ON TWO PROBLEMS OF H. NEUMANN AND J. WIEGOLD ABOUT LINKED PRODUCTS OF GROUPS

S.V. IVANOV AND A.YU. OL'SHANSKII

The authors give affirmative answers to two questions of H. Neumann and Wiegold on linked embeddings of groups.

1.

In the paper [3] of Hanna Neumann and J. Wiegold, a concept of linked product of groups was introduced and studied. A group L is called the *linked product* of groups G_1 and G_2 with kernels $K_1 \trianglelefteq G_1$ and $K_2 \trianglelefteq G_2$ (being normal subgroups of G_1 and G_2 respectively) if the following conditions are satisfied:

(i) L contains G_1 and G_2 as subgroups and $G_1 \cap G_2 = 1$ in L.

(ii) $L = \langle G_1, G_2 \rangle$, that is, L is generated by subgroups G_1 and G_2.

(iii) The normal closure G_1^L of G_1 in L meets G_2 in K_2 and the normal closure G_2^L of G_2 meets G_1 in K_1.

The question about the existence of a linked product of given groups G_1 and G_2 with given kernels K_1 and K_2 was discussed in detail in [3, 4] and answered in the affirmative in a number of special cases. These cases gave the authors of [3] a background for the following conjecture: A linked product L exists whatever nontrivial groups G_1, G_2 and their kernels K_1, K_2 are chosen except for the case where G_1 and G_2 have order 2 (we write $|G_1| = |G_2| = 2$) and one of the kernels K_1 and K_2 has order 2 while the order of the other is 1. It is easy to see that in the latter case there exists no linked product.

The following theorem confirms the conjecture.

THEOREM 1. *Let G_1 and G_2 be nontrivial groups, $K_1 \trianglelefteq G_1$ and $K_2 \trianglelefteq G_2$ their normal subgroups and neither of the two following possibilities holds:*

(i) $|G_1| = |G_2| = |K_1| = 2$ *and* $|K_2| = 1$

(ii) $|G_1| = |G_2| = |K_2| = 2$ *and* $|K_1| = 1$.

Then there exists a linked product L of the groups G_1 and G_2 with kernels K_1 and K_2.

In [3], Neumann and Wiegold posed another problem: can one find, given nontrivial groups G_1 and G_2, a simple group $S = \langle G_1, G_2 \rangle$ in which $G_1 \cap G_2 = 1$. (This is a stronger form of the previous question, since S is a simple linked product of G_1 and G_2 with kernels G_1 and G_2.) Clearly, the question is answered in the negative if $|G_1| = |G_2| = 2$. It appears that this is the single such case.

THEOREM 2. *Let G_1 and G_2 be nontrivial groups the product of whose orders is not 4.*

(a) *Then there exists a simple linked product L of the groups G_1 and G_2 (whose kernels are of course G_1 and G_2).*

(b) *Additionally suppose that G_1 and G_2 are finitely generated. Then there are uncountably many non-isomorphic simple linked products of groups G_1 and G_2.*

Both theorems can be proved simultaneously, using a technique of graded diagrams as treated in [5]. However the first theorem has a very short proof using arguments of ordinary small cancellation theory. We present this proof which, moreover, helps in understanding the idea of the proof of the second theorem. To prove the last theorem we need [5] and additional arguments of [1] or [6] in the case where the groups G_1 and G_2 have involutions. We choose [1] as it gives a shorter way. So our exposition in the third section depends heavily on [5] and [1].

<center>2.</center>

In this section, imposing defining relations on the free product $F = G_1 * G_2$, we construct a linked product L of given nontrivial groups G_1 and G_2 with kernels K_1 and K_2.

One can assume that $|G_1| \geqslant 2$ and $|G_2| \geqslant 3$ and fix elements $a \in G_1 \setminus 1$ and $b, c \in G_2 \setminus 1$, where $bc \neq 1$. Also, let k, l be some coprime integers such that $k, l \geqslant 30$. For every $x \in K_1 \setminus 1$ and $y \in K_2 \setminus 1$ impose the following relations on F:

(1) $$\left(bcabab^{-1}a^{-1}c^{-1}a^{-1}b^{-1}x\right)^k = 1$$

(2) $$\left(bcabababab^{-1}a^{-1}b^{-1}a^{-1}c^{-1}a^{-1}b^{-1}x\right)^l = 1$$

and

(3) $$\left(abcac^{-1}a^{-1}b^{-1}a^{-1}y\right)^k = 1$$

(4) $$\left(abcac^{-1}ab^{-1}a^{-1}ba^{-1}b^{-1}a^{-1}y\right)^l = 1$$

Let r_1, r_2 denote two arbitrary cyclic permutations of relators (1), (2), (3), (4) or inverse words. We can write $r_1 = u_1^m$, $r_2 = u_2^n$ in F where $m, n \geqslant 30$. It is clear that $r_1 = r_2$, if the length $|v|$ (that is, the length of the normal form in the free product F) of the maximal common prefix v of words r_1 and r_2 is greater than $2 \max(|u_1|, |u_2|)$. Otherwise $|v| \leqslant (1/9) \min(|r_1|, |r_2|)$, as $k, l \geqslant 30$. Therefore the condition $C'(1/6)$ holds for relators (1), (2), (3), (4). Hence the normal closure N of all relators (1), (2), (3), (4) in F satisfies the following conditions: $G_1 \cap N = 1$, $G_2 \cap N = 1$ and $G_1 N \cap G_2 N = 1$ (see [2, Chapter 5]), that is $G_1 \cap G_2 = 1$ in $L = F/N$.

By the definition of the relators (1), (2), (3), (4), the subgroup N is contained in the kernel of the natural homomorphism onto the direct product: $F \to (G_1/K_1) \times (G_2/K_2)$. Therefore $G_2^L \cap G_1 \leqslant K_1$ in L. On the other hand, $x^k \in G_2^L$ for each $x \in K_1$ in view of (1), as

$$bcabab^{-1}a^{-1}c^{-1}a^{-1}b^{-1} \in G_2^F.$$

By the same reasoning (see (2)) $x' \in G_2^L$, $x \in G_2^L$, as $(k, l) = 1$. Consequently, $G_2^L \cap G_1 = K_1$. In a similar way, $G_1^L \cap G_2 = K_2$, and Theorem 1 is proved. Moreover, the construction of the group L is effective by standard small cancellation arguments.

<div align="center">3.</div>

To prove Theorem 2, we need an infinite increasing sequence of rather large primes $P = (p_1, p_2, \ldots)$ (that is, $p_i > 10^{10}$ for $i = 1, 2, \ldots$).

Let $L(0) = F = G_1 * G_2$ and the set $R(0)$ of relators of rank 0 be empty.

Defining the sets $R(i)$ and the groups $L(i)$ by induction on i, we suppose that $R(i-1)$ and $L(i-1)$ have already been constructed.

Let $M(i)$ be a set of words of length i over the alphabet $(G_1 \cup G_2) \setminus 1$ maximal with respect to the following properties:

M1. If u and v are different elements of $M(i)$, then u is not conjugate to $v^{\pm 1}$ in $L(i-1)$.

M2. Any $u \in M(i)$ is not conjugate in $L(i-1)$ to a power of a word w with $|w| \leqslant \max(1, i-1)$.

M3. Each $u \in M(i)$ is not equal in $L(i-1)$ to a product of two elements of order 2.

Then we define $R(i)$ by $R(i) = R(i-1)$ if $M(i) = \emptyset$. Otherwise, $R(i) = R(i-1) \cup \{u^p \mid u \in M(i)\}$, where p is the first prime of the sequence P, which was not used at previous steps.

Now we give the group $L(i)$ by defining relators from $R(i)$:

$$L(i) = \langle F \parallel R(i) \rangle.$$

Since for any $i \geqslant 1$ we have already defined the sets $R(i)$, we can give the following limit group $L(\infty)$:

$$L(\infty) = \langle F \parallel R(\infty) = \bigcup_{i=1}^{\infty} R(i) \rangle.$$

As in [5, Section 36] it is proved that G_1 and G_2 embed properly in $L(\infty)$ and intersect trivially. (The only difference from [5] is the infinity of the set P of exponents of periods $u \in M = \bigcup_{i=1}^{\infty} M(i)$. This does not affect the proof.)

The paper [1] contains a construction of a sequence of elements $E_1, E_2, \ldots \in F$ (in the case where $|G_1| \geqslant 2$, $|G_2| \geqslant 3$) such that $|E_j| \to \infty$ if j tends to ∞ and the following statement holds.

LEMMA 1. *Let $q = t_1 E t_2 E^{-1}$, where $E = E_j$ for some $j = 1, 2, \ldots$. Assume that the word q is conjugate in $L(\infty)$ to a word u and $|t_1| + |t_2| < 0.01\,|E|$. Then the inequality $|u| < 0.01\,|E|$ involves equalities $t_1 = t_2 = u = 1$ in the group $L(\infty)$.*

As a matter of fact, [1, Lemma 6] deals with another group (not with $L(\infty)$), and it demands a stronger inequality $|u| \leqslant 1$, but the proof of [1, Lemma 6] can be repeated under the hypotheses of Lemma 1 without any change.

Now consider a nontrivial normal subgroup N of $L = L(\infty)$. The assumption that $s^2 = 1$ for every element $s \in N$ gives us one of two inclusions: $N \leqslant G_1$ or $N \leqslant G_2$ by [5, Lemmas 36.4 and 18.1]. However both inclusions are impossible as $x G_j x^{-1} \cap G_j = 1$ if $j = 1, 2$ and $x \notin G_j$. (See [1, Lemma 4] or [6, Theorem 2, item 4]).

So, one can choose $s \in N$ such that $s^2 \neq 1$. Let v be any nontrivial element of F. Choose $E = E_j$ such that $|E| > 10^3 \max(|v|, |s|)$ and consider the word $w = E s E^{-1} v$. Then w represents a nontrivial element of L, which is conjugate to a power of some word $u \in M(i)$ for some i, as it was explained in [1]. So $w^p = 1$ in L for some $p \in P$, and $v^p \in N$, because $E s E^{-1} \in N$.

One can repeat the previous construction of the word w, choosing $E' = E_k$ such that $|E'| > 100 p\,|u|$. The word w' obtained is conjugate to a power $(u')^m$ of some $u' \in M(i')$. Note that $i' > i$, because otherwise $|E'| > 100m\,|u'|$ and $w' = 1$ in L by Lemma 1.

Therefore $v^{p'} \in N$ for $p' > p$ also. This shows the inclusion $v \in N$ for every $v \in L$. Consequently, $N = L$, and the simplicity of the group L is proved.

In order to prove part (b) of Theorem 2, consider quotients $L = F/R$ and $L' = F/R'$ constructed with the help of different sequences P and P' of primes. Let p and p' be the first primes in P and P' to differ. Then for some i and $u \in M(i)$ we have $u^p = 1$ in L and $u^{p'} = 1$ in L' by definitions of L and L'. Also $u \neq 1$ in L. This can be explained exactly as in [5, Theorem 36.4]. Therefore $R \neq R'$,

and we have an uncountable set of normal subgroups $R \lhd F$ such that $L = F/R$ satisfies the conclusion of Theorem 2. Finally, note that there is only a countable set of homomorphisms of F onto any fixed countable group L, because F is a finitely generated group. Consequently, there is an uncountable set of nonisomorphic quotients $L = F/R$.

REFERENCES

[1] S.V. Ivanov, 'On periodic products of groups' (to appear).

[2] R. Lyndon and P. Schupp, *Combinatorial group theory* (Springer-Verlag, Berlin, Heidelberg, New York, 1977).

[3] H. Neumann and J. Wiegold, 'Linked products and linked embeddings of groups', *Math. Z.* **73** (1960), 1–19.

[4] B.H. Neuman and H. Neumann, 'On linked products of groups', *Acta Sci. Math.* **21** (1960), 197–205.

[5] A.Yu. Ol'shanskii, *Geometry of defining relations in groups*, (in Russian) (Nauka, Moscow, 1989).

[6] A. Yu. Ol'shanskii, 'On thrifty embeddings of groups', (in Russian), *Vestnik Moskovsk. Univ.* **2** (1989), 28–34.

Department of Mathematics and Mechanics
Moscow University
Moscow 119899
Russia