of $\mathbb{R}^d$, and $X = (x_1, \ldots, x_N)$ is an element of $(\mathbb{R}^d)^N = \mathbb{R}^{Nd}$. We also denote by $B(x_j, r)$ a ball with center $x_j \in \mathbb{R}^d$ and radius $r > 0$. Where it is not specified, the integrals are extended to all the space; if τ is a measure over $\mathbb{R}^d$, we denote by $|\tau|$ its total mass, *i.e.*,

$$|\tau| = \int_{\mathbb{R}^d} \mathrm{d}\tau.$$

We use the expression *N-transport plan for the marginal ρ* to denote a probability measure $P \in \mathcal{P}(\mathbb{R}^{Nd})$ with all the marginals equal to $\rho \in \mathcal{P}(\mathbb{R}^d)$.

If $P \in \mathcal{M}(\mathbb{R}^{Nd})$ is any measure, we define

$$P_{\mathrm{sym}} = \frac{1}{N!} \sum_{s \in S_N} \phi^s_{\#} P,$$

where S_N is the premutation group over the elements $\{1, \ldots, N\}$, and $\phi^s : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ is the function $\phi^s(x_1, \ldots, x_N) = (x_{s(1)}, \ldots, x_{s(N)})$. Note that P_{sym} is a symmetric measure; moreover, if P is a probability measure, then P_{sym} is also a probability measure.

Lemma 2.1 *Let $P \in \mathcal{M}(\mathbb{R}^{Nd})$. Then P_{sym} has marginals equal to*

$$\frac{1}{N}\sum_{j=1}^{N} \pi^j_\# P.$$

Proof Since P_{sym} is symmetric, me can calculate its first marginal:

$$\pi^1_\# P_{\mathrm{sym}} = \pi^1_\# \Big(\frac{1}{N!} \sum_{s \in S_N} \phi^s_\# P \Big) = \frac{1}{N!} \sum_{s \in S_N} \pi^1_\# (\phi^s_\# P)$$
$$= \frac{1}{N!} \sum_{s \in S_N} \pi^{s(1)}_\# P = \frac{1}{N} \sum_{j=1}^{N} \pi^j_\# P,$$

where the last equality is due to the fact that for every $j = 1, \dots, N$, there are exactly $(N-1)!$ permutations $s \in S_N$ such that $s(1) = j$. ∎

For a symmetric probability $P \in \mathcal{P}(\mathbb{R}^{Nd})$ we will use the shortened notation $\pi(P)$ to denote its marginals $\pi^j_\# P$, which are all equal.

If $\sigma_1, \dots, \sigma_N \in \mathcal{M}(\mathbb{R}^d)$, we define $\sigma_1 \otimes \cdots \otimes \sigma_N \in \mathcal{M}(\mathbb{R}^{Nd})$ as the usual product measure. In similar fashion, if $Q \in \mathcal{M}(\mathbb{R}^{(N-1)d})$, $\sigma \in \mathcal{M}(\mathbb{R}^d)$ and $1 \le j \le N$, we define the measure $Q \otimes_j \sigma \in \mathcal{M}(\mathbb{R}^{Nd})$ as

$$\int_{\mathbb{R}^{Nd}} f \, \mathrm{d}(Q \otimes_j \sigma) = \int_{\mathbb{R}^{Nd}} f(x_1, \dots, x_N) \, \mathrm{d}\sigma(x_j) \, \mathrm{d}Q(x_1, \dots, \widehat{x_j}, \dots, x_N) \tag{2.1}$$

for every $f \in C_b(\mathbb{R}^{Nd})$.

2.1 Partitions of Non-atomic Measures

Let $\sigma \in \mathcal{M}(\mathbb{R}^d)$ be a finite non-atomic measure and let $b_1, \dots, b_k$ be real positive numbers such that $b_1 + \cdots + b_k = |\sigma|$. We may want to write

$$\mathbb{R}^d = \bigcup_{j=1}^{k} E_j,$$

where the E_j's are disjoint measurable sets with $\sigma(E_j) = b_j$. This is trivial if $d = 1$, since the cumulative distribution function $\phi_\sigma(t) = \sigma((-\infty, t))$ is continuous, and one can find the E_j's as intervals. However, in higher dimension, the measure σ might concentrate over $(d-1)$-dimensional surfaces, which makes the problem slightly more difficult. Therefore, we present the following proposition.

Proposition 2.2 *Let $\sigma \in \mathcal{M}(\mathbb{R}^d)$ be a finite non-atomic measure. Then there exists a direction $y \in \mathbb{R}^d \setminus \{0\}$ such that $\sigma(H) = 0$ for all the affine hyperplanes H such that $H \perp y$.*

In order to prove Proposition 2.2, it is useful to present the following lemma.

Lemma 2.3 *Let (X, μ) be a measure space with $\mu(X) < \infty$, and $\{E_i\}_{i \in I}$ a collection of measurable sets, such that*

(i) $\mu(E_i) > 0$ *for every* $i \in I$;

(ii) $\mu(E_i \cap E_j) = 0$ *for every* $i \neq j$.

https://doi.org/10.4153/S0008414X18000664 Published online by Cambridge University Press

Then I is countable.

Proof Let $i_1, \dots, i_n$ be a finite set of indices. Then using the monotonicity of μ and the fact that $\mu(E_i \cap E_j) = 0$ if $i \neq j$,

$$\mu(X) \geq \mu\Big(\bigcup_{k=1}^{n} E_{i_k}\Big) = \sum_{k=1}^{n} \mu(E_{i_k}).$$

Hence we have that

$$\sup\Big\{\sum_{j\in J} \mu(E_j)\,\Big|\, J \subset I, J \text{ finite}\Big\} \leq \mu(X) < \infty.$$

Since all the $\mu(E_i)$ are strictly positive numbers, this is possible only if I is countable. ∎

Now we present the proof of Proposition 2.2.

Proof For $k = 0, 1, \dots, d-1$, we recall the definitions of the Grassmannian

$$\mathrm{Gr}(k, \mathbb{R}^d) = \left\{v \text{ linear subspace of } \mathbb{R}^d \mid \dim v = k\right\}$$

and the affine Grassmannian

$$\mathrm{Graff}(k, \mathbb{R}^d) = \left\{w \text{ affine subspace of } \mathbb{R}^d \mid \dim w = k\right\}.$$

Given $w \in \mathrm{Graff}(k, \mathbb{R}^d)$, we denote by $[w]$ the unique element of $\mathrm{Gr}(k, \mathbb{R}^d)$ parallel to w. If $S \subseteq \mathrm{Graff}(k, \mathbb{R}^d)$, we say that S is *full* if for every $v \in \mathrm{Gr}(k, \mathbb{R}^d)$, there exists $w \in S$ such that $[w] = v$. For every $k = 1, 2, \dots, d-1$, let $S^k \subseteq \mathrm{Graff}(k, \mathbb{R}^d)$ be the set

$$S^k = \left\{w \in \mathrm{Graff}(k, \mathbb{R}^d) \mid \sigma(w) > 0\right\}.$$

The goal is to prove that S^{d-1} is not full, while by hypothesis we know that $S^0 = \varnothing$, since σ is non-atomic.

The following key lemma leads to the proof in a finite number of steps.

Lemma 2.4 *Let $1 \leq k \leq d-1$. If S^{k-1} is not full, then S^k is not full.*

Proof Let $v \in \mathrm{Gr}(k-1, \mathbb{R}^d)$, such that for every $v' \in \mathrm{Graff}(k-1, \mathbb{R}^d)$ with $[v'] = v$, it holds $\sigma(v') = 0$. Consider the collection $W_v = \left\{w \in \mathrm{Graff}(k, \mathbb{R}^d) \mid v \subseteq [w]\right\}$. If $w, w' \in W_v$ are distinct, then $w \cap w' \subseteq v'$ for some $v' \in \mathrm{Graff}(k-1, \mathbb{R}^d)$ with $[v'] = v$, thus $\sigma(w \cap w') = 0$. Since the measure σ is finite, because of Lemma 2.3, at most countably many elements $w \in W_v$ can have positive measure, which implies that S^k is not full. ∎

This concludes the proof of Proposition 2.2. ∎

Corollary 2.5 *Given $b_1, \dots, b_k$ real positive numbers with $b_1 + \dots + b_k = |\sigma|$, there exist measurable sets $E_1, \dots, E_k \subseteq \mathbb{R}^d$ such that the following hold:*

(i) *The E_j's form a partition of $\mathbb{R}^d$*, i.e.,

$$\mathbb{R}^d = \bigcup_{j=1}^{k} E_j, \qquad E_i \cap E_j = \varnothing \text{ if } i \neq j;$$

(ii) $\sigma(E_j) = b_j$ *for every* $j = 1, \ldots, k$.

Proof Let $y \in \mathbb{R}^d \setminus \{0\}$ be given by Proposition 2.2, and observe that the cumulative distribution function

$$F(t) = \sigma\left(\left\{x \in \mathbb{R}^d \mid x \cdot y < t\right\}\right)$$

is continuous. Hence we find $E_1, \ldots, E_k$ each of the form

$$E_j = \left\{x \in \mathbb{R}^d \mid t_j < x \cdot y \le t_{j+1}\right\}$$

for suitable $-\infty = t_1 < t_2 < \cdots < t_k < t_{k+1} = +\infty$, such that $\sigma(E_j) = b_j$. ■

Corollary 2.6 *Given* $b_1, \ldots, b_k$ *non-negative numbers with* $b_1 + \cdots + b_k < |\sigma|$, *there exists measurable sets* $E_0, E_1, \ldots, E_k \subseteq \mathbb{R}^d$ *such that the following hold:*

(i) *The* E_j*'s form a partition of* $\mathbb{R}^d$, i.e.,

$$\mathbb{R}^d = \bigcup_{j=0}^{k} E_j, \qquad E_i \cap E_j = \varnothing \text{ if } i \ne j;$$

(ii) $\sigma(E_j) = b_j$ *for every* $j = 1, \ldots, k$;
(iii) *the distance between* E_i *and* E_j *is strictly positive if* $i, j \ge 1$, $i \ne j$.

Proof If $k = 1$, the results follows trivially by Corollary 2.5 applied to $b_1, |\sigma| - b_1$. If $k \ge 2$, define

$$\epsilon = \frac{|\sigma| - b_1 - \cdots - b_k}{k - 1} > 0.$$

As before, letting $y \in \mathbb{R}^d \setminus \{0\}$ be given by Proposition 2.2 and considering the corresponding cumulative distribution function, we find $F_1, \ldots, F_{2k-1}$ each of the form

$$F_j = \left\{x \in \mathbb{R}^d \mid t_j < x \cdot y \le t_{j+1}\right\}$$

for suitable $-\infty = t_1 < t_2 < \cdots < t_{2k-1} < t_{2k} = +\infty$, such that

$$\sigma(F_{2j-1}) = b_j \qquad \forall j = 1, \ldots, k,$$
$$\sigma(F_{2j}) = \epsilon \qquad \forall j = 1, \ldots, k - 1.$$

Finally, we define

$$E_j = F_{2j-1} \qquad \forall j = 1, \ldots, k,$$
$$E_0 = \bigcup_{j=1}^{k-1} F_{2j}.$$

Properties (i) and (ii) are immediate to check, while the distance between E_i and E_j, for $i, j \ge 1$, $i \ne j$, is uniformly bounded from below by

$$\min\left\{t_{2j+1} - t_{2j} \mid 1 \le j \le k - 1\right\} > 0.$$

■

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3 Condition (1.4) is Sharp

In this section we prove that condition (1.4) is the best possible; *i.e.*, given any repulsive cost function there exists $\rho \in \mathcal{P}(\mathbb{R}^d)$ with $\mu(\rho) = 1/N$ such that $C(\rho) = \infty$.

Fix ω as in Definition 1.1, and set

$$k = \int_{B(0,1)} \frac{\omega'(|y|)}{|y|^{d-1}} \, \mathrm{d}y.$$

Note that k is a positive finite constant, depending only on ω and the dimension d. In fact, integrating in spherical coordinates,

$$k = \int_0^1 \frac{\omega'(r)}{r^{d-1}} \alpha_d r^{d-1} \, \mathrm{d}r = \alpha_d \omega(1),$$

where α_d is the d-dimensional volume of the unit ball $B(0,1) \subseteq \mathbb{R}^d$.

Now define a probability measure $\rho \in \mathcal{P}(\mathbb{R}^d)$ as

$$\int_{\mathbb{R}^d} f \, \mathrm{d}\rho := \frac{1}{N} f(0) + \frac{N-1}{N} \int_{B(0,1)} f(x) \frac{\omega'(|x|)}{k\,|x|^{d-1}} \, \mathrm{d}x \qquad \forall f \in C_b(\mathbb{R}^d). \tag{3.1}$$

This measure has an atom of mass $1/N$ in the origin, and is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$. Hence the concentration of ρ is equal to $1/N$, even if for every ball B around the origin one has $\rho(B) > 1/N$.

We want to prove that any symmetric transport plan with marginals ρ has infinite cost. Let us consider, by contradiction, a symmetric plan P, with $\pi(P) = \rho$, such that

$$\int \sum_{1 \le i < j \le N} \frac{1}{\omega(|x_i - x_j|)} \, \mathrm{d}P(X) < \infty.$$

Then one would have the following geometric properties.

Lemma 3.1 (i) $P(\{(x_1, \dots, x_N) \mid \exists i \neq j, x_i = x_j\}) = 0$;
(ii) *P is concentrated over the N coordinate hyperplanes* $\{x_j = 0\}$, $j = 1, \dots, N$, i.e.,

$$\operatorname{supp}(P) \subseteq E := \bigcup_{j=1}^{N} \{x_j = 0\}.$$

Proof (i) Since $\omega(0) = 0$, recalling Definition 1.1, the cost function is identically equal to $+\infty$ in the region $\{(x_1, \dots, x_N) : \exists i \neq j, x_i = x_j\}$. Therefore, since by assumption the cost of P is finite, it must be

$$P\big(\{(x_1, \dots, x_N) \mid \exists i \neq j, x_i = x_j\}\big) = 0.$$

(ii) Define

$$\begin{aligned} p_1 &= P\big(\{x_1 = 0\}\big) \\ p_2 &= P\big(\{x_1 = 0\} \cap \{x_2 = 0\}\big) \\ &\vdots \\ p_N &= P\big((0, \dots, 0)\big). \end{aligned}$$

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Note that $p_1 = P(\{x_1 = 0\}) = \pi(P)(\{0\}) = \rho(\{0\}) = 1/N$. We claim that $p_2 = \cdots = p_N = 0$. It suffices to prove that $p_2 = 0$, since by monotonicity of the measure P, we have $p_j \geq p_{j+1}$. Since P has finite cost,

$$\int_{\mathbb{R}^{Nd}} \frac{\mathrm{d}P}{\omega(|x_1 - x_2|)}$$

must be finite. However,

$$\begin{aligned}\int_{\mathbb{R}^{Nd}} \frac{\mathrm{d}P}{\omega(|x_1 - x_2|)} &\geq \int_{\{x_1=0\}\cap\{x_2=0\}} \frac{\mathrm{d}P}{\omega(|x_1 - x_2|)} \\ &= p_2 \int_{\mathbb{R}^{2d}} \frac{\delta_0(x_1)\delta_0(x_2)}{\omega(|x_1 - x_2|)}\, \mathrm{d}x_1\, \mathrm{d}x_2,\end{aligned}$$

and hence p_2 must be zero.

By inclusion-exclusion, we have

$$P(E) = \sum_{j=1}^{N} (-1)^{j+1} \binom{N}{j} p_j = N p_1 = 1,$$

and hence P is concentrated over E.

■

In view of Lemma 3.1, letting $H_j = \{x_j = 0\}$ for $j = 1, \ldots, N$,

$$P = \sum_{j=1}^{N} P|_{H_j}.$$

For every $j = 1, \ldots, N$, there exists a unique measure Q_j over $\mathbb{R}^{(N-1)d}$ such that, recalling equation (2.1), $P|_{H_j} = Q_j \otimes_j \delta_0$, with $Q_j(\mathbb{R}^{(N-1)d}) = \frac{1}{N}$. Since P is symmetric, considering a permutation $s \in S_N$ with $s(j) = j$, it follows that Q_j is symmetric; then, considering any permutation in S_N, we see that there exists a symmetric probability Q over $\mathbb{R}^{(N-1)d}$ such that $Q_j = \frac{1}{N} Q$ for every $j = 1, \ldots, N$, *i.e.*,

$$P = \frac{1}{N} \sum_{j=1}^{N} Q \otimes_j \delta_0.$$

Projecting P to its one-particle marginal and using the definition of ρ in (3.1), we get that $\pi(Q)$ is absolutely continuous with respect to the Lebesgue measure, with

$$\frac{\mathrm{d}\pi(Q)}{\mathrm{d}\mathcal{L}^d} = \frac{\chi_{B(0,1)}(x)\omega'(x)}{k\,|x|^{d-1}}.$$

Here we get the contradiction, because

$$\begin{aligned}\int c(X) dP(X) &\geq \frac{1}{N} \int \frac{1}{\omega(|x_1 - x_2|)} \delta_0(x_1)\, \mathrm{d}x_1\, \mathrm{d}Q(x_2, \ldots, x_N) \\ &= \frac{1}{N} \int \frac{1}{\omega(|x_2|)}\, \mathrm{d}Q(x_2, \ldots, x_N) = \frac{1}{N} \int_{\mathbb{R}^d} \frac{1}{\omega(|x|)}\, \mathrm{d}\pi(Q)(x) \\ &= \frac{1}{N} \int_{B(0,1)} \frac{\omega'(|x|)}{\omega(|x|)} \frac{1}{k\,|x|^{d-1}}\, \mathrm{d}x = \frac{1}{N} \frac{\alpha_d}{k} \int_0^1 \frac{\omega'(r)}{\omega(r)}\, \mathrm{d}r = +\infty.\end{aligned}$$

4 Non-atomic Marginals

This short section deals with the case where ρ is non atomic, *i.e.*, $\mu(\rho) = 0$. In this case, the transport plan is given by an optimal transport map in Monge's fashion, which we proceed to construct.

Using Corollary 2.5, let $E_1, \dots, E_{2N}$ be a partition of $\mathbb{R}^d$ such that

$$\rho(E_j) = \frac{1}{2N} \qquad \forall j = 1, \dots, 2N.$$

Next we take a measurable function $\phi: \mathbb{R}^d \to \mathbb{R}^d$, preserving the measure ρ and defined locally such that

$$\begin{aligned} \phi(E_j) &= E_{j+2} \qquad \forall j = 1, \dots, N-2, \\ \phi(E_{2N-1}) &= E_1, \\ \phi(E_{2N}) &= E_2. \end{aligned}$$

The behaviour of ϕ on the hyperplanes that separate the E_j's is arbitrary, since they form a ρ-null set. Note that $|x - \phi(x)|$ is uniformly bounded from below by some constant $\gamma > 0$, as is clear by the construction of the E_j's (see the proof of Corollary 2.5). A transport plan P of finite cost is now defined for every $f \in C_b(\mathbb{R}^{Nd})$ by

$$\int_{\mathbb{R}^{Nd}} f \, dP = \int_{\mathbb{R}^{Nd}} f\big(x, \phi(x), \dots, \phi^{N-1}(x)\big) \, d\rho(x),$$

since

$$\int_{\mathbb{R}^{Nd}} c \, dP = \binom{N}{2} \int_{\mathbb{R}^d} \frac{1}{\omega(|x - \phi(x)|)} \, d\rho(x) \le \binom{N}{2} \frac{1}{\omega(\gamma)}.$$

5 Marginals with a Finite Number of Atoms

This section constitutes the core of the proof, as we deal with measures of general form with an arbitrary (but finite) number of atoms. Throughout this and the next section, we assume that the marginal ρ fulfills condition (1.4).

5.1 The Number of Atoms is Less than or Equal to N

Note that, if the number of atoms is at most N, then ρ must have a non-atomic part σ, due to the condition (1.4). From here on we consider

$$\rho = \sigma + \sum_{i=1}^{k} b_i \delta_{x_i},$$

where $b_1 \ge b_2 \ge \cdots \ge b_k > 0$.

We begin with the following definition.

Definition 5.1 A *partition* of σ of level $k \le N$ subordinate to $(x_1, \dots, x_k; b_1, \dots, b_k)$ is

$$\sigma = \tau + \sum_{i=1}^{k} \sum_{h=i+1}^{N} \sigma_h^i,$$

where the follwing hold:

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(i) τ, σ_h^i are non-atomic measures;
(ii) for every i and every $h \neq k$, the distance between supp σ_h^i and supp σ_k^i is strictly positive;
(iii) for every i, h, if $j \leq i$, then x_j has a strictly positive distance from supp σ_h^i;
(iv) for every i, h, $|\sigma_h^i| = b_i$, and $|\tau| > 0$.

Note that such a partition can only exist if

$$|\sigma| > \sum_{i=1}^{k} (N-i) b_i. \tag{5.1}$$

On the other hand, the following lemma proves that condition (5.1) is also sufficient to get a partition of σ.

Lemma 5.2 *Let $(b_1, \dots, b_k)$ with $k \leq N$, and*

$$|\sigma| > \sum_{i=1}^{k} (N-i) b_i.$$

Then there exists a partition of σ subordinate to $(x_1, \dots, x_k; b_1, \dots, b_k)$.

Proof Fix $(x_1, \dots, x_k)$ and for every $\varepsilon > 0$, define

$$A_\varepsilon = \bigcup_{j=1}^{k} B(x_j, \varepsilon) \quad \text{and} \quad \sigma_\varepsilon = \sigma \chi_{A_\varepsilon}.$$

Then take ε small enough such that

$$|\sigma - \sigma_\varepsilon| > \sum_{i=1}^{k} (N-i) b_i,$$

which is possibile because $\mu(\sigma) = 0$ (σ has concentration zero), and hence $|\sigma_\varepsilon| \to 0$ as $\varepsilon \to 0$. Due to Corollary 2.6, the set $\mathbb{R}^d \smallsetminus A_\varepsilon$ can be partitioned as

$$\mathbb{R}^d \smallsetminus A_\varepsilon = \Big(\bigcup_{i=1}^{k} \bigcup_{h=i+1}^{N} E_h^i \Big) \cup E,$$

with $\sigma(E_h^i) = b_i$, and $\operatorname{dist}(E_h^i, E_k^i)$ is uniformly bounded from below.

Finally, define $\sigma_h^i = \sigma \chi_{E_h^i}$, $\tau = \sigma_\varepsilon + \sigma \chi_E$. ∎

Proposition 5.3 *Suppose that $k \leq N$ and $(b_1, \dots, b_k)$ are such that*

$$|\sigma| > N b_1 - \sum_{j=1}^{k} b_j. \tag{5.2}$$

Then there exists a transport plan of finite cost with marginals

$$\sigma + \sum_{j=1}^{k} b_j \delta_{x_j}.$$

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Proof In order to simplify the notation, set $b_{k+1} = 0$. First of all we shall fix a partition of σ subordinate to $(x_1, \dots, x_k; b_1 - b_2, \dots, b_{k-1} - b_k, b_k)$. To do this we apply Lemma 5.1, since

$$\sum_{i=1}^{k-1}(N-i)(b_i - b_{i+1}) + (N-k)b_k = (N-1)b_1 - \sum_{i=2}^{k} b_i < |\sigma|\,.$$

Next we define the measures $\lambda_i = \delta_{x_1} \otimes \cdots \otimes \delta_{x_i} \otimes \sigma^i_{i+1} \otimes \cdots \otimes \sigma^i_N \in \mathcal{M}(\mathbb{R}^{Nd})$. Let us calculate the marginals of λ_i. Since $\left|\sigma^i_h\right| = b_i - b_{i+1}$ for all $h = i+1, \dots, N$, we get

$$\pi^j_\# \lambda_i = \begin{cases} (b_i - b_{i+1})^{N-i}\delta_{x_j} & \text{if } 0 \le j \le i, \\ (b_i - b_{i+1})^{N-i-1}\sigma^i_j & \text{if } i+1 \le j \le N. \end{cases}$$

Let us define, for $i = 1, \dots, k$, the measure

$$P_i = \frac{N}{(b_i - b_{i+1})^{N-i-1}}(\lambda_i)_{\mathrm{sym}},$$

where $P_i = 0$ if $b_i = b_{i+1}$. By Lemma 2.1, the marginals of P_i are equal to

$$\pi(P_i) = \frac{1}{(b_i - b_{i+1})^{N-i-1}} \sum_{j=0}^{N} \pi^j_\# \lambda_i = \sum_{j=1}^{i}(b_i - b_{i+1})\delta_{x_j} + \sum_{h=i+1}^{N} \sigma^i_h,$$

so that

$$\sum_{i=1}^{k} \pi(P_i) = \sum_{j=1}^{k} b_j \delta_{x_j} + \sum_{i=1}^{k} \sum_{h=i+1}^{N} \sigma^i_h.$$

It suffices now to take any symmetric transport plan P_τ of finite cost with marginals τ, given by the result of Section 4, and finally set

$$P = P_\tau + \sum_{i=1}^{k} P_i.$$

∎

As a corollary we obtain the following theorem.

Theorem 5.4 *If ρ has $k \le N$ atoms, then there exists a transport plan of finite cost.*

Proof Let

$$\rho = \sigma + \sum_{j=1}^{k} b_j \delta_{x_j}.$$

Note that, since $b_1 < 1/N$,

$$|\sigma| = 1 - \sum_{j=1}^{k} b_j > N b_1 - \sum_{j=1}^{k} b_j,$$

hence we can apply Proposition 5.3 to conclude the proof. ∎

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5.2 The Number of Atoms is Greater than N

Here we deal with the much more difficult situation in which ρ has $N+1$ or more atoms, *i.e.*,

$$\rho = \sigma + \sum_{j=1}^{k} b_j \delta_{x_j}$$

with $k \geq N+1$ and as before $b_1 \geq b_2 \geq \cdots \geq b_k > 0$. Note that in this case it might happen that $\sigma = 0$.

The main point is to use a double induction on the dimension N and the number of atoms k, as will be clear in Proposition 5.6. The following lemma is a simple numerical trick needed for the inductive step in Proposition 5.6.

Lemma 5.5 *Let $(b_1, \dots, b_k)$ with $k \geq N+2$ and*

$$(N-1)b_1 \leq \sum_{j=2}^{k} b_j. \tag{5.3}$$

Then there exist $t_2, \dots, t_k$ such that

(i) $t_2 + \cdots + t_k = (N-1)b_1$;
(ii) *for every $j = 2, \dots, k$, $0 \leq t_j \leq b_j$, and moreover,*

$$t_2 \geq \cdots \geq t_k,$$
$$b_2 - t_2 \geq b_3 - t_3 \geq \cdots \geq b_k - t_k;$$

(iii) $(N-2)t_2 \leq \sum_{j=3}^{k} t_j$;
(iv) $(N-1)(b_2 - t_2) \leq \sum_{j=3}^{k} (b_j - t_j)$.

Proof For $j = 2, \dots, k$ define

$$p_j = \sum_{h=j}^{k} b_j,$$

and let $\bar{j}$ be the least $j \geq 2$ such that $(N - j + 2)b_j \leq p_j$; note that $j = N+2$ works, hence $\bar{j} \leq N+2$. Define

$$t_j = b_j - \frac{p_2 - (N-1)b_1}{N} \qquad \text{for } j = 2, \dots, \bar{j}-1,$$
$$t_j = b_j - \frac{b_j}{p_{\bar{j}}} \frac{p_2 - (N-1)b_1}{N}(N - \bar{j} + 2) \qquad \text{for } j = \bar{j}, \dots, k.$$

Next we prove that this choice fulfills conditions (i)–(iv).

Proof of (i)

$$\begin{aligned}\sum_{j=2}^{k} t_j &= p_2 - \frac{p_2 - (N-1)b_1}{N}(\bar{j}-2) - \frac{p_2 - (N-1)b_1}{N}(N - \bar{j} + 2) \\ &= p_2\Big(1 - \frac{\bar{j}-2}{N} - \frac{N-\bar{j}+2}{N}\Big) + (N-1)b_1\Big(\frac{\bar{j}-2}{N} + \frac{N-\bar{j}+2}{N}\Big) \\ &= (N-1)b_1.\end{aligned}$$

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Proof of (ii) In view of the fact that $(N-1)b_1 \le p_2$ and $\bar{j} \le N+2$, it is clear that $t_j \le b_j$. If $j < \bar{j}$, we have $(N-j+2)b_j > p_j$, and hence

$$p_2 = b_2 + \cdots + b_{j-1} + p_j < (j-2)b_1 + (N-j+2)b_j.$$

Thus, since $2 \le j \le N+1$,

$$t_j = \frac{Nb_j - p_2 + (N-1)b_1}{N} > \frac{Nb_j - (N-j+2)b_j - (j-2)b_1 + (N-1)b_1}{N}$$
$$= \frac{(j-2)b_j + (N-j+1)b_1}{N} \ge 0.$$

To show that $t_j \ge 0$ for $j \ge \bar{j}$, we must prove that $[p_2 - (N-1)b_1](N-\bar{j}+2) \le Np_{\bar{j}}$, which is trivial if $\bar{j} = N-2$. Otherwise, it is equivalent to

$$-(\bar{j}-2)[p_2 - (N-1)b_1] + N[b_2 + \cdots + b_{\bar{j}-1} - (N-1)b_1] \le 0.$$

Since $2 \le \bar{j} \le N+1$, the first term is negative and $b_2 + \cdots + b_{\bar{j}-1} - (N-1)b_1 \le -(N-\bar{j}+1)b_1 \le 0$.

Using the fact that $b_2 \ge \cdots \ge b_k$, it is easy to see that $t_2 \ge \cdots \ge t_{\bar{j}-1}$ and $t_{\bar{j}} \ge \cdots \ge t_k$; note that for $j \ge \bar{j}$ we have $t_j = \alpha b_j$, for some $0 \le \alpha \le 1$. As for the remaining inequality,

$$t_{\bar{j}-1} \ge t_{\bar{j}} \iff b_{\bar{j}-1} - b_{\bar{j}} \ge \frac{p_2 - (N-1)b_1}{Np_{\bar{j}}}[p_{\bar{j}} - (N-\bar{j}+2)b_{\bar{j}}],$$

we already proved that

$$\frac{p_2 - (N-1)b_1}{Np_{\bar{j}}} \le \frac{1}{N-\bar{j}+2};$$

moreover, by definition of $\bar{j}$, we have $(N-\bar{j}+3)b_{\bar{j}-1} > p_{\bar{j}-1}$, or equivalently $(N-\bar{j}+2)b_{\bar{j}-1} > p_{\bar{j}}$. Thus,

$$\frac{p_2 - (N-1)b_1}{Np_{\bar{j}}}[p_{\bar{j}} - (N-\bar{j}+2)b_{\bar{j}}] \le \frac{p_{\bar{j}}}{N-\bar{j}+2} - b_{\bar{j}} < b_{\bar{j}-1} - b_{\bar{j}},$$

as wanted.

It is left to show that $b_2 - t_2 \ge \cdots \ge b_k - t_k$. It is trivial to check that $b_2 - t_2 = \cdots = b_{\bar{j}-1} - t_{\bar{j}-1}$, and $b_{\bar{j}} - t_{\bar{j}} \ge \cdots \ge b_k - t_k$ using $b_{\bar{j}} \ge \cdots \ge b_k$ as before. Finally,

$$b_{\bar{j}-1} - t_{\bar{j}-1} \ge b_{\bar{j}} - t_{\bar{j}} \iff \frac{p_2 - (N-1)b_1}{N} \ge \frac{b_{\bar{j}}}{p_{\bar{j}}}\frac{p_2 - (N-1)b_1}{N}(N-\bar{j}+2),$$

which is true, since $(N-\bar{j}+2)b_{\bar{j}} \le p_{\bar{j}}$ and $p_2 - (N-1)b_1 \ge 0$.

Proof of (iii) The thesis is equivalent to

$$(N-1)t_2 \le \sum_{j=2}^{k} t_j \iff (N-1)t_2 \le (N-1)b_1,$$

and this is implied by $t_2 \le b_2 \le b_1$.

Proof of (iv) The thesis is equivalent to

$$N(b_2 - t_2) \le p_2 - (N-1)b_1,$$

which is in fact an equality (see the definition of t_2). ■

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We are ready to present the main result of this section, which provides a transport plan of finite cost under an additional hypothesis on the tuple $(b_1, \dots, b_k)$. The result is peculiar because it does not involve the non-atomic part of the measure; it is in fact a general discrete construction to get a purely atomic symmetric measure having fixed purely atomic marginals.

Proposition 5.6 *Let $k > N$ and $(b_1, \dots, b_k)$ with*

$$(N-1)b_1 \le b_2 + \cdot + b_k. \tag{5.4}$$

Then for every $x_1, \dots, x_k \in \mathbb{R}^d$ distinct, there exists a symmetric transport plan of finite cost with marginals $\rho = b_1\delta_{x_1} + \cdots + b_k\delta_{x_k}$.

Proof For every pair of positive integers (N, k), with $k > N$, let $\mathfrak{P}(N, k)$ be the following proposition.

Let $(x_1, \dots, x_k; b_1, \dots, b_k)$ with $(N-1)b_1 \le b_2 + \cdots + b_k$. Then for every $(x_1, \dots, x_k)$ there exists a symmetric N-transport plan of finite cost with marginals $b_1\delta_{x_1} + \cdots + b_k\delta_{x_k}$.

We will prove $\mathfrak{P}(N, k)$ by double induction, in the following way. First we prove $\mathfrak{P}(1, k)$ for every k and $\mathfrak{P}(N, N+1)$ for every N. Then we prove

$$\mathfrak{P}(N-1, k) \wedge \mathfrak{P}(N, k-1) \implies \mathfrak{P}(N, k).$$

Proof of $\mathfrak{P}(1, k)$ This is trivial: simply take $b_1\delta_{x_1} + \cdots + b_k\delta_{x_k}$ as a "transport plan".

Proof of $\mathfrak{P}(N, N+1)$ Let us denote by A_N the $(N+1) \times (N+1)$ matrix

$$A_N = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 \end{pmatrix},$$

whose inverse is

$$A_N^{-1} = \frac{1}{N} \begin{pmatrix} -(N-1) & 1 & \cdots & 1 \\ 1 & -(N-1) & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & -(N-1) \end{pmatrix}.$$

Also define the following $(N+1) \times N$ matrix, with elements in $\mathbb{R}^d$:

$$(x_{ij}) = \begin{pmatrix} x_2 & x_3 & \cdots & x_{N+1} \\ x_1 & x_3 & \cdots & x_{N+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & x_N \end{pmatrix},$$

where the i-th row is $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{N+1})$. We want to construct a transport plan of the form

$$P = N \sum_{i=1}^{N+1} a_i (\delta_{x_{i1}} \otimes \cdots \otimes \delta_{x_{iN}})_{\text{sym}},$$

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where $a_i \geq 0$. Note that, by Lemma 2.1, the marginals of P are equal to

$$\pi(P) = \sum_{j=1}^{N+1} \Big(\sum_{\substack{i=1 \\ i \neq j}}^{N+1} a_i \Big) \delta_{x_j}.$$

Thus, the condition on the a_i's to have $\pi(P) = \rho$ is

$$A_N \begin{pmatrix} a_1 \\ \vdots \\ a_{N+1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{N+1} \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{N+1} \end{pmatrix} = A_N^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_{N+1} \end{pmatrix}.$$

Finally, observe that condition (5.3) implies that $a_1 \geq 0$, while the fact that $b_1 \geq b_2 \geq \cdots \geq b_{N+1}$ leads to $a_1 \leq a_2 \leq \cdots \leq a_{N+1}$, and hence $a_i \geq 0$ for every i, and we are done.

Inductive step Let $(b_1, \dots, b_k)$ satisfy (5.3), with $k \geq N + 2$ (otherwise we are in the case $\mathfrak{P}(N, N+1)$, which we have already proved). Take $t_2, \dots, t_k$ given by Lemma 5.5, and apply the inductive hypotheses to find the following:

- a symmetric transport plan Q_1 of finite cost in $(N-1)$ variables, with marginals

$$\pi(Q_1) = \sum_{j=2}^{k} t_j \delta_{x_j};$$

- a symmetric transport plan R of finite cost in N variables, with marginals

$$\pi(R) = \sum_{j=2}^{k} (b_j - t_j) \delta_{x_j}.$$

Define

$$Q = \frac{1}{N-1} \sum_{j=1}^{N} (Q_1 \otimes_j \delta_{x_1}).$$

Since Q_1 is symmetric, Q is symmetric. Moreover, using Lemma 5.5(i),

$$\pi(Q) = \frac{1}{N-1} \delta_{x_1} \sum_{j=2}^{k} t_j + \sum_{j=2}^{k} t_j \delta_{x_j} = b_1 \delta_{x_1} + \sum_{j=2}^{k} t_j \delta_{x_j}.$$

The transport plan $P = Q + R$ is symmetric, with marginals $\pi(P) = b_1 \delta_{x_1} + \cdots + b_k \delta_{x_k}$. ∎

In order to conclude the proof of this section, we must now deal not only with the non-atomic part of ρ, but also with the additional hypothesis of Proposition 5.6. Indeed, the presence of a non-atomic part will fix the atomic mass exceeding the inequality (5.4), as will be seen soon.

Definition 5.7 Given N, we say that the tuple $(b_1, \dots, b_\ell)$ is *fast decreasing* if

$$(N - j) b_j > \sum_{i > j} b_i \qquad \forall j = 1, \dots, \ell - 1.$$

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Remark 5.8 Note that if $(b_1, \dots, b_\ell)$ is fast decreasing, then necessarily $\ell < N$. As a consequence, given any sequence $(b_1, b_2, \dots)$, even infinite, we can select its maximal fast decreasing initial tuple $(b_1, \dots, b_\ell)$ (which might be empty, *i.e.*, $\ell = 0$).

Theorem 5.9 *If ρ is such that*

$$\rho = \sigma + \sum_{j=1}^{k} b_j \delta_{x_j}$$

with $k > N$ atoms, then there exists a transport plan of finite cost.

Proof Consider $(b_1, \dots, b_k)$ and use Remark 5.8 to select its maximal fast decreasing initial tuple $(b_1, \dots, b_\ell)$, $\ell < N$. Thanks to Proposition 5.6, we can construct a transport plan $P_{\ell+1}$ over $\mathbb{R}^{(N-\ell)d}$ with marginals $b_{\ell+1}\delta_{x_{\ell+1}} + \cdots + b_k \delta_{x_k}$, since

$$(N - \ell - 1) b_{\ell+1} \leq \sum_{j=\ell+2}^{k} b_j$$

by maximality of $(b_1, \dots, b_\ell)$, and this is condition (5.3) in this case. We extend $P_{\ell+1}$ step by step to an N-transport plan, letting

$$P_j = \frac{1}{N-j} \sum_{i=j}^{N} (P_{j+1} \otimes_i \delta_{x_j}),$$

for $j = \ell, \ell - 1, \dots, 1$.

Let $p_\ell = b_{\ell+1} + \cdots + b_k$, and $q_\ell = \frac{p_\ell}{N-\ell}$. We claim that $|P_j| = (N - j + 1) q_\ell$. In fact, by construction $|P_{\ell+1}| = p_\ell$, and inductively

$$|P_j| = \frac{1}{N-j} \sum_{i=j-1}^{N} |P_{j+1}| = \frac{N-j+1}{N-j} (N-j) q_\ell = (N-j+1) q_\ell.$$

Moreover,

$$\pi(P_j) = \sum_{i=j}^{k} q_\ell \delta_{x_i} + \sum_{i=\ell+1}^{k} b_i \delta_{x_i}.$$

This is true by construction in the case $j = \ell + 1$, and inductively

$$\pi(P_j) = \frac{1}{N-j} \delta_{x_j} |P_{j+1}| + \frac{N-j}{N-j} \pi(P_{j+1}) = \sum_{i=j}^{\ell} q_\ell \delta_{x_i} + \sum_{i=\ell+1}^{k} b_i \delta_{x_i}.$$

Note that, for every $i = 1, \dots, \ell$, $b_i \geq b_\ell > q_\ell$. We shall find, using Proposition 5.3, a transport plan of finite cost with marginals

$$\sigma + \sum_{i=1}^{\ell} (b_i - q_\ell) \delta_{x_i},$$

since condition (5.2) reads

$$N(b_1 - q_\ell) - \sum_{i=1}^{\ell} (b_i - q_\ell) = N b_1 - \sum_{i=1}^{\ell} b_i - (N - \ell) q_\ell < 1 - \sum_{i=1}^{k} b_i = |\sigma|. \qquad \blacksquare$$

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6 Marginals with Countably Many Atoms

In this section, we finally deal with the case of an infinite number of atoms, *i.e.*,

$$\rho = \sigma + \sum_{j=1}^{\infty} b_j \delta_{x_j}$$

with $b_j > 0$, $b_{j+1} \le b_j$ for every $j \ge 1$.

The main issue is topological in nature: if the atoms x_j are too close each other (for example, if they form a dense subset of $\mathbb{R}^d$) and the growth of b_j for $j \to \infty$ is too slow, the cost might diverge. With this in mind, we begin with an elementary topological result in order to separate the atoms into N groups, with controlled minimal distance from each other.

Lemma 6.1 *There exists a partition $\mathbb{R}^d = E_2 \sqcup \cdots \sqcup E_{N+1}$ such that the following hold:*

(i) *for every $j = 2, \dots, N+1$, $x_j \in \mathring{E}_j$;*

(ii) *for every $j = 2, \dots, N+1$, ∂E_j does not contain any x_i.*

Proof For $j = 3, \dots, N+1$, let $r_j > 0$ small enough such that

$$x_i \notin B(x_j, r_j) \qquad \text{for every } i = 1, \dots, N,\ i \ne j.$$

Fixing any $j = 3, \dots, N+1$, by a cardinality argument, there must be a positive real t_j with $0 < t_j < r_j$ and $\partial B(x_j, t_j)$ not containing any x_i, $i \ge 1$. We take $E_j = B(x_j, t_j)$ for $j = 3, \dots, N+1$. Note that this choice fulfills conditions (i) and (ii) for $j = 3, \dots, N+1$. Finally, we take

$$E_2 = \mathbb{R}^d \setminus \Big(\bigcup_{j=3}^{N+1} E_j \Big).$$

Clearly $x_2 \in \mathring{E}_2$, and moreover the condition (ii) is satisfied, since

$$\partial E_2 = \bigcup_{j=3}^{N+1} \partial E_j.$$

■

Consider the partition given by Lemma 6.1, and define the corresponding partition of $\mathbb{N}$ given by $\mathbb{N} = A_2 \cup \cdots \cup A_{N+1}$, where

$$A_j = \left\{ i \in \mathbb{N} \mid x_i \in E_j \right\}.$$

Next we consider, for every $j = 2, \dots, N+1$, a threshold $n_j \ge 2$ large enough such that, defining

$$\epsilon_j = \sum_{\substack{i \ge n_j \\ i \in A_j}} b_i,$$

then

$$\epsilon_2 + \cdots + \epsilon_{N+1} < \min\left\{ b_{N+1}, \frac{1}{N} - b_1 \right\}. \tag{6.1}$$

This can be done, since the series $\sum b_i$ converges, and hence for every $j = 2, \dots, N+1$, the series $\sum_{i \in A_j} b_i$ is convergent.

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For every $j = 2, \dots, N+1$ define the transport plan

$$P_j = N\left[\left(\sum_{i\in A_j, i\geq n_j} b_i\delta_{x_i}\right)\otimes\delta_{x_2}\otimes\cdots\otimes\widehat{\delta}_{x_j}\otimes\cdots\otimes\delta_{x_{N+1}}\right]_{\text{sym}},$$

and note that, by Lemma 2.1,

$$\pi(P_j) = \epsilon_j\sum_{\substack{h=2\\h\neq j}}^{N+1}\delta_{x_h} + \sum_{\substack{i\geq n_j\\i\in A_j}} b_i\delta_{x_i}.$$

Then let $P_\infty = \sum_{j=2}^{N+1} P_j$, and observe that

$$\pi(P_\infty) = \sum_{j=2}^{N+1}\left(\sum_{\substack{i=2\\i\neq j}}^{N+1}\epsilon_i\right)\delta_{x_j} + \sum_{j=2}^{N+1}\sum_{\substack{i\geq n_j\\i\in A_j}} b_i\delta_{x_i}.$$

Now let

$$\widetilde{b}_i = \begin{cases} b_i - \sum_{\substack{h=2\\h\neq i}}^{N+1}\epsilon_h & \text{if } 2\leq i\leq N+1,\\ 0 & \text{if } i\geq n_j \text{ and } i\in A_j \text{ for some } j=2,\dots,N+1,\\ b_i & \text{otherwise.}\end{cases}$$

We are left to find a transport plan of finite cost with marginals

$$\sigma + \sum_{i=1}^{\infty}\widetilde{b}_i\delta_{x_i},$$

which has indeed a finite number of atoms. Note that $\widetilde{b}_i \geq 0$ for every i, thanks to condition (6.1). Moreover, since $\widetilde{b}_1 = b_1$ and $\widetilde{b}_j \leq b_j$, $\widetilde{b}_1 \geq \widetilde{b}_j$ for every $j\in\mathbb{N}$, as is used in what follows. If

$$(N-1)\widetilde{b}_1 \leq \sum_{i=2}^{\infty}\widetilde{b}_i$$

we can conclude using Proposition 5.6. Otherwise, we proceed as in the proof of Theorem 5.9, with $\{\widetilde{b}_j\}$ replacing $\{b_j\}$. At the final stage, it is left to check that

$$N(\widetilde{b}_1 - \widetilde{q}_{k+1}) - \sum_{i=1}^{k}(\widetilde{b}_i - \widetilde{q}_{k+1}) < 1 - \sum_{i=1}^{\infty} b_i = |\sigma|.$$

Indeed this is true, since using the condition (6.1) one gets

$$N(\widetilde{b}_1 - \widetilde{q}_{k+1}) - \sum_{i=1}^{k}(\widetilde{b}_i - \widetilde{q}_{k+1}) = Nb_1 - \sum_{i=1}^{\infty} b_i + N(\epsilon_2 + \cdots + \epsilon_{N+1}) < 1 - \sum_{i=1}^{\infty} b_i.$$

Aknowledgements The author is grateful to Prof. Luigi Ambrosio and Prof. Luigi De Pascale for all their useful remarks, and wishes to thank Prof. Emmanuel Trélat for his suggestions.

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