"As Philolaos the Pythagorean Said"

Philosophy, Geometry, Freedom

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In his collection of anecdotes, *Lives, Opinions, and Remarkable Sayings of the Most Famous Ancient Philosophers,* Diogenes Laertius devotes a chapter to the life of Zeno of Elea. Zeno's reputation is based on his celebrated paradoxes, amply discussed by Aristotle: *a moving body will never reach its* (pre-defined) *telos, since it first has to cover half* (or more than half) *the remaining distance; the faster will never catch up with the slower, since it first has to get to the point from which the slower has just left.* Zeno's[1] style is laconic, like that of an Aesop fable. *Maxima e minimis*: there is no superfluous word. Everything needed to arrive at the conclusion is explicitly stated.

One of Zeno's most famous arguments is universally known by the name of "Achilles and the Tortoise."

Aristotle refers to it as the "so-called Achilles," adding, somewhat irritated, the sarcastic remark by which Zeno supposedly introduced the famous champion of speed as a tragic hero in his argument just to amuse the audience.

In fact, in Song XXII of the *Iliad,* there is a splendid metaphor vaguely reminiscent of *Achilles.* Achilles wants to kill Hector and chases him around the walls of Troy. But the outcome of the chase is altered by the treachery of the goddess Athena Pallas, who offers Hector her protection. And, *as if in a bad dream,* remarks Homer, the chase becomes a shared nightmare: although the pursuer approaches nearer and nearer to the pursued, he is unable to catch him. And likewise, the pursued cannot rid himself of his pursuer. Finally, the judgment of Zeus, the supreme god, settles matters and the spell is broken. Hector, slower and weaker, betrayed by his divine protectress, is brought face to face with

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brutal reality, (which has itself been dreamed by a blind poet). Achilles, faster and stronger, reaches him and kills him.

In Aristotle's *Poetics, the pursuit of Hector* is cited as an example of poetic license. The miraculous, the absurd, the obvious lie can carry deep poetic truth: *the true lie.*

In his commentary on Aristotle's *Physics*, Simplicius touched briefly on the epithet of *tragic hero* – which Aristotle confers upon Achilles. It would be better to speak of a tragedy transformed into a *comedy* by Zeno, remarks Simplicius, who suggests a new scenario, conferring the role of Homeric hero upon the slowest, most pathetic of all animals, the tortoise.

Aristotle, Simplicius, and, after Simplicius, an uninterrupted chain of commentators have all tried in vain to refute Zeno's paradox, which flies in the face of common sense but forces pure reasoning up against its own limits, confronting it with the absolute impossibility of refuting the paradox through logical argumentation.

Yet what does this argument demonstrate? A futile puzzle, an enigma, an amusing parlor game? Perhaps. But how can we account for the immense fascination it held and continues to hold, on even the most excellent minds?

It is notable that, Zeno's *Achilles* is supported by rigorous and acute reasoning, purely mathematical in structure. Yet most often we do not realize that what renders it so captivating is found in a *hyperouran* (supra-celestial) domain, which at first glance has no link with mathematical reasoning: must we accept or reject the existence of the infinite *in actu*?

The natural place for such a question can be found in the domain of metaphysical speculation. And to be more precise: philosophy. This non-geometrical space is that of an impalpable *substance*, but whose reality is more long-lasting, and more solid than that of objects surrounding us, an immaterial substance one usually calls *freedom*.

In his *Parmenides*, Plato reformulates and discusses Zeno's argument at length. From geometric space, Plato transposes the movement of pursuit to the space of a strange anachronic time.

In the natural process of aging, *the younger* chases *the older*, unable (how could it be otherwise?) to catch up with him in age,

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even though the younger ages faster than the older. Thus the paradoxical or even absurd assertion that faster will never catch up with the slower is turned around and reverts to something worse still than Simplicius's parody: the utterly trivial statement that the younger will never have the same age as the older.

But Plato introduces – parallel to the pursuit of the older by the younger – an inverse and reversed pursuit, in negative time, running from future to present, from the present to past. This time the older chases the younger. In this rejuvenating movement, the older is quicker, the younger slower. Plato thus simultaneously sets in motion two inverse diachronic processes: infinite aging and rejuvenation. In both cases the result is identical; in both cases the relation of equality in age will never be achieved.

Of course, negative time and inverse pursuit are no longer banalities. And the great philologist Otto Apelt devotes over a page of his commentary on *Parmenides* to express his disappointment, vehemently protesting against such an abuse of the absurd, an irresponsible game of thought dealing with the notion of inverse time, which, to his great astonishment, a man with a mind as refined and profound as Plato's devoted himself.

No platonic philologist has succeeded in deciphering the subtle structure of the metaphor so richly presented by Plato. And I know of only one of his readers who succeeds in penetrating the true core of this powerful idea of negative time – James Joyce, who presents the inverse pursuit of younger and older in one of the most seductive and cryptic passages of *Ulysses: "What relation existed between their ages?"* This time it is Stephen Dedalus and Leopold Bloom, two alter egos of Joyce himself, who play the roles of younger and older in a madly surrealistic chase through time.

Negative time concerns Plato only episodically. In a very long passage in the *Statesman*, it is the idea of negative time carrying the political utopia of a society where life is organized according to a structure that is opposed to an established social system. That is the society of *Giants*, of beings begotten by the Earth. The Giants, already elderly, emerge from the body of their mother – and their grave: Earth. The hour of their birth corresponds to the hour of death in an inverted world – our own. Their life is an uninterrupted rejuvenation and the hour of their demise occurs when they reach age zero, the age of inverse birth.

The moral of this gigantic utopia is clear: a society whose structure is the inverse of our own stands as a consistent idea in itself, one which is fully conceivable.

The game Plato imposed upon Parmenides was truly absurd, but – according to specific terms in the Platonic scenario, it was also *laborious* and above all a very *serious game*. For it is possible to interpret the double simultaneous pursuit: unlimited aging and unlimited rejuvenation both converging to the transfinite relation of equality of age (totally unattainable for one and the other).

For the doubly infinite pursuit of the *Parmenides* corresponds exactly to the essential idea of Plato's philosophy, the idea which (according to Aristotle, whose discussion is peppered with critical or even snide remarks) Plato himself referred to as the *infinite dyad* of the *greater* and the *lesser*, of *excess* and *default*.

Behind this magnificent diachronic metaphor which runs through *Parmenides* is the actual mathematical event, accompanied, in the *Achilles*, by a series of difficult but extraordinarily rigorous and vertiginously profound philosophical reflections on that sublime and extravagant subject.

For Plato understood that this cold mathematical reasoning was nothing other than the expression of what Valéry referred to, in a different context, as a great *Event of the mind*.

This concerned, in fact, the brusque intrusion – unpredictable and undesirable – of the *Irrational* in a universe ontologically closed: the universe of the Pythagorean, the universe of *Reason* itself.

In terms closer to everyday language, it asks the following question: can one speak of the *existence* of an *irrational number*, for instance, a number familiar to each one of us, known as the "square root of 2"?

Irrational number is a technical term with a neutral connotation, known by all. The term originates in antiquity, as shown by a passage of Democritus and in the dialogues of Plato that refer to *irrational lines* and *ineffable diagonals*.

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But, as can be surmised from many other passages in Plato's dialogues, the value of *equal age* – that unique and indivisible *Unity* whose precise and indivisible *topos* is located in the domain of the transfinite, that *Unity* toward which the younger and older converge simultaneously, but vainly, along two infinite trajectories – the value of equal age is irrational and ineffable, measurable precisely by the unspeakable *number* which can only be designated by its metalinguistic name, "the square root of 2" – an expression that is as ordinary as it is trivial in the language of everyday.

The geometricians of the Academy all knew the irrefutable Pythagorean demonstration perfectly, according to which the existence of such a number in the universe of the *logos* is completely impossible. Aristotle has passed down to us a variant of this demonstration: the existence of such a number implies an obvious logical contradiction that represents, in his stock of examples, the paradigm of logical absurdity itself. The Stagirite himself carefully avoided pronouncing words such as *irrational* and *ineffable* in the strictly mathematical context.

But Plato – and he alone – accepted the real existence of such an irrational arithmetic object, such as the ineffable word, completely impossible to articulate in a finite discourse of current speech, a unique and indivisible *name* of the number that must correspond, in the object-language, to the term "square root of 2" of arithmetic metalanguage.

In the *Epinomis*, the Stranger of Athens exaltedly and explicitly refers, to these ineffable *numbers* as true *miracles, which surpass the human domain*. Full of pathos, he glorifies this *work that openly displays its divine character to whomever can conceive and understand it*.

But who was capable of understanding and conceiving this divine number?

The professional geometricians of Antiquity would never have accepted the existence of such a demented number – irrational, in the most rigorous etymological sense of the word. Only Theaetetus accepted the term, but limited it exclusively to geometric objects, the length of irrational lines.

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In *Philebus,* Plato devotes long passages to the metaphysical dimension that implies the irrefutable existence of this divine, ineffable number.

This time, the actual existence of the infinite is placed at the center of his thought. Both terms, each unlimited, of the *infinite dyad* – in current mathematical literature expressed in the neutral term "section of Dedekind" – are strictly closed on the left and right by their extremes and converge to each other. They are separated from each other by just one "punctual hole," the common limit of the two – in themselves – unlimited terms.

This hole is evidently a point, an unique and indivisible Unity. A *point* – in the etymological sense of the word, so a puncture, an emptiness of being.

The elements of the two infinite terms of the dyad share a perpetual relationship of inequality, the *largest* and *smallest* and, in relation to their common limit, a relationship of *default* and *excess.* Their infinity precisely defines the *topos* and *situs* of this Unity, representing the non-being that separates both domains of being: the two terms of the dyad. *This non-being is thus knowable* – as Plato emphasizes in the *Parmenides.* He has knowledge of a measure – the correct, absolute exact measure of its place – a measure expressed as a number.

The question knowledge of this number raises pertains of course to Parmenides of Elea: *to be or not to be.* And in the *Sophist,* the Stranger of Elea promises to submit the *words of our father to all, the Great Parmenides, to a commensurate (metria) torture, so as to force him to admit that, in one way or another, we must assign being to the non-being.* Thus one must assign being in this specific space to this limit-place, full of *darkness of the non-being* (an expression that I also take from the *Sophist*), that separates the two domains of being, the set of defaults and excesses of the infinite dyad.

It is the dialectical Synthesis of the *Unity and Infinite, of the limited and unlimited which according to general opinion is certainly one of the most impossible of things to conceive, my dear Protarchus: the Same simultaneously present in the Unity and in the actually infinite multiplicity, the Same reuniting in itself the limit and unlimited. And we must not lose sight of the fact that what is found in the middle, between* *these two unlimited terms of the dyad, is a number. But this, once understood, the Unity is delivered from the infinite and bids it adieu.*

By carrying out this act, *the opposed ones, the Unequal* – the unlimited multiplicity of the Others (*ta alla* – collective singular that one could render by the term *Alterity),* of more and less, of excess and default – *and Equal,* identical and same, *manage to put an end to their hostility, and introducing the accurate measure of commensurability and harmony, produce the Number* (arithmon). *The progeny of these two [opposites], the finite Limit and Unlimited infinite, are joined in a Unity, the generation into being* (genesis eis ousian) *of the just measure by the intermediary of the limit* (tou peratos metron) – a measure whose expression is a number.

The exact measure, thus absolute equality, is manifestly defined by this in-finite dyad, by the two unlimited successions of its two terms, *defaults* and *excesses*: the two terms of the dyad, each an actual infinity of inequality constituting, by a strange process of *equalizing the unequal,* the Unity, a number – *the Unequal as a unique term, a Unity!,* exclaims Aristotle with consternation in a famous passage of his *Metaphysics.*

We could not express it any better. And, actually, in *Parmenides,* Plato himself speaks, in an analogous context, of a *phantasm of equality* (phantasma isotetos).

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I know of only two authors who have managed to decipher this transparent message, coming from a pure mathematical mind. One is the neoplatonicist Porphyry, whose commentary still bears the mark of mathematical naivete, but who nonetheless clearly reveals the place of mathematical passages of *Philebus* in Plato's famous lecture *On the Good*. The other is Georg Cantor, the founder of the theory of infinite sets, whose important work, published between 1879 and 1884, made explicit references to these passages of *Philebus* that his incredible sensitivity for the specific philology of mathematical language let him unhesitatingly recognize and identify the beginning of his own revolutionary conception of the infinite.[2]

This transmutation of non-being, which, regarding the infinite pursuit of the limit, of the One and Equal by the infinity of Others, of excesses and defaults of the unequal, is also found at the center of the dialectical discussion. In the same tone inspired by this *divine madness that goes beyond good common sense of mortals* – that he discusses at length in his *Phaedrus*[3] – Plato speaks of an anachronic movement, a sudden metamorphosis: the *Instantaneous, by its very nature a marvelously strange thing, whose place is found outside time.*

The privileged place Plato reserves for mathematics in the education of the free spirit originates in the importance he gives to forming a political thought based on the idea of justice and fairness. According to him, only mathematics, the *divine geometry*, can lead the spirit toward the *hyperouran* reality where, through the "eyes of the soul," it can acquire knowledge of pure, universal and eternal ideas of Good and Beautiful, Virtue and Justice. This transcendent world is closed to the tyrant.

The central problem of political science is finding the correct measure between good and evil and this is knowledge, Plato writes, that only mastery of the *metretike* can offer: determining the correct measure by the procedure of excess and default. The *metretike* is a branch of geometry, but this geometry is not the same as that used by surveyors and masons. It is a science Plato himself calls *divine geometry*, the only one that can give us knowledge of truth, beauty, virtue – of their exact, pure measure. And the *mathematical pleasure (mathematon hedonas)*, by which we participate in this pure knowledge, is *not mixed with pain* but, it is nevertheless *possessed by a very small number of people.*[4] This science, this art of measuring – as presented in *Philebus* and in the *Statesman* – is presented as the only way to attain demonstration of the Exact-in-itself (*auto t'akribes apodeixin)* and the embodiment of the correct measure (*to metrion*).

Plato was above all a political thinker. Since his time, being a political thinker means putting in the center of thought this *metaphysical dimension* of life, and mostly of human History, going beyond the anecdotal boundaries of the trivial event and finding oneself beyond the epic confines of the *res gestae* of the chronicles of the memorialists. So much so that if we had to sum up the specific characteristics that define the singular position and quality of what is generally called the *European mind*, I would dare say that it is the development of musical polyphony but also, precisely and above all, the permanent presence of philosophical reflection oriented toward the metaphysical dimension of the human condition, of the *man-being*.

Only this activity of intense thought allows humankind to distinguish itself from animals to become a *political being* or, as another admirable expression of Aristotle has it, an *orthogonal being*.

Certainly Plato's intention in going to Syracuse was to convince Dionys to institute the reign of *isonomy*, of justice and fairness, and renounce the *bestial life* which Plato considered to be that of a tyrant. *Most* (hoi polloi) *of those in power, my dear Callicles, are villains. And the powerful ones are, by and large, evil perverts* – he writes in the *Gorgias*.

In his *Third Letter* addressed to Dionys, Plato relates an episode that corresponds exactly to the message in his dialogues. *You asked me* – writes Plato to Dionys – *if I remember having advised you to repopulate the ancient Hellenic cities. Yes, I have not forgotten it, because I did – and still do – believe that it is the best we can do. But I asked you if it was the only advice I had given you, and not something else again. To which you replied, snickering insolently and affectedly, in anger for also having advised you not to undertake all that, having acquired a philosophical education for you, or, if not, that it was better not to undertake any action. As for me, I answered you that it was exactly so. But you retorted: Education in geometry, right? Or what else? And I answered you nothing, fearing that the answer coming from my mind would lead you to prevent my imminent return to Athens.*

The contents of the *Third Letter* may be seen as representative of Plato's conception of the place of geometry in philosophical teaching, and thus in political education. Although the authenticity of its anecdotes may be questioned, this passage from the *Third Letter* is in perfect harmony with what Plato explicitly expresses in the *Republic*, in the *Laws* and in other dialogues, such as *Philebus*, the *Statesman* and *Phaedo* – where the long narration of the Second Navigation (*deuteros plous*) alternates with recurring reflections of a mathematical nature.

Along the same line of thought, I would like to bring up another story here – an undoubtedly apocryphal anecdote, but one that is surely noteworthy, almost always quoted in presentations on Plato's life and philosophy.

It concerns the famous inscription Plato supposedly had engraved above the entrance of the Academy, forbidding access to those ignorant in geometry. The principal source of this anecdote is the Byzantine author Tzetzes. But, in the usual quotations, Tzetzes's text – taken from Joannes Philopon's commentaries to Aristotle's book *On The Soul* – is rendered in a fragmented, even mutilated manner. For, having related the legendary, definitely enigmatic inscription, Tzetzes adds an interpretation to make it understandable. The real message becomes transparent – remarks Tzetzes – if the term "geometry" is understood in the sense Plato gave it: equivalent of *Justice, Fairness, Equality.*

Actually, this also provides the key for deciphering the somewhat enigmatic expression in *Gorgias,* where Plato refers to the idea of *geometric equality* between human beings.[5]

Between the *esprit de géométrie* and the *esprit fin* of metaphysical speculation, there exists a deeper link which deals with the propedeutic role of geometry in the education of a free citizen.

The irrational and ineffable number is presented by the Stranger of Athens as the work of a divinity, implicitly as that of a *subject,* not an empirical subject, a human, but a *subject* situated in a non-geometric space above and beyond the spheres in which individual human agents go about their activities.

Parmenides in Plato's *Parmenides,* the Stranger of Elea in the *Sophist* and in the *Statesman,* and Socrates in *Philebus* speak of a strange generation of the being, from achronic metamorphosis from non-being to being. Such an ontological inversion naturally cannot be identical to a natural diachronic genesis, such as in the evolution of a plant. Before they had access to being, these irrational numbers were already present in the epistemic domain of knowledge. The cognitive subject already possessed them, but assigned to them – not without very strong theoretical arguments – the value of the non-being.

Yet it turned out, in the end, to be as impossible to demonstrate as it is to refute their non-being. The simultaneous impossibility of demonstrating and refuting existence and non-existence unexpectedly opens a space where are absent all objective forces capable of deciding what is undecided and undecidable: to be or not to be.

This undecidability represents the manifest violation of the logical axiom of the excluded middle, expressed by Parmenides of Elea, which imposes a strict alternative to every pair of contradictory assertions.

In his *Parmenides,* Plato forces Parmenides himself to push that undecidability to its final conclusions, with astonishing clarity, by the conjunction of the two negations: *neither being nor non-being.* A bizarre and paradoxical state, if ever there was one. In terms of the examples of the *Achilles* and the pursuit of the younger and the older, this means that the simultaneous arrival of the two at the limit-point of *equal age* is theoretically undecidable.

No logical error is committed in stating that the faster will never catch up with the slower, and no logical error is committed by stating the contrary. But if the axiom of the excluded middle is inviolable, this indecision is intolerable. The decision is not within the competence of the inferential reasoning of logic. It requires a *subject*. And subject implies *freedom.*

It is the subject and only the *free subject* who is qualified to assign being or non-being to an ineffable number. It is up to the subject alone to decide, faced with the alternative, to either accept or reject the existence of infinity *in actu.*

And if ever the existence of irrational numbers one day managed to be accepted, the event of its passage from non-being to being is uniquely the result of choice and decision of a free subject.[6]

Georg Cantor was perfectly aware of the strictly metaphysical and even ethical implications of the act by which the ontic state of being is assigned to the actual infinity and to irrational numbers: *Mathematics is a science whose essence is freedom.* Cantor's aphorism is but one version of Hegel's well-known thesis: *the essence of the spirit is liberty.*

This decisive role of *Ethos,* in which the *Logos* of mathematical knowledge is immersed, may, at first glance, surprise by its strangeness. But this link only demonstrates the unity and universality of the mind: *no man is an island.* Mathematics is too an organic, inalienable part of the same mind, whose essence is freedom.[7]

Hegel uses a seemingly paradoxical formulation: *Man is free. He does not know it. So he is not free.* True, the actual reality of what people call Freedom is as sure as that of a block of granite or a rose. But the difference between the two realities lies not only in opposition between the palpable substance of granite and the immaterial impalpability of freedom.

By any other name a rose smells as sweet: When Juliet declaimed these magnificent words to Romeo she spoke the truth. The rose is red and sweet-smelling; even if no Juliet or no botanist exists to observe and analyze it. A block of granite does not require that a geologist be aware of its existence. The existence of material things precedes their knowledge.

All is inverted in the realm of the mind. There, knowledge precedes the thing, which does not exist unless it is known as such. The irrational number exists solely because there is a subject that has cognizance of it – that knows it to be irrational and consciously assigns to it being. The ontic value of being is assigned to this irrational number because its presence within the space filled with the substance of thought can be directly established by direct cognition, the object and agent of which are one and the same unique thought.

Freedom is perhaps the most important of these realities: subjects are free if, and only if, they are conscious of their own freedom. This cognitive process is evidently extremely sophisticated in structure because it implies the presence of the same, sole and indivisible subject in a double simultaneous hypostasis: that of the subject – agent of knowledge – and that of object as object of knowledge.

Knowledge of freedom is not a discovery, as one speaks of discovering America, of an exotic insect, of a new elementary particle or star. Acquiring this knowledge means passing from a state of conscience to the state of consciousness of self. It is this process, paradoxical perhaps, of cognition as self-cognition of the subject by the subject that Hegel designates as the *Phenomenology of the Mind*.

Acquiring this knowledge follows the trajectory of phenomenology of the mind: the state of awareness is the ascension of the mind as it becomes conscious of its own freedom. This movement may be qualified as "natural," because it belongs to the intimate nature of subject – just as weight is a natural property of matter – but out of all the things in the world, it is this internal movement toward knowledge of self by self that is the sole characteristic differentiating and defining the subject.

The slow, tortured effort of the mind, that requires ascension of the subject to the state of the self-consciousness of its own freedom is the specific work of the mind that is called philosophy.

The presence of free subject behind the act of mathematical knowledge is expressed in the *Epinomis* by the mythical epistemological category of divine being. Its work was to create a new world of numbers in which irrational numbers find their natural place.

But the presence of the subject is also manifested in another area of Greek mathematics, this time more intensely and more explicitly: in geometry.

Geometry, which the Ancients left to us in the form of the magnificent cathedral that is Euclid's *Elements*, was the first edifice of knowledge constructed according to strict rules of logical inference. At its core are propositions called *axioms*. These are true propositions, but the value of truth is assigned to them, with no demonstration, by the transcendent subject of geometry.

The presence of subject is inescapable. The axioms are not derived by mechanical inference from other propositions, nor does truth come to them automatically by a necessity mediated by logical deduction. Their truth is not the genetic result of a logical heredity. If they are accepted as true, it is solely due to a direct act of assignment whose agent is the subject of geometry. And the presence of this subject is evidenced by a unique word, spelled out at the beginning of the list of postulates: the passive third person perfect imperative of the verb *to require, demand.* The verb itself "to demand, require" – already expresses the imperative act, which can be due only by a subject who demands, requires, gives the order to do or execute a specific action. But the verb is in the third person, so this order does not emanate from you or me, but from a third party. This order is not in the present tense, now, today; the order is there, subsisting from a distant, unspecified past of the past perfect. The order is given in the passive form; its agent is impersonal. Therefore expresses a constraint imposed for an eternity by a transcendental unknown and impersonal subject.

But – and this is of the utmost importance – those obliged to submit to the order are not humans, but objects constituted of an immaterial geometric substance: points, straight lines, circles. The orders they receive concern their mutual and specifically geometric relations: to cross each other or not, to be exterior to a segment of a straight line or to be located in its interior, between its extremes.

This unique word: *Eitestho*, expresses that over the world of points, straight lines, and circles has forever been hovering an order that forces them to conduct themselves rigorously as expressed in the texts of the postulates.

The term does not reappear in the text of the *Elements*. Its presence is a unique occurrence and its natural place is found in the domain of metalanguage. In the object-language of triangles, parallel lines and circles, it is never uttered. It does not belong to the primitive term of the vocabulary of geometric objects.

Due to the fifth and last postulate, the network of theorems logically stemming from the postulates is called *Euclidean geometry.*

Euclid's postulate concerns a pair of straight coplanar lines which are not co-orthogonal, that is two straight lines located on the same plane, in oblique position to each other, having no common perpendicular. The order given by the *Verb at the beginning* is: *if a third straight line intersects one of them at a right angle, and the other at an interior acute angle, then both straight lines will meet.* This incidence of two coplanar lines obliquely inclined to each other is neither a natural necessity of the physical universe, nor an intrinsic necessity to the statement. It is a necessity imposed by the constraint of the order arising from the transcendental subject of geometry. And straight lines obey, faithfully following this order.

Even though the incidence of lines obliquely inclined to each other appears glaringly obvious, it can be replaced by equivalent sentences. For example, if we hypothesize that there *is no maximal triangle or maximal square*, that we can therefore construct triangles and squares as large as desired. Or: *the plane may be tiled with equal squares*, so that at each point of intersection exactly four squares meet. Finally, it is enough to give the order that there *exists one square whose four vertexes are all rectangular*. To sum up: *if one square is Euclidean – everything*, the entire universe, The *Whole, is necessarily Euclidean*. This time, we are dealing with logical necessity: the universality of the rectangularity of squares can be rigorously demonstrated.

How is a non-orthogonal square indeed possible? How can we conceive of tiling a plane surface with equal squares, so that at each point of intersection three, five, six or eight squares meet, but never four? And isn't it totally impossible to imagine the absurdity of a surface tiled with orthogonal pentagons, so that at each point of intersection four pentagons cross exactly, or perhaps four orthogonal heptagons?

Nor does the fifth postulate, or any of the propositions equivalent to Euclid's postulate appear in the *Elements* written at the time of Plato and Aristotle. In his *Posterior Analytics*, Aristotle emphasizes that trivial statements should not be explicitly mentioned among the *archai*, that they are so obvious that it seems completely natural to refrain from mentioning them explicitly.

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But here, in one way or another, the geometricians affiliated with the Academy realized that the demonstration of one of the fundamental theorems, basically very simple, even obvious, concerning parallel straight lines – for example: *parallel lines are equidistant* – is marred by a grave error, that of the *vicious circle.*

They also quickly realized that, to be executed correctly, the demonstration *requires* that one of the auxiliary propositions, absolutely essential for its demonstration, be accepted without demonstration as an axiom or postulate (either term will do). The Greeks designated these accepted propositions without demonstration by the term *arché.*

This conclusion has been preceded by arduous work. First of all, the goal was to eliminate error by looking for new strategies of demonstration, namely indirect demonstration, i.e. *reductio ad absurdum*. Thus they chose the fundamental theorem of Euclidean geometry: *the sum of the angles of a triangle is equal to two right angles* – and they tried to demonstrate it by reducing to absurd the opposite hypothesis.

By stating a non-Euclidean hypothesis: *the sum of the angles of a triangle is not equal to two right angles* – but *larger* or *smaller* than two right angles – the goal was to end up with a logical contradiction. These three hypotheses of *a sum of smaller, equal, greater than two right angles* are explicitly mentioned in the *Posterior Analytics.* This undertaking resulted in recurring failures. They were unable to find any two contradictory propositions among the logical consequences of the non-Euclidean hypothesis. They were unable to demonstrate the absurdity of the non-Euclidean hypothesis.

Although they did not realize it, their failure was actually a real victory. For in analyzing the logical consequences of the non-Euclidean hypothesis, a number of consequences became apparent: namely, strange non-Euclidean theorems, but of great interest nevertheless, due to their specific geometric content.[8] In the *Corpus Aristotelicum* there are eighteen fragments of different size, historical relevance, and theoretical value that represents the verbal remains of this strange undertaking: non-Euclidean fossiles of the future, hidden under the geological layers of the Aristotelian *corpus*.

Aristotle cites these non-Euclidean propositions without qualifying them anywhere as absurdities, without even qualifying them with the predicate "false." He quotes with his usual calmness, in the text of just one proposition, with no discrimination, both the Euclidean triangle and its opposite, the non-Euclidean triangle. We can discern no distinction concerning the value of their truth. The two contradictory chains of hypotheses and theorems are presented in his text in a state of indecision and undoubtedly the undecidability of the alternative, "Euclidean or non-Euclidean," was already an acquired idea, firmly established in its time.

The undecidability of this alternative is impressively expressed in the large passages of *The Great Ethics* and *Eudemian Ethics,* devoted to the problem of *human freedom.*

Here human beings are defined by their singular position of rational beings, capable of deciding to undertake one action or its opposite, and choosing between good and evil.

Deciding when confronted with an alternative necessarily requires a subject, for in itself it is undecidable by any empirical means or logical reasoning. It is up to the subject, and him or her alone as agent, to decide what is – inherently – absolutely undecided.

Having no term equivalent with our "freedom," Aristotle obtains help from a significant expression: "preferential choice" *(prohairesis),* usually translated as "free will." But to allow the reader to understand the central idea of his analysis, he writes that we need a concrete example, intuitive and palpable. The example he cites as a rhetoric *parallel* is not from the field of daily ethical or political *praxis*, but from geometry (*paraballontes epi ton en geometrian*).

The opposition – evidently undecidable by empirical or logical means – that he cites is that of the Euclidean and non-Euclidean triangle. The first act, the decision, depends evidently on the subject as agent of the *praxis*.

Once the decision is made, the rest necessarily follows. Thus, if a Euclidean triangle as *arché* is chosen, the result will necessarily be Euclidean theorems; but, if the non-Euclidean triangle is chosen, the necessary consequences that result are non-Euclidean theorems.

* * *

One of the consequences of the non-Euclidean *arché* that Aristotle cites in his *Eudemian Ethics* is truly astounding: a square whose four angles are flat, thus each equal to two right angles. This is a logical consequence, thus absolutely necessary, of an *arché* that states that the sum of the angles of a triangle is greater than two right angles – an *arché* opposed by formal contradiction to the Euclidean *arché* of a triangle whose angles are equal to two right angles.

But what of this square figure, a necessary consequence of the *arché*, the subject that the geometric *praxis* he has decided to choose, whose vertexes are smooth straight lines without any bends? It is simply a maximal square on the non-Euclidean plane, called *elliptic plane*, a square whose perimeter is a straight line, of finite length and closed unto itself like a circle.

The figure of this square itself is undoubtedly a geometric monster. However, Aristotle shows neither irritation nor surprise: he mentions it without interrupting, even for an instant, the continuous flux of his discourse, as calmly as if he were describing an exotic fish or bird.

More surprising than the figure of the strange square, consisting of a unique straight line closed unto itself, on which are marked four points to be read as "vertexes A,B,C,D" – is that none of the ancient or modern commentators, and they are legion, has stopped for an instant to question what the word "square" means in this context. Is it possible to speak, in the text of the same, of the rectilinear figure of a perhaps somewhat boring but honest orthogonal square, and of an exceedingly extravagant square whose sides are all equal segments of straight lines, but whose sum is equal to the total finite length of the straight lines of the plane, where the square is placed, all these straight lines being closed unto themselves? Is it not rather a circle? Is this not a glaring error, or perhaps an obvious flaw, due to the intervention of a zealous but ignorant copyist who substituted the word "circle," which was supposed to be in Aristotle's manuscript, for the word "square"? No, the text poses no palaeographical problem and, moreover, the theorem cited by Aristotle is absolutely correct, the square is the maximal square of a non-Euclidean, so-called elliptic plane.

It is truly strange: no discrimination between the two triangles, two squares, Euclidean and non-Euclidean, can be found in the text. Not the slightest allusion indicating a preference or fondness for the Euclidean triangle; no trace of repugnance or disdain for the non-Euclidean triangle. Aristotle's discourse is calm, serene, harmonious and characterized by a disturbing tranquillity.

But moreover, none of his commentators ever asks what a geometric example, especially such a strange one, is doing in an ethical reflection on human freedom? Why does Aristotle twice insist on the fact that the rhetoric parallel taken from geometry can serve as an intuitively palpable example for illustrating the *arché* of an ethical idea, that of free choice, essentially freedom of subject? And how to explain the absence of examples from the domain of daily *praxis*, ethical or political?

In my opinion, the strongest of the passages where Aristotle comments on the undecidability of Euclidean and non-Euclidean alternatives is found in *Problems*. *Problem* XXX 7 is entirely devoted to it.

Aristotle poses the question of the *Problem* as follows: *Why do we not feel joy* (khairomen) *in contemplating or waiting in hope* (elpizein) *that the sum of the angles be equal to two right angles*?

The question is bizarre: how to *wait in hope* for something that is a known theorem, a theorem that has long been rigorously demonstrated in geometry textbooks and whose first demonstration was preserved for us in his *Metaphysics* by Aristotle himself?

But the answer given in *Problem* XXX 7 is even more puzzling: *Because we experience the same pleasure* (hédone) *when the sum of*

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angles of a triangle is not equal to two right angles, but, for instance, is greater than them.

This astonishing answer is motivated by the following remarks: in life we experience pleasure only if victory comes to us and never when it comes to our opponents. The example cited to illustrate "our cause" is that of the naval battle of Salamis. But, adds Aristotle somewhat vaguely, with things that are naturally themselves, truthful, we always experience joy in accepting them as they are.

In short, with the battle of Salamis we are biased, and experience immediate joy hoping that tomorrow victory will be won by our fleet. But we contemplate the geometric naumachy with total impartiality, and we will place the laurel wreath of truth on the head of the victor with equal joy, whether he is Euclidean or non-Euclidean.

The analogy of the battle of Salamis is eloquent because, whether or not it happened, whether victory falls to one or another of the adversaries, will depend solely on the decision made by a subject, the admiral. The metaphor of the naumachy evidently implies that the levee of the undecidable, anachronic opposition between Euclidean and non-Euclidean also depends solely on an admiral, who cannot be but the transcendent subject of geometry.

And the subject of geometry, like any subject, is free.[9] The freedom of the subject is defined by Aristotle as a preferential choice, but also as a decision taken *in absence of any constraint.* It is opposed to, and limited by, necessity. *Only a madman could decide to choose the impossible*! – he repeats on many occasions.

In the long political reflection around the concept of freedom following Aristotle's *Ethics*, the decisive juncture is brought by Spinoza's *Ethics*. This is the first work in the history of thought where the word "Ethics" is defined by the title of its last chapter, having as object the *power of reason or human freedom.*

Spinoza openly condemns the confusion between *free will* and *freedom.* The preferential choice between the two opposed terms is merely an arbitrary act unrelated to true *freedom* and Spinoza himself defines freedom not as the opposite of necessity, but as the opposite of the arbitrary. His concept is summarized in the sublime wording of the *protasis* of theorem XXXVI of part five of his *Ethics*: *Mentis Amor intellectualis erga Deum – seu libertas.*

The tyrant is not free even if he makes his decision in the absence of any constraint. For it is not necessity that is opposed to freedom, but the arbitrary of caprice, and freedom *virtus est quod ex fortitudine animi oritur.*

As opposed to preferential choice, freedom is accompanied by necessity. Or, as Camus expresses it: *The free spirit likes that which is necessary.*

Georg Cantor, who saw mathematics as a domain of freedom, also speaks, on the subject of creation of infinite sets, of an inherent necessity which he cannot oppose, and against which resistance opposed by its numerous adversaries – then the majority of the mathematical world – would quickly reveal itself to be nothing but vanity.

The works of the arbitrary are ephemeral and reversible. The works of freedom are perennial and irreversible.

The choice the free subject makes is in fact not made in a space void from constraint. The opposite is true: to be free means opposing one's self to the established constraint.

The theorems of non-Euclidean geometry have been developed little by little over two millenniums, but the subject of geometry has always rejected these theorems, attributing to them the logical value of *false*, and challenged them as patent absurdities. However, it was impossible for him to escape from and to get rid of them.

In a state of pure negativity, non-Euclidean geometry was inevitably present in the state of *unhappy conscience*, as the incarnation of the geometric non-being.

The constraint exerted by the tangible presence of the well-established Euclidean world prevented him from deciding in favor of a non-Euclidean universe; the Euclidean world represented such an obvious necessity that only a madman could have denied it.

And yet, at the beginning of the nineteenth century, the monstrous and absurd world of non-Euclidean geometry made its appearance as the simultaneous work of three mathematicians, each of them working for himself, independently, in three different places: Gauss in Göttingen, Lobatchevsky in Kazan, and Johann Bolyai in Marosvasarhely.

Ontologically, its creation represented the passage from non-being to being that Plato spoke of in *Parmenides* and in the *Sophist*. Johann Bolyai, one of these founders, expressed this thought in the form of a splendid aphorism: *From nothing, I created a new, different world.*

But this new world, different from the Euclidean one, did not come out from the void, but from a nothingness full of its own non-being – specifically non-Euclidean.

* * *

Epistemologically, this act of transmutation into being of the non-Euclidean world of non-being constitutes one of the largest and most decisive scientific revolutions. Yes, a revolution, the term is correct: it was a revolution in the most profound political sense of the word.

There was no practical need that would have required the formulation of non-Euclidean theorems, the construction of maximal squares whose perimeter is a unique straight line closed upon itself. Nor was there a specifically geometric problem whose solution could have stimulated the formulation of non-Euclidean theorems.

The non-Euclidean text presents itself as the answer to a question that no one had ever asked, as a supply to a non-existent demand. No one asked for these horrible monstrosities, but once there, no one could voluntarily relinquish them. With their presence came the inescapable feeling that they represented a necessity.

The sole necessity that for millenniums prevented the elimination and forgetting of its eccentric and patently absurd theorems was the absolute necessity of the geometric subject to become aware of its own freedom, even in the domain of a science as cold and severe as mathematics.

Non-Euclidean geometry is not the only one of the important mathematical conceptions that owes its existence to freedom. Its singular position is due to the decisive role that it played in the phenomenology of the mathematical mind; it is thanks to the creation of non-Euclidean geometry that the transcendent subject of mathematics has raised itself to attain explicit self-knowledge

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of its own freedom, and thanks to the non-Euclidean event that the geometric mind has become aware of its own freedom.

In carrying out the non-Euclidean revolution, the world of mathematical knowledge demonstrated that its cosmic trajectory is definite, determined by the same lines of force that filled the space of universal philosophical thought, which is simultaneously and implicitly political.

* * *

At the end of the nineteenth century, the penetration of non-Euclidean geometry into the *universitas scientiarum* came up against vehement protests from the majority of mathematicians and philosophers from all schools. One of its staunchest partisans, the mathematician Felix Klein of the University of Göttingen, claimed *gleiche bürgerliche Rechte* for non-Euclidean geometry, the same rights of citizenship as Euclidean geometry enjoyed in the world of the *episteme*. The political message of the metaphor is manifest. Henri Poincaré upheld Klein by quoting, in one of his works, Klein's expression in the original German: *Very good, Monsieur Klein:* "gleichberechtigt!" *That's the word!*

That is precisely the word that had been used, at the beginning of the nineteenth century, by Carl Friedrich Gauss, the Prince of Mathematicians, when he pleaded for the gates of mathematics to be opened to what are still known as "imaginary" numbers – figments of the mind, neither bigger nor smaller than zero, false and sophistical numbers, freak amphibians of being and non-being – putting an end to centuries of ostracism by mathematicians and philosophers who had invoked precisely the same arguments that had been used to keep non-Euclidean geometry out.

At the time when Gauss was claiming *equal rights for* (mathematical) *citizenship* for these numbers, the political metaphor was not a term bandied about lightly on the whim of a moment's inspiration. On the contrary, it was a weighty term, and a topical one, that carried the weight of a vehement combat which dominated political life in all regions of Germany.

It concerned the emancipation of the Jews and the question that incited such passion was concerning whether or not they should be given equal rights to citizenship.

Gauss was a firm monarchist, although a partisan of constitutional monarchy, the unique form of government that, in his opinion, was capable of guaranteeing equality and rights of all citizens. Like Felix Klein, he was naturally an unconditional partisan of the emancipation of the Jews and their equal rights to citizenship. Considering this, Felix Klein's metaphor allows us into the depths of political significance of the non-Euclidean event.

* * *

When Zeno, to and against all evidence, stated that the faster would never catch up with the slower, he did so because he was conscious of himself as a free subject making a choice, free to decide – without fear of being punished by the legislation of logical thought. His act was not only that of a brilliant logician, but also, above all, that of a mind illuminated by the very philosophical knowledge of the conscience of its own freedom, as opposed to the constraint of empirical evidence, but also, in conformity with his mathematical philosophy, to the constraint wielded by tyranny in his own life.

Diogenes Laertius spends only a few words to remind us of his arguments, but devotes the main part of his presentation to biographical anecdotes about Zeno. *Noble mind in both philosophy and politics* – writes Diogenes – *he mounted a conspiracy to deliver his city from tyranny.* Doubtless the tyrant was none too interested in his mathematical demonstrations and moreover he did not let himself become impressed by conclusions that were more ridiculous than absurd.

The name of the tyrant is rather uncertain. But we do know he was strong and powerful, stronger and more powerful than Zeno, the philosopher. The stronger, quicker one quickly caught up with the weaker, slower one. The author of the arguments against movement and tyranny was subjected by the quick and strong victor to a terrible torture, then put to death. Diogenes compares Zeno to Aristogenos the tyrannicide.

* * *

Federigo Enriques, one of the most eminent geometricians of the first half of our century, in 1936 published a brilliant work on the history of mathematical thought in classical antiquity, entitled *Pythagoriciens et Éléates.*

After presenting detailed and profound arguments, Enriques concludes his work, briefly commenting on the anecdote regarding Zeno's demise: "His demise was heroic," he writes. To the sarcastic question of the tyrant, who witnessed his torture: "tell me now what philosophy teaches you?," Zeno, according to Enriques, replied: "To despise tyrants."

Enriques does not indicate the source of his information, but it can be found in Tertullians's *Apologetics.* Zeno's response to the tyrant would have been, according to Tertullian: "*contemptum mortis.*" The anecdote, told in the erudite work of the great Italian mathematician, represents his own answer to the question: *what use is philosophy?* And the name of the tyrant that philosophy taught Enriques to despise requires no doxographical researches.

Of course Enriques's domain of research was strictly limited to the study of algebraic curves and surfaces; philosophy and, to a greater extent, politics went far beyond his competency as a specialist in algebraic geometry.

A few years after Enriques's work was published, the tyrant, whose name we all know, imposed upon Italy the famous *leggi razziali* which had already prevailed for a long time in the country of which he was the most loyal ally. Italian cultural institutions were forced to purge themselves of members whom this law classified among elements of the "inferior race."

Federigo Enriques and, the other great Italian mathematician, Tullio Levi-Civita were among the most venerated members of the *Accademia dei Lincei*, the oldest and one of the most distinguished scientific societies of intellectual history in the Western world. This academy was given the choice of excluding Enriques and Levi-Civita or of being dissolved.

The choice that confronted the *Lincei* was difficult, dramatic. The decisive meeting was preceded by long, tumultuous discussions. The exclusion of Levi-Civita and Enriques seemed to be, for a number of the *Lincei*, a minor act of political tactics: saving a secular, famous institution seemed necessary according to the good sense of political wisdom. A simple formality, which would do nothing to alter the esteem and veneration of their colleagues toward Enriques and Levi-Civita, an insignificant act which would cast no shadow on their personal relations, their old and solid friendship. Even Enriques and Levi-Civita conceded the strength of these arguments.[10] The venerable society found itself up against a coercion whose weight was precisely perceived and felt by all its members. The gravity of the situation was obvious. They had to choose between dishonor and human dignity.

The decision made by the *Lincei* was clear and firm: *"No to infamy!"* It was made in the presence and against the coercion wielded by a tyranny that did not allow for the jokes of the scientists.

The next day the *Accademia dei Lincei* was dissolved.

It was a unique event in the history of scientific institutions of the time. The exclusion of Albert Einstein and the famous chemist Fritz Haber, another Nobel Prize winner, from the Academy of Berlin occurred easily; provoking no convulsions in the moral consciousness of the important experts of scientific research of the Reich. Indeed, many of them publicly declared their satisfaction.

Even among those who experienced some discomfort, it was the idea of saving the institution that prevailed. Remember, however, that they acted under the threat of obvious coercion. Between dishonor and survival of their institution, they chose dishonor. They cloaked themselves with dishonor and ruined their institution.

For the *Lincei*, the idea of human dignity was more important that the destiny of a scientific society, even one as distinguished and venerable as the ancient *Accademia dei Lincei*.

But what prompted the *Lincei* to turn away from dishonor? Certainly not their competence in the specialized areas of scientific research. What the dissolution of the Academy did demonstrate was their total incompetence in the domain of everyday political maneuvering. The reason for the *Lincei*'s refusal, their saying "no," can certainly not be found in their laboratories, but solely in this immaterial *substance* called "Philosophy" and whose teaching was clearly and decisively formulated by the mathematician Enriques in his erudite study on the arguments of Zeno.

* * *

Zeno's *Achilles* was certainly challenging, an act of intellectual provocation.

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Certainly Zeno realized that, if it is logically undecidable that the faster catches up or does not catch up with the slower, he deals solely with the *freedom of the subject* in deciding the undecidable by stating the impossibility of catching up with the slower, for it is surely impossible to refute such a statement by dianoetic means of logical inference – as paradoxical and as absurd as they may be or appear.

The essential part of Zeno's achievement was to have become aware of the freedom of the subject in mathematics. For to say "no!," to the obvious, to a long-established system, to what everyone accepts, is one of the most spectacular, most daring, even riskiest demonstrations of freedom, source at once of countless risks and dangers to the existence of the subject itself, but also one of the most sublime and richest sources of thought, with hugely efficient results.

It is precisely this awareness of freedom, the freedom to say "No" – to the uncontested evidence, never contested by anyone – that gives Zeno's argument the irresistible fascination that his thought has exerted on minds forever.

The extravagant text of *Achilles* is – even though it is formulated in cryptic language and must be read and understood as the first proclamation of liberty – the essence of mathematical thought. It is this implicit message that guaranteed him effectiveness and immortality. The explicitly formulated statement in the argument is irrefutable, but may be accepted or rejected, depending on the choice of the individual subject. But the important, decisive *Idea* that the argument vehicles is not the mathematical anecdote of faster and slower – this Idea is illegible in his text; it is perpetually hidden in the argument. Its force is indefinable. This is the idea of Freedom.

In his *Eudemian Ethics*, Aristotle recollects the words of one of Zeno's contemporaries: ... *as Philolaos the Pythagorean said, ideas are more powerful than ourselves.*

According to widespread opinion, ideas of freedom and human dignity are sublime, but appear physically too weak when confronted by the little tyrants of Syracuse and especially by the larger ones of Rome, Berlin, and Moscow. Nothing could be further from the truth. Tyrants disappear, but ideas of freedom and human dignity are everlasting. And this is an empirical truth, experienced since very ancient times. To conclude, I would like to recollect the words by which Hegel briefly summarized the conclusions of his *Phenomenology of the Mind*: *"This indivisible substance of absolute freedom lifts itself onto the throne of the world, without any power being able to resist."*[11]

Translated by Jon Kaplansky

Notes

* The author has translated and paraphrased (in italics) quotations from the Greek himself (*N.d.l.R.*)

1. Certainly a provocative personality, we must recognize (if we can trust Plato's portrait of him in the first part of *Parmenides*) an intellectual dandyish quality in the *handsome and elegant* Zeno, who, at forty years old, is the lover of Parmenides, with whom he travels with to Athens. Parmenides, a handsome man himself, imposing in his noble presence, white beard and hair, is just over sixty-five.
2. In his *Academicorum historia*, Phiolodemus speaks of Plato as the architect of the *metrologia* – a very successful terminological choice thanks to its semantic proximity to the current technical expression *theory of measure*.
3. Alfred North Whitehead, the great mathematician and philosopher, speaks of mathematics as "divine madness" – certainly a metastasis of *Epinomis* and *Phaedrus*.
4. At least on this point, Aristotle agrees with Plato. In his *Nicomedean Ethics* he comes back to the idea that mathematical pleasures are not mixed with pain and in his *Topics* he refers to the act of contemplation of the incommensurability of the diagonal of the square, as example of a pleasure that does not know the opposition of pain, exactly like the act of carnal love, whereas spiritual love is linked to its opposite, hate. And, in his *Metaphysics*, he expresses himself even more categorically, when he writes that *those who state that mathematical sciences have nothing to say about Beauty and Goodness are certainly wrong*.
5. Such a narrow link between geometry and political thought may very well give the impression of being far-fetched and naive, but it has experienced a true renaissance centuries later. Under the somewhat strange title: "*Géomètre,*" d'Alembert wrote an article in *l'Encyclopédie*, which he, along with Diderot, published. Completely unexpectedly, we read in it the following text, which has the tone of a political manifesto: "Geometry is perhaps the only way to stir up, little by little, in certain countries in Europe, the yoke of oppression and ignorance under which they suffer. Give birth, if possible, to geometri-

cians among these peoples. Soon the study of geometry will lead to true Philosophy, which by the generalized and immediate light it will shed will soon be stronger than all forces of superstition."

In a geometry textbook published in 1817 in Erlangen by the German mathematician, Georg Simon Ohm, we find at the beginning of the phalanx of severely structured theorems – the following text, certainly quite unusual in the work of a specialist: "Geometry, only geometry is capable of instilling men with the spirit of independence, it alone can preserve the biases of a spiritual despotism." Actually, the masterly and unique work that is Greek geometry is the supreme product of the first society founded on the principles of democracy. Without Greek democracy – no *Elements* by Euclid. "There is no royal road in geometry," Euclid of Alexandria is purported to have answered Ptolemy, King of Egypt. Yes, in the world of geometry, the young slave of Menon and the King of Egypt must obey the same universal laws of reason.

6. One of the greatest mathematicians of the nineteenth century, Leopold Kronecker, never accepted the existence of irrational numbers, and still today there are eminent mathematicians who are openly reserved and even averse to the idea, only accepting a limited class of irrational numbers whose existence may be founded on certain constructions.

7. Philosophy knows of and admits of no direct application, in mathematics or elsewhere. No theorem may be demonstrated by means of the philosophemes. Behind no theory or mathematical conception can one identify the teaching of a precise philosopher.

The channels through which philosophical thought nonetheless has a decisive influence on the body of mathematical thought are topologically complex; the network of capillaries through which the currents of philosophical thought irrigate the universe of mathematical knowledge is delicately structured, and next to nothing is known about the fine details.

The sole existing visible indications are the aphoristic declarations of a large number of mathematicians, as well as of some authors, not mathematicians, who understood exactly this interpenetration and interdependence: first Plotin, Nicolas de Cuse, Marsilio Ficino and two important poets – Novalis, and above all, the amazing Edgar Allan Poe.

8. I will cite but two. In the *Posterior Analytics* there is the theorem called "elliptical geometry" – valid only in one of the two large branches of non-Euclidean geometry: *if the sum of the angles of a triangle is greater than two right angles, then the parallels will meet*, that is: in this non-Euclidean surface there are no straight lines which are not incident, all the straight lines meet. In *On The Heavens*, we find the non-Euclidean theorem, valid in both types of non-Euclidean geometry, elliptical and hyperbolic: *if it is impossible for the sum of the angles of a triangle to be equal to two right angles* – that is if it is impossible for a triangle to be Euclidean – *then the diagonal is commensurable to the side of the square*. This theorem is notable not only for its fundamental character, the richness and sophistication of its specific geometric content, but perhaps above all, for the way its hypothesis is formulated: *impossible* – for a triangle – *to be Euclidean*! The amazing character of this expression is truly astonishing because the modal predicate *impossible* is never attributed by Aristotle to a non-Euclidean triangle.

9. For readers interested in details about these non-Euclidean fragments which Aristotle preserved for us, allow me to recommend my book, *Aristotele e I fondamenti assiomatici della geometria. Prolegomeni alla comprensione dei frammenti non-euclidei del "Corpus Aristotelicum,"* Milan, Vita e Pensiero 1998.
10. In an erudite work, *La comunità matematica Italiana di fronte alle leggi razziali,* published in Cosenza in 1991, Pietro Nastasi provided a summary both detailed and brilliant in its historical and political sensitivity toward the events that preceded and followed the decision of the Academy.
11. "Diese ungetheilte Substanz der absoluten Freiheit erhebt sich auf den Thron der Welt, ohne dass irgend eine Macht ihr Widerstand zu leisten vermögte" – Berlin, Ed. 1842, p. 428.