MATRIX RATIONAL COMPLETIONS SATISFYING GENERALIZED INCIDENCE EQUATIONS

E. C. JOHNSEN

1. Introduction. Let us consider the following problem. Let there be v elements $x_1, \ldots, x_v$ and v sets $S_1, \ldots, S_v$ such that every set contains exactly k distinct elements and every pair of sets has exactly λ distinct elements in common. To avoid trivial situations we shall in general assume that $0 < \lambda < k < v - 1$. This is known as a v, k, λ *configuration* or *design*. We can give an equivalent characterization of a configuration in terms of a matrix $A = [a_{ij}]$, called its *incidence matrix*, by writing the elements $x_1, \ldots, x_v$ in a row and the sets $S_1, \ldots, S_v$ in a column and setting $a_{ij} = 1$ if x_j is in S_i and $a_{ij} = 0$ if x_j is not. This matrix A, of order v, is composed entirely of 0's and 1's and by the conditions of the problem is easily seen to satisfy the matrix equation

$$AA^T = (k - \lambda)I + \lambda J \equiv B, \tag{1.1}$$

where A^T is the transpose of A, I is the identity matrix of order v, and J is the matrix consisting entirely of 1's of order v. Equation (1.1) is known as the *incidence equation* for a v, k, λ design. Ryser showed in **(7)** that $\lambda(v - 1) = k(k - 1)$ and that A is *normal*, i.e., $AA^T = A^TA = B$.

Considering A as a matrix with rational entries, equation (1.1) with $AA^T = AIA^T$ asserts that B is *rationally congruent* to the identity. Here the Minkowski-Hasse theory for the rational congruence of two integral symmetric matrices may be applied. Using the fundamental Minkowski-Hasse theorem, Bruck and Ryser **(2)** proved a non-existence theorem for finite projective plane designs. Later, Chowla and Ryser **(3)**, using elementary methods and not the Minkowski-Hasse theory, proved a non-existence theorem for general v, k, λ designs which included the Bruck-Ryser theorem as a special case. Still later, Shrikhande **(9)** applied the Minkowski-Hasse theory to equation (1.1) for general v, k, λ to obtain results which exclude the same values of these parameters as the theorem of Chowla and Ryser. So far, this general theorem accounts for all known excluded configurations.

We now go further into this situation and ask two questions. When is there a normal rational solution to the incidence equation? Given a design-consistent

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start of r complete rows or columns of 0's and 1's, when is this start rationally completable to a normal solution to the incidence equation? The first question was answered for a special case of finite projective planes by Albert **(1)**. He proved for plane cases of orders $n = a^2 + b^2$, where a and b are integers, that a normal rational A satisfying equation (1.1) exists. The proof was constructive and from it A could be obtained. Later, both questions were answered completely by Hall and Ryser **(5)**. They extended Albert's conclusion to all design cases whose matrices B are rationally congruent to the identity, i.e., all cases not excluded by Chowla and Ryser **(3)**. In that same paper they also proved a stronger theorem, which stated that if the matrix B for any design case is rationally congruent to the identity and we have a 0, 1 entry start in r complete rows (columns), $0 < r < v$, whose row (column) inner products are consistent with those of a design, then that start can be rationally completed to a matrix A of order v which is a normal solution to the incidence equation. These proofs made some use of the combinatorial aspects of the problem, but for the most part they involved matrix theory.

The question "If $AA^T = B$, what is A^TA?" now leads to a further consideration of this situation. Ryser **(8)**, in investigating the integral solutions to (1.1), gave a hint to the answer to this question in the proof of his Theorem 2.2. His proof indicated that if a rational matrix A satisfies (1.1), then A^T satisfies a generalization of (1.1) where B is replaced by a matrix in a class $\mathfrak{B}$ of matrices over the rationals of which B is a member. In this larger context we first generalize the preliminary and main results in **(5)**. We then conclude with a result whose conjecture is readily motivated by this work, namely, that if the matrices in $\mathfrak{B}$, for a given v, k, λ, are rationally congruent to the identity and we have a rational entry start in r rows and s columns, $0 < r < v$, $0 < s < v$, which satisfies certain necessary consistency conditions with respect to two arbitrary matrices B_1 and B_2 in $\mathfrak{B}$, then that start can be rationally completed to a matrix A of order v such that $AA^T = B_1$ and $A^TA = B_2$. When the rational entries in the start are 0's and 1's and $B_1 = B_2 = B$, then we get a new rational completion result related to v, k, λ designs. In what follows, capital letters will generally denote matrices. We denote rational congruence by $\sim$ and rational incongruence by $\nsim$. For the most part we shall be concerned specifically with the field of rationals.

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2. Basic theorems. The main result of this section, Theorem 2.2, is a slight generalization of a fairly well known theorem proved in Jones **(6**, p. 14, Theorem 7**)**. However, we shall need the generalization to obtain our main working result, Corollary 2.3. We give here an equivalent statement of the theorem in Jones **(6)**.

THEOREM 2.1. *Let X and Y be two $m \times n$ matrices and U be an $n \times n$ matrix,*

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all with entries in a field F of characteristic $\neq 2$ such that $XUX^T = YUY^T = Z$, where U is non-singular and symmetric and

$$\text{rank } X = \text{rank } Y = \text{rank } Z = m \leqslant n.$$

Then there exists an $n \times n$ matrix Q with entries in F for which $QUQ^T = U$ and $XQ = Y$.

We use this result to obtain the following generalization.

THEOREM 2.2. *Let X and Y be two $m \times n$ matrices and U be an $n \times n$ matrix, all with entries in a field F of characteristic $\neq 2$ such that $XUX^T = YUY^T = Z$, where U is non-singular and symmetric and*

$$\text{rank } X = \text{rank } Y = \text{rank } Z.$$

Then there exists an $n \times n$ matrix Q with entries in F for which $QUQ^T = U$ and $XQ = Y$.

Proof. Let the common rank of X, Y, and Z be $r \geqslant 0$. If $r = 0$, then $X = Y = 0$ and we may take $Q = I$. Now suppose $r \geqslant 1$. Since Z is symmetric, there exists an $r \times r$ principal submatrix Z_r of order and rank $r \leqslant \min(m, n)$. Then we have

$$X_r UX_r^T = Y_r UY_r^T = Z_r,$$

where X_r and Y_r are the corresponding $r \times n$ submatrices of X and Y, respectively. Since

$$r = \text{rank } Z_r \leqslant \text{rank } X_r, \text{rank } Y_r \leqslant r,$$

we have

$$\text{rank } X_r = \text{rank } Y_r = \text{rank } Z_r = r \leqslant n.$$

Hence, by Theorem 2.1 there is an $n \times n$ matrix Q with entries in F for which $QUQ^T = U$ and

$$X_r Q = Y_r. \tag{2.1}$$

We shall now show that if $\mathbf{x}$ and $\mathbf{y}$ are corresponding row vectors of X and Y, respectively, where $\mathbf{x}$ is not in X_r and $\mathbf{y}$ is not in Y_r, then $\mathbf{x}Q = \mathbf{y}$ as well. Let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be the row vectors of X_r and $\mathbf{y}_1, \ldots, \mathbf{y}_r$ be those of Y_r. Now $\mathbf{x}$ is linearly dependent upon $\mathbf{x}_1, \ldots, \mathbf{x}_r$ and $\mathbf{y}$ is linearly dependent upon $\mathbf{y}_1, \ldots, \mathbf{y}_r$. Let

$$\mathbf{x} = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{r} \beta_i \mathbf{y}_i,$$

where $\alpha_i, \beta_i \in F$. Then for each $\mathbf{x}_j$ and $\mathbf{y}_j$,

$$\sum_{i=1}^{r} \alpha_i \mathbf{x}_i U\mathbf{x}_j^T = \mathbf{x}U\mathbf{x}_j^T = \mathbf{y}U\mathbf{y}_j^T = \sum_{i=1}^{r} \beta_i \mathbf{y}_i U\mathbf{y}_j^T = \sum_{i=1}^{r} \beta_i \mathbf{x}_i U\mathbf{x}_j^T$$

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since $XUX^T = YUY^T$; hence

$$\sum_{i=1}^{r} (\alpha_i - \beta_i)\mathbf{x}_i U\mathbf{x}_j^T = 0, \qquad j = 1, \ldots, n. \tag{2.2}$$

Now (2.2) is a homogeneous system of linear equations with matrix

$$[\mathbf{x}_i U\mathbf{x}_j^T]^T = X_r UX_r^T = Z_r,$$

which is non-singular. Hence (2.2) has only the trivial solution $\alpha_i - \beta_i = 0$ or $\alpha_i = \beta_i$, $i = 1, \ldots, r$. Thus

$$\mathbf{x}Q = \sum_{i=1}^{r} \alpha_i \mathbf{x}_i Q = \sum_{i=1}^{r} \beta_i \mathbf{y}_i = \mathbf{y}.$$

This together with (2.1) yields $XQ = Y$.

We shall be using this result in the following form.

COROLLARY 2.3. *Let X and Y be two $m \times n$ matrices with entries in a formally real field F such that $XX^T = YY^T$. Then there exists an $n \times n$ orthogonal matrix Q with entries in F such that $XQ = Y$.*

Proof. We take $U = I$ in Theorem 2.2. Let $\mathbf{x}_1, \ldots, \mathbf{x}_m$ be the row vectors of X with components in a formally real field F. Now the rank of the Gramian $[(\mathbf{x}_i, \mathbf{x}_j)]$ of these vectors is equal to the dimension of the space generated by the vectors; see **(4)** for a discussion of Gramians. Hence for a formally real field F, if $XX^T = YY^T = Z$, then rank X = rank Y = rank Z and we have the corollary by Theorem 2.2.

Corollary 2.3 has a geometric interpretation, particularly if F is the real or the rational field. Let F be one of these fields, and let $V_n(F)$ denote an n-dimensional vector space over F with the standard inner product relative to its basis. Let us call two sets of vectors in $V_n(F)$ *isometric* if they can be put into a one-to-one correspondence which preserves inner products. Corollary 2.3 then states that for each pair of isometric sets of vectors in $V_n(F)$ there is a motion (rotation or reflection) of $V_n(F)$ with a matrix representation over F which carries either set into the other.

3. The class of matrices $\mathfrak{B}$.

We define a $v \times v$ matrix

$$B(\mathbf{u}), \qquad \mathbf{u} = (u_1, \ldots, u_v),$$

as

$$B(\mathbf{u}) \equiv \lambda\mathbf{u}^T\mathbf{u} + (k - \lambda)I = [\lambda u_i u_j + (k - \lambda)\delta_{ij}],$$

where the u_i's are rational and satisfy

$$\sum_{i=1}^{v} u_i^2 = \mathbf{u}\mathbf{u}^T = v,$$

δ_{ij} is the Kronecker delta, and where $(v - 1)\lambda = k(k - 1), 0 < \lambda < k < v - 1$, and v, k, λ are integers. We note that $B(-\mathbf{u}) = B(\mathbf{u})$ and that $B(\mathbf{x}) = B(\mathbf{u})$

implies $\mathbf{x} = \pm\mathbf{u}$. Let us denote the class of all such matrices $B(\mathbf{u})$ for a fixed set of v, k, λ parameter values by $\mathfrak{B}$. We note that in the incidence equation (1.1), $B \equiv B(\mathbf{1})$, where $\mathbf{1} = (1, \ldots, 1)$. The equations $XX^T = B(\mathbf{u})$ and $Y^TY = B(\mathbf{w})$ where X and Y are of order v and have rational entries are then generalizations of the incidence equation.

LEMMA 3.1. *Any two matrices in $\mathfrak{B}$ are rationally orthogonally congruent.*

Proof. Let $B(\mathbf{u})$ and $B(\mathbf{w})$ be arbitrary in $\mathfrak{B}$. Now $\mathbf{u}\mathbf{u}^T = \mathbf{w}\mathbf{w}^T = v$. Hence by Corollary 2.3 there exists a rational orthogonal matrix Q of order v such that $\mathbf{u}Q = \mathbf{w}$. Then by a straightforward computation

$$Q^TB(\mathbf{u})Q = B(\mathbf{w}).$$

By Lemma 3.1 either all matrices in $\mathfrak{B}$, or none, are rationally congruent to the identity I. Hence we may write without ambiguity either $\mathfrak{B} \sim I$ or $\mathfrak{B} \nsim I$.

LEMMA 3.2. *Every $B(\mathbf{u})$ in $\mathfrak{B}$ is non-singular;*

$$B^{-1}(\mathbf{u}) = (k - \lambda)^{-1}[I - \lambda k^{-2}\mathbf{u}^T\mathbf{u}] \tag{3.1}$$

and

$$\det B(\mathbf{u}) = k^2(k - \lambda)^{v-1}. \tag{3.2}$$

Proof. We have

$$\begin{aligned}[\lambda\mathbf{u}^T\mathbf{u} + (k - \lambda)I](k - \lambda)^{-1}[I - \lambda k^{-2}\mathbf{u}^T\mathbf{u}]\\ = (k - \lambda)^{-1}[\lambda - \lambda^2k^{-2}v - \lambda k^{-2}(k - \lambda)]\mathbf{u}^T\mathbf{u} + I,\end{aligned}$$

and replacing $k - \lambda$ by $k^2 - \lambda v$, we see that $B(\mathbf{u})$ is non-singular and that its inverse is given by (3.1). Since, by Lemma 3.1, any two matrices in $\mathfrak{B}$ are orthogonally congruent, all the matrices in $\mathfrak{B}$ have the same determinant.By computing the determinant of $B = (k - \lambda)I + \lambda J$ we easily find the value given in (3.2).

THEOREM 3.3. *Let X be a $v \times v$ rational matrix satisfying $XX^T = B(\mathbf{u})$. Then $X^TX = B(\mathbf{w})$, where $\mathbf{w} = \epsilon k^{-1}\mathbf{u}X$, $\epsilon = 1$ or -1.*

Proof. We have $XX^T = B(\mathbf{u})$ where $B(\mathbf{u})$, and hence also X, is non-singular. Hence we may write

$$X^TB^{-1}(\mathbf{u}) = X^{-1},$$

which by Lemma 3.2 becomes

$$X^T[I - \lambda k^{-2}\mathbf{u}^T\mathbf{u}] = (k - \lambda)X^{-1}. \tag{3.3}$$

Multiplying each side of (3.3) on the right by X yields

$$X^TX - \lambda k^{-2}(\mathbf{u}X)^T(\mathbf{u}X) = (k - \lambda)I, \tag{3.4}$$

and setting $\mathbf{w} = \epsilon k^{-1}\mathbf{u}X$, $\epsilon = 1$ or -1, in (3.4) we obtain

$$X^TX = \lambda\mathbf{w}^T\mathbf{w} + (k - \lambda)I.$$

Now

$$\mathbf{w}\mathbf{w}^T = k^{-2}\mathbf{u}XX^T\mathbf{u}^T = k^{-2}\mathbf{u}[\lambda\mathbf{u}^T\mathbf{u} + (k-\lambda)I]\mathbf{u}^T = k^{-2}v[\lambda v + k - \lambda] = v.$$

Hence $X^TX = B(\mathbf{w}) \in \mathfrak{B}$, which proves the theorem.

COROLLARY 3.4. *For a $v \times v$ rational matrix X, any two of the following three conditions imply the third:*

(a) $$XX^T = B(\mathbf{u}),$$

(b) $$X^TX = B(\mathbf{w}),$$

(c) $$\mathbf{u}X = \epsilon_1 k\mathbf{w}, \qquad \epsilon_1 = 1 \text{ or } -1,$$

and likewise for (a), (b), *and*

(d) $$\mathbf{w}X^T = \epsilon_2 k\mathbf{u}, \qquad \epsilon_2 = 1 \text{ or } -1.$$

Proof. By Theorem 3.3, (a) and (b) imply (c), and (a) and (c) imply (b). Now assume (b) and (c). Applying Theorem 3.3 to X^T we see that there exists a $\mathbf{z}$ such that

(3.5) $$XX^T = B(\mathbf{z}).$$

Applying the theorem again to (3.5) and (b), we have

(3.6) $$\mathbf{z}X = \epsilon_* k\mathbf{w}, \qquad \epsilon_* = 1 \text{ or } -1,$$

and combining (3.6) and (c), we have

$$\mathbf{z}X = \pm\mathbf{u}X.$$

Since X is non-singular, $\mathbf{z} = \pm\mathbf{u}$, whence by (3.5) $XX^T = B(\mathbf{u})$, which is (a). We obtain the same result for the conditions (a), (b), and (d) by interchanging X and X^T, and $\mathbf{u}$ and $\mathbf{w}$, in conditions (a), (b), and (c).

COROLLARY 3.5. *Let X be a $v \times v$ rational matrix satisfying* (a) *and* (b) *in Corollary* 3.4. *Then X satisfies* (c) *and* (d) *there with $\epsilon_1 = \epsilon_2$.*

Proof. By Corollary 3.4 we know that $\mathbf{u}X = \epsilon_1 k\mathbf{w}$ and $\mathbf{w}X^T = \epsilon_2 k\mathbf{u}$ where $\epsilon_1, \epsilon_2 = 1$ or -1. Then

$$\begin{aligned}\epsilon_1\epsilon_2 k^2\mathbf{u} = \epsilon_1 k\mathbf{w}X^T &= \mathbf{u}XX^T = \mathbf{u}[\lambda\mathbf{u}^T\mathbf{u} + (k-\lambda)I] \\ &= (\lambda v + k - \lambda)\mathbf{u} = k^2\mathbf{u}.\end{aligned}$$

Since $\mathbf{u} \neq (0, \ldots, 0)$, $\epsilon_1\epsilon_2 = 1$ or $\epsilon_1 = \epsilon_2$.

We may now prove the following general theorem for $\mathfrak{B}$.

THEOREM 3.6. *Suppose $\mathfrak{B} \sim I$, and let $B(\mathbf{u})$ and $B(\mathbf{w})$ be arbitrary in $\mathfrak{B}$. Then there exists a $v \times v$ rational matrix A such that $AA^T = B(\mathbf{u})$ and $A^TA = B(\mathbf{w})$.*

Proof. Since $B(\mathbf{u}) \sim I$, there exists a $v \times v$ rational matrix C such that $CC^T = B(\mathbf{u})$. By Theorem 3.3, $C^TC = B(\mathbf{x})$ for some $B(\mathbf{x})$ in $\mathfrak{B}$. Now

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$\mathbf{x}\mathbf{x}^T = \mathbf{w}\mathbf{w}^T = v$. Hence by Corollary 2.3, there exists a $v \times v$ rational orthogonal matrix Q such that $\mathbf{x}Q = \mathbf{w}$. Let $A = CQ$. Then

$$AA^T = CQ(CQ)^T = CC^T = B(\mathbf{u}).$$

By Corollary 3.4 we have

$$\mathbf{u}A = \mathbf{u}CQ = \epsilon k\mathbf{x}Q = \epsilon k\mathbf{w}, \qquad \epsilon = 1 \text{ or } -1,$$

and hence $A^TA = B(\mathbf{w})$.

The following result of Hall and Ryser **(5)** then becomes a corollary to this theorem.

COROLLARY 3.7. *Suppose $B \sim I$. Then there exists a $v \times v$ rational matrix A such that $AA^T = A^TA = B$.*

4. Rational completions. Let $\mathbf{x} = (x_1, \ldots, x_v)$, where the x_i's are rational, satisfy $\mathbf{x}\mathbf{x}^T = v$. We then define $\mathbf{x}_m{}^i = (x_1, \ldots, x_m)$ and $\mathbf{x}_m{}^t = (x_{v-m+1}, \ldots, x_v)$, where i and t suggest "initial" and "terminal" with reference to the subset of m entries taken from $\mathbf{x}$. Then $B(\mathbf{x}_m{}^i)$ and $B(\mathbf{x}_m{}^t)$ will denote the initial principal and terminal principal $m \times m$ submatrices of $B(\mathbf{x})$ respectively. We further let $\mathbf{x}_m$, without any attached i or t, denote any m-tuple of rational entries such that $\mathbf{x}_m \mathbf{x}_m{}^T \leqslant v$ and define

$$B(\mathbf{x}_m) \equiv \lambda \mathbf{x}_m{}^T\mathbf{x}_m + (k - \lambda)I_m.$$

We define a *$\Gamma_s{}^r$-array* as the configuration from a $v \times v$ matrix consisting of the first r rows and the first s columns. We let Γ^r denote the $r \times v$ row submatrix and Γ_s the $v \times s$ column submatrix of $\Gamma_s{}^r$.

We can now prove a general rational completion theorem for a given consistent rational start of s complete columns of entries.

THEOREM 4.1. *Let $\mathfrak{B} \sim I$ and let $B(\mathbf{u})$ and $B(\mathbf{w})$ be arbitrary in $\mathfrak{B}$. Suppose A_s, $0 < s < v$, is a $v \times s$ rational matrix satisfying*

$$A_s{}^TA_s = B(\mathbf{w}_s{}^i).$$

Then

$$\mathbf{u}A_s = \epsilon k\mathbf{w}_s{}^i, \qquad \epsilon = 1 \text{ or } -1, \tag{4.1}$$

is a necessary and sufficient condition that there exist a $v \times v$ rational matrix A containing A_s as its first s columns and satisfying $AA^T = B(\mathbf{u})$ and $A^TA = B(\mathbf{w})$.

Proof. Let (4.1) be satisfied. Since $\mathfrak{B} \sim I$, there exists by Theorem 3.6 a $v \times v$ rational matrix C such that $CC^T = B(\mathbf{u})$ and $C^TC = B(\mathbf{w})$, where we may choose the signs of the entries of C such that by Corollary 3.4 $\mathbf{u}C = \epsilon k\mathbf{w}$, whence

$$\mathbf{u}C_s = \epsilon k\mathbf{w}_s{}^i.$$

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Let

$$P = [C_s, \mathbf{u}^T] \qquad \text{and} \qquad R = [A_s, \mathbf{u}^T].$$

Then

$$P^TP = \begin{bmatrix} C_s^TC_s & (\mathbf{u}C_s)^T \\ \mathbf{u}C_s & \mathbf{u}\mathbf{u}^T \end{bmatrix} = \begin{bmatrix} A_s^TA_s & (\mathbf{u}A_s)^T \\ \mathbf{u}A_s & \mathbf{u}\mathbf{u}^T \end{bmatrix} = R^TR.$$

Hence by Corollary 2.3 there exists a $v \times v$ rational orthogonal matrix Q such that $R^TQ = P^T$ and so $QP = R$. Let $A = QC$. Then A contains A_s as its first s columns,

$$A^TA = C^TQ^TQC = C^TC = B(\mathbf{w}),$$

and since

$$\mathbf{u}A = \mathbf{u}QC = \mathbf{u}C = \epsilon k\mathbf{w},$$

we have, by Corollary 3.4, $AA^T = B(\mathbf{u})$. This shows that the condition is sufficient. The necessity of the condition follows trivially from Corollary 3.4 since $A = [A_s, *]$, $AA^T = B(\mathbf{u})$, and $A^TA = B(\mathbf{w})$ imply $\mathbf{u}A = \epsilon k\mathbf{w}$ implies $\mathbf{u}A_s = \epsilon k\mathbf{w}_s^{\,i}$, where $\epsilon = 1$ or -1.

It is clear that Theorem 4.1 could also have been stated in terms of a row start instead of a column start. When this is done, the following result of Hall and Ryser **(5)** becomes a corollary to the theorem.

COROLLARY 4.2. *Let $B \equiv B(\mathbf{1}) \sim I$. Suppose $A^r, 0 < r < v$, is an $r \times v$ matrix of 0's and 1's such that*

$$A^rA^{rT} = B(\mathbf{1}_r^{\,i}). \tag{4.2}$$

Then there exists a $v \times v$ rational matrix A having A^r as its first r rows such that $AA^T = A^TA = B$.

Proof. The condition for the rational completability of the row start to the desired matrix is that

$$\mathbf{1}A^{rT} = \epsilon k\mathbf{1}_r^{\,i}, \qquad \epsilon = 1 \text{ or } -1.$$

Now by (4.2) every row of A^r has exactly k 1's and $v - k$ 0's; hence the condition is satisfied.

Under the same necessary and sufficient condition as in the theorem we may sacrifice the satisfaction of one of the generalized incidence equations and, instead, obtain a rational solution to the other equation, which contains a prescribed $\Gamma_s^{\,r}$-array.

COROLLARY 4.3. *Let $\mathfrak{B} \sim I$ and let $B(\mathbf{u})$ be arbitrary in $\mathfrak{B}$. Suppose $A_s^{\,r}$ is a $\Gamma_s^{\,r}$-array, $0 < r < v$, $0 < s < v$, such that*

$$A^rA^{rT} = B(\mathbf{u}_r^{\,i})$$

and

$$A_s^T A_s = B(\mathbf{w}_s).$$

Then

(4.3) $$\mathbf{u}A_s = \epsilon k \mathbf{w}_s, \qquad \epsilon = 1 \text{ or } -1,$$

is a necessary and sufficient condition that there exist a $v \times v$ rational matrix A containing A_s^r as a Γ_s^r-array and satisfying $AA^T = B(\mathbf{u})$.

Proof. Let (4.3) be satisfied, and let $\tilde{A} = [A_s, X]$ be a matrix such that $\tilde{A}\tilde{A}^T = B(\mathbf{u})$, which is guaranteed to exist by Theorem 4.1. Set

$$A_s = \begin{bmatrix} W \\ Z \end{bmatrix}, \qquad X = \begin{bmatrix} Y \\ V \end{bmatrix}, \qquad \text{and } \tilde{A}^r = [W, Y],$$

where W and Y have r rows. Let $A^r = [W, U]$. Now

$$\tilde{A}^r \tilde{A}^{rT} = A^r A^{rT},$$

whence

$$WW^T + YY^T = WW^T + UU^T$$

or

$$YY^T = UU^T.$$

Hence by Corollary 2.3 there exists a $(v - s) \times (v - s)$ rational orthogonal matrix Q such that $YQ = U$. Set $A = [A_s, XQ]$. Then A contains A_s^r as a Γ_s^r-array and

$$AA^T = A_s A_s^T + XQQ^T X^T = A_s A_s^T + XX^T = B(\mathbf{u}).$$

This shows that the condition is sufficient. The necessity of the condition follows from Theorem 3.3 and Corollary 3.4 since $A = [A_s, *]$ and $AA^T = B(\mathbf{u})$ imply that there exists an $\mathbf{x}$ such that $A^T A = B(\mathbf{x})$ and $\mathbf{u}A = \epsilon_1 k\mathbf{x}$, where $\mathbf{x}_s^i = \epsilon_2 \mathbf{w}_s$, $\epsilon_1, \epsilon_2 = 1$ or -1, which implies that

$$\mathbf{u}A_s = \epsilon_1 k \mathbf{x}_s^i = \epsilon_1 \epsilon_2 k \mathbf{w}_s = \epsilon k \mathbf{w}_s, \qquad \epsilon = \epsilon_1 \epsilon_2 = 1 \text{ or } -1.$$

This corollary readily suggests the following theorem.

THEOREM 4.4. *Let $\mathfrak{B} \sim I$ and let $B(\mathbf{u})$ and $B(\mathbf{w})$ be arbitrary in $\mathfrak{B}$. Suppose A_s^r is a Γ_s^r-array, $0 < r < v$, $0 < s < v$, such that*

$$A^r A^{rT} = B(\mathbf{u}_r^i)$$

and

$$A_s^T A_s = B(\mathbf{w}_s^i).$$

Then

(4.4) $$\mathbf{u}A_s = \epsilon k \mathbf{w}_s^i$$

and

(4.5) $$\mathbf{w}A^{rT} = \epsilon k \mathbf{u}_r^i,$$

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$\epsilon = 1$ or -1, are necessary and sufficient conditions that there exist a $v \times v$ rational matrix A containing A_s^r as a Γ_s^r-array and satisfying $AA^T = B(\mathbf{u})$ and $A^TA = B(\mathbf{w})$.

Proof. Let (4.4) and (4.5) be satisfied. Also let

$$A = [W, U] \quad \text{and} \quad A_s = \begin{bmatrix} W \\ Y \end{bmatrix}.$$

Since $\mathfrak{B} \sim I$, there exists by Theorem 4.1 a $v \times v$ rational matrix

$$\tilde{A} = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

satisfying $\tilde{A}\tilde{A}^T = B(\mathbf{u})$ and $\tilde{A}^T\tilde{A} = B(\mathbf{w})$, where $\tilde{A}^r = [W, X]$ and $\tilde{A}_s = A_s$. Then by Corollaries 3.4 and 3.5 $\mathbf{u}\tilde{A} = \epsilon_* k\mathbf{w}$ and $\mathbf{w}\tilde{A}^T = \epsilon_* k\mathbf{u}$, $\epsilon_* = 1$ or -1. Now

$$\epsilon_* \mathbf{w}\tilde{A}^{rT} = k\mathbf{u}_r{}^i = \epsilon \mathbf{w} A^{rT}$$

or

$$\epsilon_* (\mathbf{w}_s{}^i W^T + \mathbf{w}_{v-s}^t X^T) = \epsilon(\mathbf{w}_s{}^i W^T + \mathbf{w}_{v-s}^t U^T). \tag{4.6}$$

If $\epsilon_* = \epsilon$, then (4.6) becomes

$$\epsilon_* \mathbf{w}_{v-s}^t X^T = \epsilon \mathbf{w}_{v-s}^t U^T, \tag{4.7}$$

and if $\epsilon_* = -\epsilon$ we have from

$$\epsilon_* k\mathbf{w}_s{}^i = \mathbf{u}\tilde{A}_s = \mathbf{u}A_s = \epsilon k \mathbf{w}_s{}^i$$

that $\mathbf{w}_s{}^i = (0, \ldots, 0)$ in $s > 0$ components, so that again (4.6) becomes (4.7). Now

$$\tilde{A}^r\tilde{A}^{rT} = A^r A^{rT},$$

which becomes $WW^T + XX^T = WW^T + UU^T$, or

$$XX^T = UU^T. \tag{4.8}$$

Hence for

$$P = \begin{bmatrix} X \\ \epsilon_* \mathbf{w}_{v-s}^t \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} U \\ \epsilon \mathbf{w}_{v-s}^t \end{bmatrix}$$

of size $(r + 1) \times (v - s)$ we have from (4.7) and (4.8) that

$$PP^T = RR^T.$$

By Corollary 2.3 there then exists a $(v - s) \times (v - s)$ rational orthogonal matrix Q_{v-s} such that $PQ_{v-s} = R$. We now construct the $v \times v$ rational orthogonal matrix

$$Q = I_s \dotplus Q_{v-s},$$

where I_s is the $s \times s$ identity matrix. We set $A = \tilde{A}Q$. Then A contains A_s^r as a Γ_s^r-array and

$$AA^T = \tilde{A}QQ^T\tilde{A}^T = \tilde{A}\tilde{A}^T = B(\mathbf{u}).$$

Now

$$\begin{aligned}\mathbf{u}A &= \mathbf{u}\tilde{A}Q = \epsilon_* k\mathbf{w}Q \\ &= k[\epsilon_* \mathbf{w}_s{}^i I_s, \epsilon_* \mathbf{w}^t_{v-s} Q_{v-s}] \\ &= k[\epsilon_* \mathbf{w}_s{}^t, \epsilon\mathbf{w}^t_{v-s}] \\ &= \epsilon k\mathbf{w},\end{aligned}$$

since either $\epsilon_* = \epsilon$ or $\mathbf{w}_s{}^i = (0, \ldots, 0)$. Hence by Corollary 3.4 we have $A^TA = B(\mathbf{w})$. This shows that the conditions are sufficient. The necessity of the conditions follows trivially since for an A containing $A_s{}^r$ as a $\Gamma_s{}^r$-array where $AA^T = B(\mathbf{u})$ and $A^TA = B(\mathbf{w})$, we must have by Corollaries 3.4 and 3.5 that $\mathbf{u}A = \epsilon k\mathbf{w}$ and $\mathbf{w}A^T = \epsilon k\mathbf{u}$, whence $\mathbf{u}A_s = \epsilon k\mathbf{w}_s{}^i$ and $\mathbf{w}A^{rT} = \epsilon k\mathbf{u}_r{}^i$ where $\epsilon = 1$ or -1.

This theorem yields the following corollary, which is of interest in the study of v, k, λ designs.

COROLLARY 4.5. *Let $B \equiv B(\mathbf{1}) \sim I$. Suppose $A_s{}^r$ is a $\Gamma_s{}^r$-array, $0 < r < v$, $0 < s < v$, composed of 0's and 1's such that*

$$A^rA^{rT} = B(\mathbf{1}_r{}^i) \tag{4.9}$$

and

$$A_s{}^TA_s = B(\mathbf{1}_s{}^i). \tag{4.10}$$

Then there exists a $v \times v$ rational matrix A containing $A_s{}^r$ as a $\Gamma_s{}^r$-array and satisfying $AA^T = A^TA = B$.

Proof. The conditions for the rational completability of the $\Gamma_s{}^r$-array to the desired matrix are that

$$\mathbf{1}A_s = \epsilon k\mathbf{1}_s{}^i$$

and

$$\mathbf{1}A^{rT} = \epsilon k\mathbf{1}_r{}^i,$$

where $\epsilon = 1$ or -1. Now by (4.9) and (4.10), every row of A^r and every column of A_s has exactly k 1's and $v - k$ 0's; hence the conditions are satisfied.

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The Ohio State University and
The University of California, Santa Barbara