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SOCLES OF VERMA MODULES IN QUANTUM GROUPS

A.V. JEYAKUMAR AND P.B. SARASIJA

In this paper the Verma modules $M_\varepsilon(\lambda)$ over the quantum group $U_\varepsilon(sl(n+1), \mathbb{C})$, where ε is a primitive ℓth root of 1 are studied. Some commutation relations among the generators of U_ε are obtained. Using these relations, it is proved that the socle of $M_\varepsilon(\lambda)$ is non-zero.

0. INTRODUCTION

A quantum group $U_q = U_q(g)$ is a q-deformation of the classical universal enveloping algebra U of a complex semi-simple Lie algebra g, where q is an indeterminate. The representations of U_q have recently occupied the attention of many mathematicians (see for example, [1, 2, 3, 4]). When q is a root of unity, the representation theory of U_q has a close bearing on the modular representation theory of semi-simple, simply connected algebraic groups and affine Lie algebras.

In [1], De Concini and Kac defined the notion of Verma modules over U_q and U_ε (where ε is a primitive ℓth root of 1, ℓ is an odd integer) analogous to the classical Verma modules. In this paper, we study the Verma module $M_\varepsilon(\lambda)$ over $U_\varepsilon = U_\varepsilon(g)$, where $g = sl(n+1)$, and in particular prove that the socle of $M_\varepsilon(\lambda)$ over U_ε is non-zero.

1. PRELIMINARIES

1.1. Let us fix some notations which are standard (see for example, [1]).

For a fixed $n \in \mathbb{N}$, let $(a_{ij})_{1\leqslant i,j\leqslant n}$ be the cartan matrix of type A_n.

Let q be an indeterminate and let $A = \mathbb{C}[q, q^{-1}]$ with the quotient field $\mathbb{C}(q)$. For any integer $M \geqslant 0$, we define

$$[M] = \frac{q^M - q^{-M}}{q - q^{-1}} \in A, \quad [M]! = [M]\,[M-1]\ldots[1],$$

and

$$\begin{bmatrix} M \\ j \end{bmatrix} = \frac{[M]!}{[j]!\,[M-j]!} \quad \text{for } j \in \mathbb{N}, \quad \begin{bmatrix} M \\ 0 \end{bmatrix} = 1.$$

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Let U_q be the $\mathbb{C}(q)$ algebra with 1, defined by the generators E_i, F_i, $K_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$ with the relations:

(a) $$K_iK_i^{-1} = K_i^{-1}K_i = 1, \quad K_iK_j = K_jK_i,$$

(b) $$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q^{-a_{ij}}F_j,$$

(c) $$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q - q^{-1}},$$

(d) $$E_iE_j = E_jE_i \text{ if } a_{ij} = 0,$$

(e) $$E_i^2E_j - (q + q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \text{ if } a_{ij} = -1,$$

(f) $$F_iF_j = F_jF_i \text{ if } a_{ij} = 0,$$

(g) $$F_i^2F_j - (q + q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \text{ if } a_{ij} = -1.$$

Then U_q is a Hopf algebra over $\mathbb{C}(q)$ which is called the quantum group associated to the matrix (a_{ij}), with comultiplication $\triangle$, antipode S and counit ν defined by

$$\triangle E_i = E_i \otimes 1 + K_i \otimes E_i, \quad \triangle F_i = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$
$$\triangle K_i = K_i \otimes K_i$$
$$SE_i = -K_i^{-1}E_i, \quad SF_i = -F_iK_i, \quad SK_i = K_i^{-1}$$
$$\nu E_i = 0, \quad \nu F_i = 0, \quad \nu K_i = 1.$$

Also introduce the elements

$$[K_i; n] = \frac{(K_iq^n - K_i^{-1}q^{-n})}{q - q^{-1}} \quad \text{in} \quad U_q.$$

1.2. It is well known that one can introduce a root system associated to the matrix (a_{ij}). We briefly describe the construction here. For details refer to [**1**, **5**].

Let P be a free abelian group with basis ω_i, $i = 1, 2, \ldots, n$ (P is usually called the lattice of weights). Let P^+ denote the subgroup of non-negative integral combinations of $\omega_1, \omega_2, \ldots, \omega_n$ and any element of P^+ is called a dominant weight. Define the following elements in P:

$$\rho = \sum_{i=1}^{n} \omega_i, \qquad \alpha_j = \sum_{i=1}^{n} a_{ij}\omega_i \quad (j = 1, \ldots, n)$$

let $$Q = \sum_i Z\alpha_i, \qquad Q_+ = \sum_i Z_+\alpha_i.$$

Define a bilinear pairing $P \times Q \to Z$ by

(1.2.1) $$(\omega_i \mid \alpha_j) = \delta_{ij}.$$

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Then $(\alpha_i \mid \alpha_j) = a_{ij}$, so that we get a symmetric Z-valued bilinear form on Q such that $(\alpha \mid \alpha) \in 2Z$.

Define automorphisms r_i of P by $r_i\omega_j = \omega_j - \delta_{ij}\alpha_i$ $(i, j = 1, 2, \ldots, n)$.

Then $r_i\alpha_j = \alpha_j - a_{ij}\alpha_i$. Let W be the (finite) subgroup of $GL(P)$ generated by $r_1, r_2, \ldots, r_n$. Then Q is W-invariant and the pairing $P \times Q \to Z$ is W-invariant. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $R = W\Pi$ and denote $R \cap Q_+$ by R^+. Then R is a root system corresponding to the cartan matrix (a_{ij}) with Weyl group W and R^+ the system of positive roots. Clearly p is half the sum of positive roots. We introduce a partial ordering of P by $\lambda \geqslant \mu$ if $\lambda - \mu \in Q_+$. Let w_0 be the unique element of W such that $w_0(R^+) = -R^+$.

1.3. Let U_A be the A-subalgebra of U_q generated by the elements E_i, F_i, $K_i^{\pm 1}$, $[K_i; 0]$ $(i = 1, 2, \ldots, n)$. Let U_A^+ (respectively U_A^-) be the A-subalgebra of U_A generated by the E_i (respectively F_i) and U_A^0 the subalgebra generated by the K_i and $[K_i; 0]$.

1.4. We shall show how to choose a canonical basis for U_q from the given set of generators (for details see [**1**, **5**, **6**]).

We note that we can define an anti-automorphism ω of U_q defined by

$$\omega E_i = F_i \qquad \omega F_i = E_i, \qquad \omega K_i = K_i^{-1}, \qquad \omega q = q^{-1}. \tag{1.4.1}$$

For any i, $1 \leqslant i \leqslant n$, there is a unique algebra automorphism T_i of U_q such that

$$\begin{aligned} &T_iE_i = -F_iK_i, \quad T_jE_i = -E_jE_i + q^{-1}E_iE_j \text{ if } a_{ji} = -1 \\ &\text{and } T_j(E_i) = E_i \text{ if } a_{ij} = 0 \end{aligned} \tag{1.4.2}$$

$$\begin{aligned} &T_iF_i = -K_i^{-1}E_i, \quad T_jF_i = -F_jF_i + qF_iF_j \text{ if } a_{ji} = -1 \\ &\text{and } T_j(F_i) = F_i \text{ if } a_{ij} = 0 \end{aligned} \tag{1.4.3}$$

$$T_iK_j = K_jK_i^{-a_{ij}}, \quad T_i\omega = \omega T_i. \tag{1.4.4}$$

Let $w \in W$ and let $r_{i_1} \ldots r_{i_k}$ be a reduced expression of w. Then the automorphism $T_w = T_{i_1} \ldots T_{i_k}$ of U_q is independent of the choice of the reduced expression of w.

Fix a reduced expression $r_{i_1} r_{i_2} \ldots r_{i_N}$ of the longest element of W, where $N = |R^+|$. Then this gives us an enumeration of the elements of R^+:

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = r_{i_1}\alpha_{i_2}, \ldots, \beta_N = r_{i_1} \ldots r_{i_{N-1}}\alpha_{i_N}.$$

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We define the root vectors:

$$E_{\beta_s} = T_{i_1}T_{i_2}\dots T_{i_{s-1}}E_{i_s},$$
$$F_{\beta_s} = T_{i_1}T_{i_2}\dots T_{i_{s-1}}F_{i_s} \quad \text{which is the same as } \omega E_{\beta_s}.$$

For $j = (j_1, j_2, \dots, j_N) \in Z_+^N$ let

$$E^j = E_{\beta_1}^{j_1}\dots E_{\beta_N}^{j_N}, \quad F^j = \omega E^j. \tag{1.4.5}$$

The elements $F^j K_1^{m_1}\dots K_n^{m_n}E^r$ where $j, r \in Z_+^N$, $(m_1 \dots m_n) \in Z^n$ form a basis of U_q over $\mathbb{C}(q)$.

1.5. Given $\varepsilon \in \mathbb{C}^*$, we now consider the specialisation $U_\varepsilon = U_A/(q-\varepsilon)U_A$. We take ε in such a way that $\varepsilon^2 \neq 1$.

Then U_ε is an algebra over $\mathbb{C}$ with generators E_i, F_i, $K_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$ (identifying these vectors with their images), and defining relations,

(a′) $$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

(b′) $$K_iE_jK_i^{-1} = \varepsilon^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = \varepsilon^{-a_{ij}}F_j,$$

(c′) $$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{\varepsilon - \varepsilon^{-1}},$$

(d′) $$E_i^2E_j - \left(\varepsilon + \varepsilon^{-1}\right)E_iE_jE_i + E_jE_i^2 = 0 \text{ if } a_{ij} = -1,$$

(e′) $$F_i^2F_j - \left(\varepsilon + \varepsilon^{-1}\right)F_iF_jF_i + F_jF_i^2 = 0 \text{ if } a_{ij} = -1,$$

(f′) $$E_iE_j = E_jE_i = 0, \quad F_iF_j = F_jF_i = 0 \text{ if } a_{ij} = 0.$$

1.6. We denote by U_ε^+, U_ε^-, U_ε^0 the images of U_A^+, U_A^-, and U_A^0 in U_ε. The automorphism T_i of U_q defined in (1.4) clearly induces an automorphism T_i of U_ε. The vectors E^j, F^j et cetera of (1.4.5) can then be taken to represent their images in U_ε. Then the elements E^j, $j \in Z_+^N$ form a basis of U_ε^+ over $\mathbb{C}$, and the elements $F^jK_1^{m_1}\dots K_n^{m_n}E^r$ where $j, r \in Z_+^N$ and $(m_1 \dots m_n) \in Z^n$ form a basis of U_ε over $\mathbb{C}$.

2. Some commutation relations

2.1. We shall now introduce certain basic relations among the generators of U_ε corresponding to the positive roots.

Consider the following sequence of elements in U_ε.

$$\begin{gathered} E_2,\ T_2T_1(E_2),\ T_{i+1}(E_i) \quad i = 1, 2, \dots, n, \\ T_{i-2}(E_i) \quad i = 3, \dots, n,\ T_{i+2}T_{i+1}(E_i) \quad i = 1, \dots, n-2, \\ \dots,\ T_nT_{n-1}\dots T_2(E_1). \end{gathered} \tag{2.1.1}$$

For convenience we shall write the above terms in the same order.

(2.1.2)
$$E_2,\, E_1,\, E_{i\,i+1}, \quad i=1,2,\dots,n,\; E_{i\,i-2}, \quad i=3,\dots,n,$$
$$E_{i\,i+1\,i+2}, \quad i=1,\dots,n-2,\dots,E_{1\,2\dots n}.$$

The subscripts correspond to the various positive roots: For example the subscript 12 corresponds to $\alpha_1+\alpha_2$, and 123 corresponds to $\alpha_1+\alpha_2+\alpha_3$.

For A_2 and A_3 these elements are E_2, E_1, $E_{1\,2}$ (see [6]) and E_2, E_1, E_3, $E_{1\,2}$, $E_{2\,3}$, $E_{1\,2\,3}$ respectively.

2.2. Using the identities (1.4.2) we obtain the following commutation formulas among the elements defined in (2.1.2).

$$E_{s\,s+1\dots k}E_{k+1\,k+2\dots \ell}=\varepsilon E_{k+1\dots \ell}E_{s\,s+1\dots k}+\varepsilon E_{s\,s+1\dots \ell},\ 1\leqslant s,\ k\leqslant n,\ k+1\leqslant \ell\leqslant n;$$
$$E_{1\,2\dots k}E_{s\,s+1\dots \ell}=E_{s\,s+1\dots \ell}E_{1\,2\dots k},1\leqslant k\leqslant n,\ 1<s,\ \ell<k;$$
$$E_{s\,s+1\dots k}E_{\ell\,\ell+1\dots k}=\varepsilon^{-1}E_{\ell\,\ell+1\dots k}E_{s\,s+1\dots k},\ 1\leqslant s,\ k\leqslant n,\ s<\ell\leqslant k;$$
$$E_{s\,s+1\dots k}E_{s\,s+1\dots \ell}=\varepsilon E_{s\,s+1\dots \ell}E_{s\,s+1\dots k},\ 1\leqslant s,\ k\leqslant n,\ s\leqslant \ell<k;$$
$$E_{s\,s+1\dots k}E_{\ell\,\ell+1\dots m}=E_{\ell\,\ell+1\dots m}E_{s\,s+1\dots k}+\left(\varepsilon^{-1}-\varepsilon\right)E_{r\,r+1\dots p}E_{s\,s+1\dots m},$$
$$1\leqslant s,\ k<n,\ s\neq k,\ s<\ell\leqslant k<m\leqslant n,\ \ell=k=r=p,\ \ell\leqslant r,\ p\leqslant k,\ r\neq p;$$
$$E_{1\,2\dots k}E_{k+2}=E_{k+2}E_{1\,2\dots k},\quad 1<k<n-1;$$
$$E_iE_j=E_jE_i,\qquad i,j=1,2\dots n,\quad i\neq j,\ a_{ij}=0.$$

The above commutation formulas give rise, by induction, to commutation formulas between the basis element of U_ε^+.

$$E^m_{s\,s+1\dots k}E^p_{k+1\dots \ell}=\varepsilon^{mp}E^p_{k+1\dots \ell}E^m_{s\,s+1\dots k} + \sum_{j=1}^{\min(m,p)}\in^{((p-j)m+j)}[j]!\begin{bmatrix}p\\ j\end{bmatrix}\begin{bmatrix}m\\ j\end{bmatrix}E^j_{s\,s+1\dots \ell}E^{p-j}_{k+1\dots \ell}E^{m-j}_{s\,s+1\dots k};$$
$$1\leqslant s,\ k\leqslant n,\ k+1\leqslant \ell\leqslant n;$$
$$E^m_{s\,s+1\dots k}E^u_{\ell\,\ell+1\dots t}=E^u_{\ell\,\ell+1\dots t}E^m_{s\,s+1\dots k} + \sum_{j=1}^{\min(m,u)}(-1)^{j+1}\left(\varepsilon^{-1}-\varepsilon\right)^j\varepsilon^{j-1}[j]!\begin{bmatrix}m\\ j\end{bmatrix}\begin{bmatrix}u\\ j\end{bmatrix}E^j_{r\,r+1\dots p}E^{m-j}_{s\,s+1\dots k}E^{u-j}_{\ell\,\ell+1\dots t}E^j_{s\,s+1\dots t};$$
$$1\leqslant s,\ k<n,\ s\neq k,\ s<\ell\leqslant k<t\leqslant n,\ \ell=k=r=p,\ \ell\leqslant r,\ p\leqslant k,\ r\neq p;$$
$$E^m_{s\,s+1\dots K}E^p_{\ell\,\ell+1\dots k}=\varepsilon^{-mp}E^p_{\ell\,\ell+1\dots k}E^m_{s\,s+1\dots k},\ 1\leqslant s,\ k\leqslant n,\ s<\ell\leqslant k;$$
$$E^m_{1\,2\dots k}E^p_{s\,s+1\dots \ell}=E^p_{s\,s+1\dots \ell}E^m_{1\,2\dots k},\ 1\leqslant k\leqslant n,\ 1<s,\ \ell<k;$$
$$E^m_{s\,s+1\dots k}E^p_{s\,s+1\dots \ell}=\varepsilon^{mp}E^p_{z\,z+1\dots \ell}E^m_{s\,s+1\dots k},\ 1\leqslant s,\ k\leqslant n,\ s\leqslant \ell<k;$$

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$E^m_{1\,2\ldots k}E^p_{k+2} = E^p_{k+2}E^m_{1\,2\ldots k}$, $1 < k < n-1$;
$E^m_i E^p_j = E^p_j E^m_i$ if $a_{ij} = 0, i \neq j, \quad i,j = 1\ldots n$.

By using the relations $\omega E_i = F_i$, $\omega\varepsilon = \varepsilon^{-1}$ we obtain similar relations among the F_i's.

3. Verma modules

3.1. The notion of Verma modules over U_q and U_ε was introduced by De Concini and Kac in [1]. In the rest of the paper, we shall be concerned only with Verma modules over U_ε, where ε is a primitive ℓth root of unity. We recapitulate the definition below:

For each $\lambda \in P$ the Verma module $M_\varepsilon(\lambda)$ over U_ε is the vector space $M_\varepsilon(\lambda)$ in which there exists a non-zero distinguished vector v_λ such that $U^+_\varepsilon v_\lambda = 0$, $Kv_\lambda = \varepsilon^{(\lambda|\alpha)}v_\lambda$, $K \in U^0_\varepsilon$ where $(\ |\ )$ is the pairing from $P \times W \to Z$ defined in (1.2) and $\{F^j v_\lambda \left(j \in Z^N_+\right)\}$ is a basis of $M_\varepsilon(\lambda)$. Let $L_\varepsilon(\lambda)$ denote the unique irreducible quotient of $M_\varepsilon(\lambda)$ by its unique maximal submodule.

Then we have

$$Kv_\lambda = \varepsilon^{(\lambda|\alpha)}v_\lambda. \tag{3.1.1}$$

Also for each $h = 1, 2, \ldots, N$, $F_h v_\lambda$ is a weight vector of weight $\lambda - \alpha_h$ as easily seen below.

$$\begin{aligned} KF_h v_\lambda &= \varepsilon^{-(\alpha|\alpha_h)}F_h K v_\lambda \\ &= \varepsilon^{-(\alpha|\alpha_h)}\varepsilon^{(\lambda|\alpha)}F_h v_\lambda \qquad (\text{since } (\alpha_h \mid \alpha) = (\alpha \mid \alpha_h)) \\ &= \varepsilon^{-(\lambda-\alpha_h|\alpha)}F_h v_\lambda. \end{aligned}$$

3.1.2. This shows that for any $r \in Z_+$, $F^r_h v_\lambda$ is a weight vector of weight $\lambda - r\alpha_h$ and therefore each $F^j v_\lambda \left(= F^{j_1}_i \ldots F^{j_N}_N v_\lambda\right)$ is a weight vector of weight $\lambda - \sum\limits_{h=1}^{N} j_h\alpha_h$.

3.2 Verma Modules over some subalgebras of U_ε.

We first define the subalgebras U_r, U^+_r, U^-_r, of U_ε generated by

$$\{F^j, \prod_{i=1}^{n} K_i^{m_i}, E^r, 0 < j_i, r_i < \ell^r, \quad (m_1 \ldots m_n) \in Z^n\},$$

$$\{E^r, \prod_{i=1}^{n} K_i^{m_i}, 0 < r_i < \ell^r, \quad (m_1 \ldots m_n) \in Z^n\},$$

$$\{F^j, 0 \leqslant j_i < \ell^r\} \quad \text{respectively.}$$

The set

$$\left\{F_1^{j_1}\dots F_N^{j_N}\,K_1^{m_1}\dots K_n^{m_n}\,E_1^{r_1}\dots E_N^{r_N},\,0\leqslant j_i,\,r_i<\ell^r,\,(m_1\dots m_n)\in Z^n\right\} \tag{3.2.1}$$

is a basis of U_r and the set

$$\{F_1^{j_1}\dots F_N^{j_N},\,0\leqslant j_i<\ell^r\}\quad \text{is a basis of } U_r^-. \tag{3.2.2}$$

We can then define the Verma modules $M_{\varepsilon,r}(\lambda)$ of weight λ over U_r analogously to $M_\varepsilon(\lambda)$ over U_ε, that is, there exists a non-zero vector (say) $\widehat{v}_\lambda$ such that $U_r^+\widehat{v}_\lambda=0$, $K\widehat{v}_\lambda=\varepsilon^{(\lambda|\alpha)}\widehat{v}_\lambda$ for $K\in U_r^0$ and $\{F^j\widehat{v}_\lambda,\,0\leqslant j_i<\ell^r\}$ form a basis of $M_{\varepsilon,r}(\lambda)$.

There is a natural injective homomorphism $f_r\colon M_{\varepsilon,r}(\lambda)\to M_\varepsilon(\lambda)$ given by

$$f_r(F^j\widehat{v}_\lambda)=F^jv_\lambda. \tag{3.2.3}$$

3.3. We next introduce certain elements defined by I_r of U_ε^-, which play an important role in our future study of the socles of Verma modules and homomorphisms between Verma modules.

For each positive integer r, let $I_r=F_1^{\ell^r-1}\dots F_N^{\ell^r-1}$ which is an element of U_r^-.

It then follows that I_rv_λ is a weight vector of U_rv_λ of weight $\lambda-2(\ell-1)\rho$, where ρ is half the sum of the positive roots.

In fact,

(3.3.1)

$$\begin{aligned}
KI_rv_\lambda &= K\,F_1^{\ell^r-1}F_2^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda\\
&=\varepsilon^{(\lambda-(\ell^r-1)\alpha_1+\dots+\alpha_N|\alpha)}F_1^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda \qquad \text{from [3.1.2]}\\
&=\varepsilon^{(\lambda-2(\ell^r-1)\rho|\alpha)}F_1^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda\\
&=\varepsilon^{(\lambda+2\rho|\alpha)}F_1^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda \qquad [\text{since } \varepsilon^{\ell^r}=1]\\
&=\varepsilon^{(\lambda-2\ell\rho+2\rho|\alpha)}F_1^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda\\
&=\varepsilon^{(\lambda-2(\ell-1)\rho|\alpha)}F_1^{\ell^r-1}\dots F_N^{\ell^r-1}v_\lambda.
\end{aligned}$$

In particular, when $\lambda=0$, we see that $I_r\widehat{v}_0$ is a weight vector of $M_{\varepsilon,r}(0)$ with minimal weight $-2(\ell-1)\rho$.

We observe for later use that I_r is an integral of U_r^-. In fact, for $\alpha\in R^+$ and $a\in\mathbb{N}$ such that $0<a<\ell^r$, $R_\alpha^aI_r$ and $I_rF_\alpha^a$ are in U_r^-. Hence $F_\alpha^aI_r\widehat{v}_0$ and $I_rF_\alpha^a\widehat{v}_0$ are weight vectors of $M_{\varepsilon,r}(0)$ with weight $-2(\ell-1)\rho-a\alpha$. By the minimality of the weight $-2(\ell-1)\rho$, it follows that $F_\alpha^aI_r=I_rF_\alpha^a=0$. This shows that I_r is an integral of U_r^-, in other works $uI_r=\nu(u)I_r$ for all $u\in U_r^-$, where $\nu\colon U_r^-\to\mathbb{C}$ is the augmentation function.

3.4 A HOMOMORPHISM BETWEEN TWO VERMA MODULES. $M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ is a map ϕ: $M_\varepsilon(\lambda) \to M_\varepsilon(\mu)$ such that ϕ is a vector space homomorphism and $\phi(uv) = u\phi(v)$, $u \in U_\varepsilon$, $v \in M_\varepsilon(\lambda)$.

LEMMA 3.4.1. *If $M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ are Verma modules over the quantum group U_ε, and there is an injective U_ε module homomorphism ϕ: $M_\varepsilon(\lambda) \to M_\varepsilon(\mu)$, then $\lambda = \mu$ and ϕ is multiplication by some element of $\mathbb{C}$.*

PROOF: Let v_λ, v_μ be non-zero highest weight vectors of $M_\varepsilon(\lambda)$, $M_\varepsilon(\mu)$ respectively. Since v_λ generates $M_\varepsilon(\lambda)$, ψ is determined by $\psi(v_\lambda)$. Say $\psi(v_\lambda) = uv_\mu$, $u \in U_\varepsilon^-$. Now by definition, U_ε^- is the union of the subalgebras U_r^- for $r = 1, 2, \ldots$ and so there is some r for which $u \in U_r^-$. Since I_r is an integral for U_r^-,

$$\nu(u)I_r v_\mu = I_r u v_\mu = I_r \psi(v_\lambda) = \phi(I_r v_\lambda)$$

where ν: $U_r^- \to \mathbb{C}$ is the augmentation function and $I_r v_\lambda$ is an element of the basis for $M_\varepsilon(\lambda)$, so is non-zero, and therefore $\nu(u) \neq 0$. But $\psi(v_\lambda)$ must have weight λ, so uv_μ has weight λ, which contradicts $\nu(u) \neq 0$ unless $\lambda = \mu$. Since v_μ spans the μ-weight space of $M_\varepsilon(\mu)$, $\psi(v_\lambda) = cv_\mu = cv_\lambda$ for some $c \in \mathbb{C}$, and ϕ is just multiplication by c. ∎

4. SOCLE OF VERMA MODULES

Denote the socle of the U_ε module $M_\varepsilon(\lambda)$ by $\mathrm{Soc}(M_\varepsilon(\lambda))$ and the socle of the U_r module $M_{\varepsilon,r}(\lambda)$ by $\mathrm{Soc}(M_{\varepsilon,r}(\lambda))$.

Since for any $r > 0$, $M_{\varepsilon,r}(\lambda)$ is finite dimensional, clearly $\mathrm{Soc}(M_{\varepsilon,r}(\lambda)) \neq 0$. It is interesting to note that even for the infinite dimensional module $M_\varepsilon(\lambda)$, its socle is non-zero. We proceed to prove this in this section.

LEMMA 4.1. *If $0 \neq u \in U_r^-$ for some $r \in \mathbb{N}$, then $U_r u$ contains $\mathbb{C}I_r$.*

PROOF: We shall order the positive roots $\alpha(1), \alpha(2), \ldots, \alpha(N)$ in such a way that if $\alpha(i) + \alpha(j) = \alpha(k)$ then $k < i, j$.

If $0 < a < \ell^r$ then clearly

$$F_{\alpha(1)}^{\ell^r-1} F_{\alpha(1)}^a = F_{\alpha(1)}^{\ell^r-1+a} = 0.$$

We shall prove by induction on i, with $1 \leqslant i \leqslant N$, that $F_{\alpha(1)}^{\ell^r-1} \ldots F_{\alpha(i)}^{\ell^r-1} F_\alpha^a = 0$ whenever $\alpha \in \{\alpha(1), \ldots, \alpha(i)\}$ and $0 < a < \ell^r$.

Suppose there exists some i, $2 \leqslant i \leqslant N$, such that

$$F_{\alpha(1)}^{\ell^r-1} F_{\alpha(2)}^{\ell^r-1} \ldots F_{\alpha(i-1)}^{\ell^r-1} F_\alpha^a = 0, \tag{4.1.1}$$

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whenever $\alpha \in \{\alpha(1), \alpha(2), \dots \alpha(i-1)\}$ and $0 < a < \ell^r$.

Now, suppose that there is some $\alpha \in \{\alpha(1), \alpha(2), \dots, \alpha(i)\}$ and choose a such that $0 < a < \ell^r$.

If $\alpha = \alpha(i)$, then $F_{\alpha(i)}^{\ell^r-1} F_\alpha^a = 0$, and so

$$F_{\alpha(1)}^{\ell^r-1} F_{\alpha(2)}^{\ell^r-1} \dots F_{\alpha(i)}^{\ell^r-1} F_\alpha^a = 0.$$

If $\alpha \neq \alpha(i)$, then the commutation relations defined in (2.2) imply that

$$F_{\alpha(1)}^{\ell^r-1} \dots F_{\alpha(i)}^{\ell^r-1} F_\alpha^a$$

is a sum of elements of the form

$$F_{\alpha(1)}^{\ell^r-1} \dots F_{\alpha(i-1)}^{\ell^r-1} F_\beta^b u$$

with $\beta \in \{\alpha(1), \dots, \alpha(i-1)\}$, $0 < b < \ell^r$, $u \in U_\varepsilon$ and each element of this form equals 0 by (4.1.1). So (4.1.1) holds for all i.

Using this equation together with the commutation relations defined in (2.2), if $1 \leqslant i \leqslant N$ and $0 < a < \ell^r$, then

$$\begin{aligned} F_{\alpha(1)}^{\ell^r-1} F_{\alpha(2)}^{\ell^r-1} \dots F_{\alpha(i-1)}^{\ell^r-1} F_{\alpha(i)}^a \\ - \varepsilon^{-1(i-1)(\ell^r-1)} F_{\alpha(i)}^a F_{\alpha(1)}^{\ell^r-1} \dots F_{\alpha(i-1)}^{\ell^r-1} = 0 \end{aligned} \tag{4.1.2}$$

and so if $1 \leqslant i \leqslant N$ and $0 < a, b < \ell^r$ then

$$\begin{aligned} F_{\alpha(i)}^a F_{\alpha(1)}^{\ell^r-1} F_{\alpha(2)}^{\ell^r-1} \dots F_{\alpha(i-1)}^{\ell^r-1} F_{\alpha(i)}^b \\ = \varepsilon^{-(i-1)(\ell^r-1)a} F_{\alpha(1)}^{\ell^r-1} \dots F_{\alpha(i-1)}^{\ell^r-1} F_{\alpha(i)}^{a+b} \\ = 0 \text{ if } a + b \geqslant \ell^r. \end{aligned}$$

Suppose u is a non-zero element of U_r^-. Then by the basis of U_r^- the element u is of the form

$$F_{\alpha(1)}^{a(1)} F_{\alpha(2)}^{a(2)} \dots F_{\alpha(N)}^{a(N)} \text{ with } 0 \leqslant a(1), \dots, a(N) < \ell^r.$$

By repeated use of (4.1.2)

$$\begin{aligned} \mathbb{C} F_{\alpha(N)}^{\ell^r-1-a(N)} \dots F_{\alpha(1)}^{\ell^r-1-a(1)} u = \mathbb{C} F_{\alpha(N)}^{\ell^r-1} \dots F_{\alpha(1)}^{\ell^r-1} \\ = \mathbb{C} I_r \text{ as required.} \end{aligned}$$

□

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COROLLARY 4.2. *Let r be a positive integer.*

$$I_{r+1} \in U_\varepsilon I_r.$$

PROOF: Lemma 4.1 implies that $\mathbb{C}I_{r+1} \subseteq U_{r+1}I_r$, so $I_{r+1} \in U_{r+1}I_r \subseteq U_\varepsilon I_r$. ∎

COROLLARY 4.3.

(i) *If M is a non-zero U_r submodule of $M_{\varepsilon,r}(\lambda)$ and $\widehat{v}_\lambda \in M_{\varepsilon,r}(\lambda)$, then $I_r\widehat{v}_\lambda \in M$.*

(ii) *If M is a non-zero U_ε submodule of $M_\varepsilon(\lambda)$ and $v_\lambda \in M_\varepsilon(\lambda)$, then $I_r v_\lambda \in M$ for all r.*

PROOF:

(i) By the basis of $M_{\varepsilon,r}(\lambda)$, M contains some vector $u\widehat{v}_\lambda$ with $u \in U_r^-$. By Lemma 4.1, $I_r\widehat{v}_\lambda \in \mathbb{C}I_r\widehat{v}_\lambda \subseteq U_r u\widehat{v}_\lambda \subseteq M$.

(ii) By the basis of $M_\varepsilon(\lambda)$, M contains some vector uv_λ with $u \in U_\varepsilon^-$, hence $u \in U_r^-$ for some r.

By Lemma 4.1, $I_r v_\lambda \in \mathbb{C}I_r v_\lambda \subseteq U_\varepsilon u v_\lambda \subseteq M$. ∎

COROLLARY 4.4. $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ *is simple.*

PROOF: $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ is a non-zero U_r submodule of $M_{\varepsilon_r}(\lambda)$ and by Corollary 4.3 (i) the submodule $U_rI_r\widehat{v}_\lambda$ is contained in every simple component of $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ and hence $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ itself is simple. ∎

LEMMA 4.5. *Let $\lambda \in P^+$, the set of dominant weights. Then for all $r > 0$, the highest weight of* $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ *is $w_o(\lambda - 2(\ell - 1)\rho)$ and hence is independent of r.*

PROOF: From (3.3.1), the lowest weight of $M_{\varepsilon,r}(\lambda)$ is $\lambda - 2(\ell - 1)\rho$ for all $r > 0$. From Corollary 4.3(i), we have seen that any non-zero submodule of $M_{\varepsilon,r}(\lambda)$ contains $I_r\widehat{v}_\lambda$. Hence $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ contains $I_r\widehat{v}_\lambda$ whose weight is $\lambda - 2(\ell - 1)\rho$. Therefore the lowest weight of $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ is $\lambda - 2(\ell - 1)\rho$ for all $r > 0$ and hence the highest weight of $\mathrm{Soc}\,(M_{\varepsilon,r}(\lambda))$ is $w_o(\lambda - 2(\ell - 1)\rho) = w_0(\lambda + 2\rho)$, which is independent of r. Hence the result. ∎

We shall proceed to prove our main result concerning the socle of the Verma modules.

THEOREM 4.6. $\mathrm{Soc}\,(M_\varepsilon(\lambda))$ *is non-zero for all $\lambda \in P^+$.*

PROOF: Let v_λ, $\widehat{v}_\lambda$ be non-zero highest weight vectors of the Verma modules $M_\varepsilon(\lambda)$ over U_ε and $M_{\varepsilon,r}(\lambda)$ over U_r respectively. Let M be an arbitrary non-zero U_ε submodule of $M_\varepsilon(\lambda)$. Then by Corollary 4.3(ii), $I_r v_\lambda \in U_r u v_\lambda \subseteq M$ for all r and hence $U_\varepsilon I_r v_\lambda \subseteq M$. Now, let I denote the submodule $\bigcap\limits_{r>0} U_\varepsilon I_r v_\lambda$ of $M_\varepsilon(\lambda)$.

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Replacing M by each simple component of $\mathrm{Soc}(M_\varepsilon(\lambda))$, it immediately follows that $\mathrm{Soc}(M_\varepsilon(\lambda)) \supseteq I$.

We proceed to prove that $I \neq (0)$. Since $M_{\varepsilon,r}(\lambda)$ is finite dimensional, $\mathrm{Soc}(M_{\varepsilon,r}(\lambda)) \neq 0$. By Corollary 4.3(i), $\mathrm{Soc}(M_{\varepsilon,r}(\lambda))$ is simple and we can take $\mathrm{Soc}(M_{\varepsilon,r}(\lambda))$ to be isomorphic to the simple U_r module $L_{\varepsilon,r}(\mu)$ (where μ is $w_o(\lambda - 2(\ell-1)\rho)$). Also by Corollary 4.3(i), $\mathrm{Soc}(M_{\varepsilon,r}(\lambda))$ contains $I_r\widehat{v}_\lambda$. Therefore there is some x_r in U_r such that $x_rI_r\widehat{v}_\lambda$ is in the highest weight space of $\mathrm{Soc}(M_{\varepsilon,r}(\lambda))$. In other words, $x_rI_r\widehat{v}_\lambda \in (M_{\varepsilon,r}(\lambda))^\mu$, the μth weight space of $M_{\varepsilon,r}(\lambda)$. Now let f_r be the injective U_r module homomorphism from $M_{\varepsilon,r}(\lambda)$ to $M_\varepsilon(\lambda)$ described in (3.2.3), then $f_r(\widehat{v}_\lambda) = v_\lambda$.

So, $x_rI_rv_\lambda = f_r(x_rI_r\widehat{v}_\lambda) \in M_\varepsilon(\lambda)^\mu$.

This shows that for each r,

$$U_\varepsilon I_r v_\lambda \cap (M_\varepsilon(\lambda))^\mu \neq (0)$$

and is a finite dimensional $\mathbb{C}$-vector space (since $(M_\varepsilon(\lambda))^\mu$ is finite dimensional).

From Corollary (4.2), we have the descending chain of submodules

$$U_\varepsilon I_1 v_\lambda \cap (M_\varepsilon(\lambda)) \supseteq U_\varepsilon I_2 v_\lambda \cap (M_\varepsilon(\lambda))^\mu \supseteq \dots .$$

Hence its intersection which is just $I \cap M_\varepsilon(\lambda)^\mu$ is non-zero which implies that $I \neq 0$. Since $\mathrm{Soc}(M_\varepsilon(\lambda)) \supseteq I \neq 0$, it follows that $\mathrm{Soc}(M_\varepsilon(\lambda)) \neq 0$.

Hence the theorem. □

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Department of Mathematics
Madurai Kamaraj University
Madurai 625 021
India