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Galois groups of chromatic polynomials

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Abstract

The chromatic polynomial $P(G,\lambda)$ gives the number of ways a graph G can be properly coloured in at most λ colours. This polynomial has been extensively studied in both combinatorics and statistical physics, but there has been little work on its algebraic properties. This paper reports a systematic study of the Galois groups of chromatic polynomials. We give a summary of the Galois groups of all chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and all chromatic polynomials of non-clique-separable θ-graphs of order at most 19. Most of these chromatic polynomials have symmetric Galois groups. We give five infinite families of graphs: one of these families has chromatic polynomials with a dihedral Galois group and two of these families have chromatic polynomials with cyclic Galois groups. This includes the first known infinite family of graphs that have chromatic polynomials with the cyclic Galois group of order 3.

1. *Introduction*

The chromatic polynomial $P(G,\lambda)$ gives the number of proper colourings of a graph G in at most λ colours. It was first introduced by Birkhoff [**7**] in an attempt to prove algebraically the four-colour theorem, that is, that every planar graph is four colourable. Although this attempt was unsuccessful, the chromatic polynomial has been extensively studied both in graph theory [**17, 45**] and in statistical mechanics where the Potts model partition function generalises this polynomial. An overview of the relationship between the chromatic polynomial in graph theory and the partition function in statistical mechanics is given in [**1**]. The limit points of the roots of the partition function give the locations of possible physical phase transitions [**31, 39–41**]. Thus, there has been a large amount of research about complex roots of chromatic polynomials of families of graphs, in particular identifying zero-free regions and zero-dense regions in the complex plane for these families [**3–6, 11, 15, 18, 19, 29, 31–38**].

Although there has been considerable research in the study of *chromatic roots*, that is, the roots of chromatic polynomials, until recently there had been very little research in the algebraic study of these roots. The main exception is the proofs that the non-integer Beraha numbers $B_i = 2 + 2\cos(2\pi/i)$, $i \geqslant 5$, [**2, 43**] (excluding possibly B_{10}) are not chromatic roots [**31, 42**]. Sokal [**41**, Footnote 13] commented that the algebraic theory of chromatic roots was 'as yet rather undeveloped'. But a polynomial is by nature an algebraic object and the chromatic polynomial's roots (excuse the pun) are algebraic, so it is natural to study the algebraic properties of chromatic roots. Recently there has been an increasing interest in the algebraic study of chromatic roots instigated by our own work in [**23–26**] and by a work group initiated by the Combinatorics and Statistical Mechanics programme at the Isaac Newton Institute in 2008 [**10**].

A fundamental algebraic property of any polynomial is its factorisation. Thus, as a start of an algebraic study of the chromatic polynomial we introduced the concept of the *chromatic*

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factorisation of a graph G, that is, where the chromatic polynomial of G can be expressed as

$$P(G,\lambda) = \frac{P(H_1,\lambda)P(H_2,\lambda)}{P(K_r,\lambda)},$$

where $r \geqslant 0$ and the graphs H_1 and H_2 each have chromatic number at least r (see [**24**–**26**]). The graphs H_1 and H_2 are called the *chromatic factors* of G. We say a graph is *clique-separable* if it can be obtained by identifying an r-clique in H_1 with an r-clique in H_2, for some H_1 and H_2. Any clique-separable graph has a chromatic factorisation. Graphs are said to be *chromatically equivalent* if they have the same chromatic polynomial. A graph is said to be a *strongly non-clique-separable graph* if it is not chromatically equivalent to any clique-separable graph. In [**24**, **25**] we identified strongly non-clique-separable graphs that have chromatic factorisations.

In this paper we look at the Galois groups of chromatic polynomials. The Galois group of a polynomial provides information about symmetries of the roots. The first explicit connection with Galois groups was our own work in [**22**] and joint work in [**10**]. As the Galois group is arguably the most central object in any algebraic study of polynomials, it is surprising that the connections between Galois groups of chromatic polynomials and graphs have not been previously explored.

We say the graphs G and H are *Galois equivalent* if $P(G,\lambda)$ and $P(H,\lambda)$ have the same Galois group. We are particularly interested in identifying families of Galois equivalent graphs that have common structural properties.

There are two main contributions in this paper. The first contribution is to give a summary of our computations of Galois groups of the irreducible non-linear factors of chromatic polynomials of all strongly non-clique-separable graphs of order at most 10 and of θ-graphs of order at most 19. The Galois groups of irreducible factors of chromatic polynomials of any clique-separable graph of order at most 10 can easily be obtained from this data.

In most (over 90%) cases the Galois groups are symmetric Galois groups. Recently, it was shown that the Galois group of the multivariate Tutte–Whitney polynomial is always a direct product of symmetric groups [**8**]. The chromatic polynomial is an evaluation of this polynomial. As most chromatic polynomials of strongly non-clique-separable graphs of order at most 10 have symmetric Galois groups, it is interesting to consider cases where the Galois group is not the symmetric Galois group, which leads to our second contribution.

The second contribution of this paper is to give some new infinite families of Galois equivalent graphs. Three of these families have graphs with chromatic polynomials that have non-symmetric Galois groups, namely, the cyclic groups, $C(3) \cong A_3$ and $C(4)$, of orders 3 and 4 respectively, and the dihedral group $D(4)$. (Table A.1 gives the notation used for Galois groups in this article. We use the notation given in [**12**, Appendix A].) It is easy to show that the Galois group of the chromatic polynomial of a cycle of order n is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$, the multiplicative group of units of $\mathbb{Z}/(n-1)\mathbb{Z}$ (see [**10**, **22**]). If $n = p+1$, where p is prime, then the Galois group is the cyclic group $C(p-1)$ of order $p-1$ (see [**10**, **22**]). Thus, there are examples of chromatic polynomials with cyclic Galois groups of order $p-1$. However, we found no chromatic polynomial of degree at most 10 that has a cyclic group of odd order (excluding the trivial group) as the Galois group. This led to the question of whether there are chromatic polynomials with the Galois group $C(n)$, n odd. In this paper, we give an affirmative answer for the case where $n = 3$ by providing an infinite family of graphs that have chromatic polynomials with Galois group $C(3) \cong A_3$. The smallest (in terms of order) of these graphs has order 11.

This article is organised as follows. Section 2 gives some basic properties of chromatic polynomials and a brief overview of Galois theory. We give a summary of the Galois groups of non-linear factors of chromatic polynomials of graphs of order at most 10 in Section 3 and of the Galois groups of non-linear factors of chromatic polynomials of θ-graphs of order at most 19 in Section 4. The θ-graphs have some nice divisibility relations which we describe briefly. We then give a description of five infinite families of graphs in Section 5. Each family is shown to have Galois group S_1, S_2, $C(3) \cong A_3$, $C(4)$ and $D(4)$ respectively.

2. *Background*

First we give some relations and properties of the chromatic polynomial that will be used in this article. Let $G = (V, E)$ be the graph with vertex set V and edge set E. The *deletion-contraction* relation states that for any edge $e \in E$

$$P(G, \lambda) = P(G\backslash e, \lambda) - P(G/e, \lambda)$$

where $G\backslash e$ is the graph obtained by deleting the edge e in G and G/e is the graph obtained by identifying the endpoints of e and discarding any multiple edges and loops introduced by the identification.

The *addition-identification* relation states for any $uv \notin E$

$$P(G, \lambda) = P(G + uv, \lambda) + P(G/uv, \lambda)$$

where $G + uv$ is the graph obtained by adding the edge uv to G and G/uv is the graph obtained by identifying the vertices u and v and discarding any multiple edges introduced by the identification.

The chromatic polynomial of the complete graph on n vertices is

$$P(K_n, \lambda) = \lambda(\lambda - 1)\ldots(\lambda - n + 1)$$

and the chromatic polynomial of the cycle on n vertices is

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

A *θ-graph* is a graph that can be obtained from three disjoint paths $u_0, u_1, \ldots, u_a$, $v_0, v_1, \ldots, v_b$ and $w_0, w_1, \ldots, w_c$, $a, b, c \geqslant 1$, by identifying vertices u_0, v_0 and w_0 and identifying vertices u_a, v_b and w_c. We call this graph $\theta_{a,b,c}$.

The chromatic polynomial is a monic polynomial with integer coefficients [**28, 30, 42**] and so $P(G, \lambda) \in \mathbb{Q}[\lambda]$. The *splitting field*, L, of a polynomial $p(\lambda) \in \mathbb{Q}[\lambda]$ is the smallest extension field of $\mathbb{Q}$ containing all the roots of $p(\lambda)$. The *Galois group* of $p(\lambda)$ is the group of automorphisms of L that point-wise fix the elements of $\mathbb{Q}$. The Galois group is isomorphic (under the *Galois correspondence*) to a subgroup of the symmetric group acting on the roots, so in this form it is a permutation group.

If $p(\lambda)$ has only linear factors, then all the roots lie in $\mathbb{Q}$ so the splitting field of $p(\lambda)$ is $\mathbb{Q}$ and the Galois group of $p(\lambda)$ is the trivial group S_1. Thus chromatic polynomials that have only integer roots have the trivial Galois group. For example, chromatic polynomials of complete graphs have the trivial Galois group.

If $p(\lambda)$ has non-linear factors, then the splitting field is a non-trivial extension field of $\mathbb{Q}$. If $p(\lambda)$ has a single non-linear factor $p'(\lambda)$, then the Galois group of $p(\lambda)$ is the Galois group of $p'(\lambda)$. For example, the Galois group of $P(C_4, \lambda) = \lambda(\lambda - 1)(\lambda^2 - 3\lambda + 3)$ is the Galois group of the non-linear factor $\lambda^2 - 3\lambda + 3$. Here the splitting field is $\mathbb{Q}(\sqrt{-3})$. The automorphisms of $\mathbb{Q}(\sqrt{-3})$ that fix $\mathbb{Q}$ are the identity map and the bijection $\sqrt{-3} \leftrightarrow -\sqrt{-3}$ and so the Galois group is isomorphic to the symmetric group S_2.

If $p(\lambda)$ has more than one non-linear factor, then the roots of these factors may interact in a non-trivial way. For example, the chromatic polynomial of the cycle on 7 vertices is

$$P(C_7, \lambda) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 3\lambda + 3)(\lambda^2 - \lambda + 1)$$

which has non-integer roots $\{(3 \pm \sqrt{-3})/2, (1 \pm \sqrt{-3})/2\}$. Each of the quadratic factors has Galois group S_2. However, the extension field $\mathbb{Q}(\sqrt{-3})$ contains all these roots and is the splitting field for this polynomial. So again the Galois group is S_2.

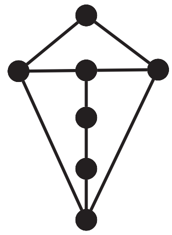

FIGURE 1. *Graph with chromatic polynomial* $\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)(\lambda^2-3\lambda+4)$.

The chromatic polynomial of the graph in Figure 1 also has two non-linear factors, $\lambda^2-3\lambda+3$ and $\lambda^2-3\lambda+4$. However the roots of the second factor do not lie in $\mathbb{Q}(\sqrt{-3})$. In this case the roots are $\{(3\pm\sqrt{-3})/2, (3\pm\sqrt{-7})/2\}$. Here the splitting field is $\mathbb{Q}(\sqrt{-3}, \sqrt{-7})$. The automorphisms of $\mathbb{Q}(\sqrt{-3}, \sqrt{-7})$ that fix $\mathbb{Q}$ are the identity map, $\sigma_1 = \sqrt{-3} \leftrightarrow -\sqrt{-3}$, $\sigma_2 = \sqrt{-7} \leftrightarrow -\sqrt{-7}$ and $\sigma_3 = \sigma_1 \circ \sigma_2$ and so the Galois group is isomorphic to the Klein group which has order 4.

3. *Galois group computation*

Tables A.2–A.11 in Appendix A give a list of all Galois groups of non-linear factors of chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and the number of times they occur. We give a list of all chromatic polynomials of strongly non-clique-separable graphs of order at most 8 with the Galois groups of their irreducible non-linear factors in [**21**]. This list includes a list of graph numbers corresponding to the numbering in McKay's collection of simple connected graphs [**20**].

We use the group notation given in [**12**, Appendix A]. A summary of this notation for the groups we encounter is given in Table A.1. Tables A.2–A.4 give the number of strongly non-clique-separable graphs (up to isomorphism) of order $n = 4, 5, 6$ and the number of corresponding chromatic polynomials with given Galois groups. All chromatic polynomials of strongly non-clique-separable graphs of order at most 6 have at most one non-linear factor and so the Galois group of this factor is the Galois group of the chromatic polynomial.

Some chromatic polynomials of strongly non-clique-separable graphs of order >6 have more than one non-linear factor. In Tables A.5–A.11 we give a list of the Galois groups of all non-linear factors. For each entry, we give the number of strongly non-clique-separable graphs (up to isomorphism) and the number of corresponding chromatic polynomials with these Galois groups. In the case where there is more than one non-linear factor, we give the order and the generators of the Galois group of the chromatic polynomial. For example, in Table A.5 we have three graphs of order 7 that have two non-linear factors. There are two graphs with chromatic polynomials that have two quadratic factors. Each of these factors has Galois group S_2. The first of these graphs is the cycle C_7 which has chromatic polynomial $\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)(\lambda^2-\lambda+1)$ which has the Galois group of order 2 generated by $(1,2)(3,4)$. The second graph is the graph in Figure 1 which has chromatic polynomial $\lambda(\lambda-1)(\lambda-2)(\lambda^2-3\lambda+3)(\lambda^2-3\lambda+4)$ which has a Galois group of order 4 with generators $(1,2)$ and $(3,4)$. The chromatic polynomial of the third graph has a quadratic and a cubic factor that have Galois groups S_2 and S_3 respectively. This graph has chromatic polynomial $\lambda(\lambda-1)(\lambda^2-4\lambda+5)(\lambda^3-4\lambda^2+7\lambda-5)$ which has a Galois group of order 12 with generators $(1,2)$, $(3,4,5)$ and $(3,4)$. When the chromatic polynomial has only one irreducible non-linear factor there is no entry given in the Generators column of the tables.

PARI/GP 2.3.0 [**27**] was used to compute the Galois groups of all the irreducible non-linear factors of the chromatic polynomials. In the case where the chromatic polynomial has a single irreducible non-linear factor, the Galois group of this factor is the Galois group of the entire polynomial. When the chromatic polynomial has more than one irreducible non-linear factor, we use Magma V2.14-11 [**44**] to compute its Galois group, as Pari is not able to compute Galois groups of reducible polynomials. In this case we give the order and the generators of the groups found by Magma.

There are only three strongly non-clique-separable graphs of order at most 3: the complete graphs K_1, K_2 and K_3. The chromatic polynomials of these graphs all have the trivial Galois group.

Any chromatic polynomial that factorises into linear factors in $\mathbb{Z}[\lambda]$ has the trivial Galois group. The class of chordal graphs is a class of graphs that have only integer roots [**30**]. The *quasi-chordal graphs* are the graphs that are chromatically equivalent to the chordal graphs. D'Atonna, Mereghetti and Zamparini [**14**] showed that there are 224 non-chordal graphs of at most 9 vertices that have no complex chromatic roots: 206 of these are quasi-chordal graphs. In Section 5.1 we give an infinite family of quasi-chordal graphs that are not chordal graphs.

Chromatic polynomials of degree at most 10 with Galois groups corresponding to each of the transitive permutation groups of even degree $\leqslant 6$ and to each of the transitive permutation groups of odd degree $\leqslant 5$, excluding A_3 and $C(5)$, were found. However, chromatic polynomials exist with Galois group $C(3) \cong A_3$. An example of such a chromatic polynomial was found by Peter Cameron using the ring of cliques structure. We have also found a non-trivial infinite family of graphs that have chromatic polynomials with Galois group $C(3) \cong A_3$. Each graph in this family has a chromatic polynomial with a different irreducible cubic factor that has Galois group $C(3) \cong A_3$.

As any strongly non-clique-separable graph is connected and contains no cut-vertex, its chromatic polynomial is divisible by $\lambda(\lambda - 1)$. As the degree of the chromatic polynomial is the order of the graph, only strongly non-clique-separable graphs of order $\geqslant 9$ may have an irreducible septic factor. It is therefore not surprising that some transitive permutation groups of degree $\geqslant 7$ do not correspond to any Galois group of a chromatic polynomial of degree at most 10.

The majority of chromatic polynomials of degree at most 10 have the symmetric group S_l, $l \in [1, n-2]$, as their Galois group. In Table A.6 about 91% of chromatic polynomials of order 8 with a single non-linear irreducible factor have Galois group S_l, $1 \leqslant l \leqslant 6$, and in the case of chromatic polynomials with more than one irreducible factor, the Galois groups of all but one of these factors are the symmetric groups. In Tables A.7–A.8 over 94% of chromatic polynomials of order 9 with a single non-linear irreducible factor have Galois group S_l, $1 \leqslant l \leqslant 7$, and in over 90% of the cases where the chromatic polynomial has more than one non-linear factor, all its non-linear factors have symmetric Galois groups. In Tables A.9–A.11 almost 99% of chromatic polynomials of order 10 with a single non-linear irreducible factor have Galois group S_l, $1 \leqslant l \leqslant 8$, and in about 90% of the cases where the chromatic polynomial has more than one non-linear factor, all its non-linear factors have symmetric Galois groups. Excluding the symmetric groups, the dihedral groups and cyclic groups appear to occur most frequently in these tables.

4. *Galois groups of chromatic polynomials of θ-graphs*

In Section 1 we commented that the Galois group of the chromatic polynomial of the cycle graph of order n is isomorphic to $(\mathbb{Z}/(n-1)\mathbb{Z})^*$. As the Galois groups of the chromatic polynomials of cycle graphs are well understood, it seemed natural to look at the Galois groups of the chromatic polynomials of θ-graphs. A θ-graph can be obtained from a cycle graph by adding a path that connects two vertices in the cycle. Interestingly, when certain conditions are satisfied the chromatic polynomial of a θ-graph is divisible by the chromatic polynomial of a cycle graph.

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In this section we look at the Galois groups of chromatic polynomials of θ-graphs of order at most 19. The chromatic polynomials of all but two of these graphs have an irreducible factor with a symmetric Galois group of order $\geqslant 3$.

Tables B.1 and B.2 in Appendix B give a list of all Galois groups of the non-linear factors of chromatic polynomials of $\theta_{a,b,c}$-graphs of order at most 19 where $a, b, c \geqslant 2$. The list of θ-graphs is given as a semi-colon-separated list where a, b, c refers to the θ-graph $\theta_{a,b,c}$, $a \geqslant b \geqslant c$. The Galois group of each irreducible non-linear factor of each chromatic polynomial was calculated using Magma.

Table B.1 gives the number of θ-graphs with chromatic polynomials having a given Galois group where the chromatic polynomial has a single non-linear irreducible factor. These correspond to 75 of the 147 $\theta_{a,b,c}$-graphs, $a, b, c \geqslant 2$, of order $n \leqslant 19$. All but two of these chromatic polynomials have Galois group $S_{n-\chi(\theta_{a,b,c})}$ where $\chi(\theta_{a,b,c})$, the chromatic number of the graph, is 2 if a, b and c have the same parity and is 3 otherwise. There are two exceptions: the chromatic polynomials of $\theta_{2,3,3}$ and $\theta_{2,3,5}$ that have Galois group $D(4)$ and $2S_4(6)$ respectively.

Table B.2 gives the number of θ-graphs with chromatic polynomials having more than one irreducible non-linear factor. These correspond to 72 of the 147 $\theta_{a,b,c}$-graphs, $a, b, c \geqslant 2$, of order $n \leqslant 19$. This table gives a list of the Galois groups of all the irreducible non-linear factors of each chromatic polynomial. In each case one of the Galois groups is the symmetric Galois group S_l for $3 \leqslant l \leqslant n - \chi(\theta_{a,b,c})$. The other non-linear irreducible factors are all factors of chromatic polynomials of cycle graphs. This is not surprising as by applying a single addition-identification relation to any θ-graph we can express the chromatic polynomial as

$$P(\theta_{a,b,c}, \lambda) = \frac{P(C_{a+1}, \lambda)P(C_{b+1}, \lambda)P(C_{c+1}, \lambda)}{P(K_2, \lambda)P(K_2, \lambda)} + \frac{P(C_a, \lambda)P(C_b, \lambda)P(C_c, \lambda)}{P(K_1, \lambda)P(K_1, \lambda)}. \tag{1}$$

We consider $C_2 \cong K_2$.

From (1) it is clear that if any of the following occur the chromatic polynomial of $P(\theta_{a_0,a_1,a_2}, \lambda)$ is divisible by the chromatic polynomial of some cycle graph:

- $P(C_{a_i+1}, \lambda)$ divides $P(C_{a_j}, \lambda)$;
- $P(C_{a_i}, \lambda)$ divides $P(C_{a_j+1}, \lambda)$;
- for some cycle C, the chromatic polynomial $P(C, \lambda)$ divides $P(C_{a_i}, \lambda)$ and $P(C_{a_j+1}, \lambda)$,

for some $i, j \in \{0, 1, 2\}$. If more than one of these cases occur, $P(\theta_{a_0,a_1,a_2}, \lambda)$ is divisible by the chromatic polynomials of at least two cycle graphs.

In [**22**, pp. 97–100] we gave a number of families of θ-graphs that have chromatic polynomials that are divisible by chromatic polynomials of cycle graphs of lower degree and showed that these divisibility properties can be proved by applying some simple operations on the graphs.

Table B.3 in Appendix B gives the Galois groups of the chromatic polynomials of cycles of order $4 \leqslant n \leqslant 9$ and the θ-graphs of order at most 19 (excluding $\theta_{2,7,9}$) that have chromatic polynomials that can be expressed as

$$P(C_n, \lambda)P(\lambda) \quad \text{or} \quad P(C_n, \lambda)(\lambda - 2)P(\lambda)$$

where $P(\lambda)$ is an irreducible factor that has the symmetric Galois group. The triplet, a, b, c, is considered to be unordered. For example, 2, 4, 5 corresponds to the graph $\theta_{a,a+1,b}$ and 2, 3, 7 corresponds to the graph $\theta_{a,2^{k-1}a+1,b}$. The graph $\theta_{2,7,9}$ is not included in Table B.3. The chromatic polynomial of this graph is

$$\begin{aligned} P(\theta_{2,7,9}, \lambda) = &\frac{P(C_3, \lambda)P(C_8, \lambda)P(C_{10}, \lambda)}{P(K_2, \lambda)P(K_2, \lambda)} \\ &+ \frac{P(C_2, \lambda)P(C_7, \lambda)P(C_9, \lambda)}{P(K_1, \lambda)P(K_1, \lambda)}. \end{aligned}$$

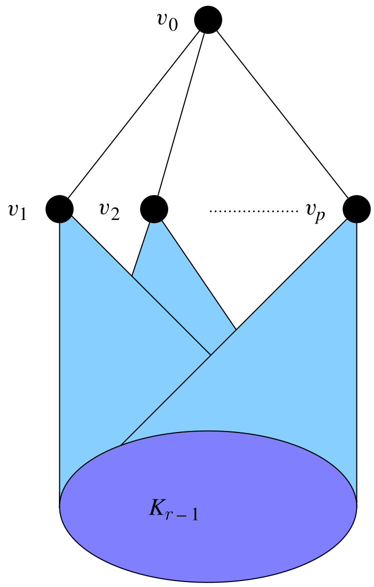

FIGURE 2. *Graph* $G_{p,0,r}$.

In this case $P(C_4, \lambda)$ divides $P(\theta_{2,7,9}, \lambda)$ as $P(C_4, \lambda)$ divides both $P(C_{10}, \lambda)$ and $P(C_9, \lambda)$. Cases where the chromatic polynomial of the θ-graph is divisible by the chromatic polynomials of two cycles, C_{n_1} and C_{n_2}, of order $\geqslant 4$ are not included in Table B.3.

Every chromatic polynomial in Table B.2 has only one irreducible factor that is not the factor of some cycle graph. This factor has the symmetric Galois group, which leads to the following question.

PROBLEM 1. Can the chromatic polynomial $P(\theta_{a,b,c}, \lambda)$, excluding $P(\theta_{2,3,3}, \lambda)$ and $P(\theta_{2,3,5}, \lambda)$, always be expressed

$$P(\theta_{a,b,c}, \lambda) = \prod_{i=0}^{j} \frac{P(C_{s_i}, \lambda)}{P(K_{r_i}, \lambda)} p(\lambda)$$

where $j > 0$, $s_i \geqslant 2$, $r_i > 0$ and $p(\lambda)$ is an irreducible polynomial with Galois group S_{n-l} where $\chi(\theta_{a,b,c}) \leqslant l < j$?

5. *Infinite families of Galois equivalent graphs*

In this section we will give some families of graphs that have chromatic polynomials with Galois groups S_1, S_2, $C(3) \cong A_3$, $C(4)$, and $D(5)$.

Let $G_{p,0,r}$ be the graph obtained by identifying a clique of order $r-1$, $r \geqslant 2$, in $p \geqslant 2$ copies of K_r and connecting the vertices that are not identified in this $(r-1)$-clique, which we label $v_1, v_2, \ldots, v_p$, to vertex v_0 (see Figure 2). The vertices in the intersecting clique we label $v_{p+1}, v_{p+2}, \ldots, v_{p+r-1}$. The graph $G_{p,q,r}$ is the graph $G_{p,0,r}$ with $q \in [0, r-1]$ additional edges, which connect vertex v_0 to q distinct vertices, $v_{p+1}, v_{p+2}, \ldots, v_{p+q}$.

First we show the following theorem.

THEOREM 1. *The chromatic polynomial*

$$P(G_{p,0,r}, \lambda) = P(K_r, \lambda)((\lambda - r)^p + (r-1)(\lambda - r + 1)^{p-1})$$

for $p \geqslant 2$ *and* $r \geqslant 2$.

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Proof. The proof uses induction on p.
Suppose $p=2$, then we have

$$\begin{aligned}
P(G_{2,0,r},\lambda) &= P(G_{2,0,r}+(v_1,v_2),\lambda)+P(G_{2,0,r}/(v_1,v_2),\lambda)\\
&= \frac{P(K_{r+1},\lambda)P(K_3,\lambda)}{P(K_2,\lambda)}+\frac{P(K_r,\lambda)P(K_2,\lambda)}{P(K_1,\lambda)}\\
&= P(K_r,\lambda)((\lambda-r)(\lambda-2)+(\lambda-1))\\
&= P(K_r,\lambda)(\lambda^2-(r+1)\lambda+2r-1)\\
&= P(K_r,\lambda)(\lambda^2-2r\lambda+r^2+(r-1)\lambda-r^2+2r-1)\\
&= P(K_r,\lambda)((\lambda-r)^2+(r-1)\lambda-(r-1)^2)\\
&= P(K_r,\lambda)((\lambda-r)^2+(r-1)(\lambda-r+1)).
\end{aligned}$$

Suppose $p>2$, then

$$\begin{aligned}
P(G_{p,0,r},\lambda) &= P(G_{p,0,r}\backslash(v_0,v_p),\lambda)-P(G_{p,0,r}/(v_0,v_p),\lambda)\\
&= \frac{P(G_{p-1,0,r},\lambda)P(K_r,\lambda)}{P(K_{r-1},\lambda)}-\frac{P(K_{r+1},\lambda)^{p-1}}{P(K_r,\lambda)^{p-2}}\\
&= (\lambda-r+1)P(G_{p-1,0,r},\lambda)-(\lambda-r)^{p-2}P(K_{r+1},\lambda)\\
&= (\lambda-r+1)P(G_{p-1,0,r},\lambda)-(\lambda-r)^{p-1}P(K_r,\lambda)
\end{aligned}$$

which by induction becomes

$$\begin{aligned}
&P(G_{p,0,r},\lambda)\\
&\quad= (\lambda-r+1)P(K_r,\lambda)((\lambda-r)^{p-1}+(r-1)(\lambda-r+1)^{p-2})-(\lambda-r)^{p-1}P(K_r,\lambda)\\
&\quad= P(K_r,\lambda)((\lambda-r+1)(\lambda-r)^{p-1}+(\lambda-r+1)(r-1)(\lambda-r+1)^{p-2}-(\lambda-r)^{p-1})\\
&\quad= P(K_r,\lambda)((\lambda-r)^p+(\lambda-r)^{p-1}+(r-1)(\lambda-r+1)^{p-1}-(\lambda-r)^{p-1})\\
&\quad= P(K_r,\lambda)((\lambda-r)^p+(r-1)(\lambda-r+1)^{p-1}). \qquad\square
\end{aligned}$$

THEOREM 2. *The chromatic polynomial*

$$P(G_{p,q,r},\lambda)=P(K_r,\lambda)((\lambda-r)^p+(r-q-1)(\lambda-r+1)^{p-1})$$

for $p\geqslant 2$, $0\leqslant q\leqslant r-1$ *and* $r\geqslant 2$.

Proof. The proof uses induction on q. When $q=0$, the result follows from Theorem 1.
Suppose $q>0$. Then

$$\begin{aligned}
P(G_{p,q,r},\lambda) &= P(G_{p,q,r}\backslash(v_0,v_{p+q}),\lambda)-P(G_{p,q,r}/(v_0,v_{p+q}),\lambda)\\
&= P(G_{p,q-1,r},\lambda)-\frac{P(K_r,\lambda)^p}{P(K_{r-1},\lambda)^{p-1}}\\
&= P(G_{p,q-1,r},\lambda)-P(K_r,\lambda)(\lambda-r+1)^{p-1},
\end{aligned}$$

which by induction becomes

$$\begin{aligned}
P(G_{p,q,r},\lambda) &= P(K_r,\lambda)((\lambda-r)^p+(r-q)(\lambda-r+1)^{p-1})-P(K_r,\lambda)(\lambda-r+1)^{p-1}\\
&= P(K_r,\lambda)((\lambda-r)^p+(r-q-1)(\lambda-r+1)^{p-1}). \qquad\square
\end{aligned}$$

It is clear from Theorem 2 that the Galois group of $P(G_{p,q,r},\lambda)$ is the Galois group of the factor $(\lambda-r)^p+(r-q-1)(\lambda-r+1)^{p-1}$. If this polynomial is irreducible, the Galois group is a subgroup of S_p. We consider some families of graphs $G_{p,q,r}$ where $p=2$, 3 and 4. In the cases where $p=3,4$, we identify some families of Galois equivalent graphs that have chromatic

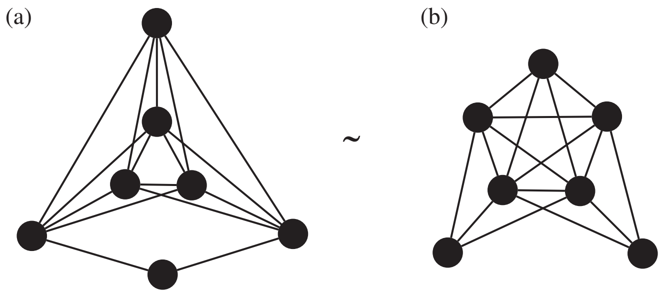

FIGURE 3. *A pair of chromatically equivalent graphs.*

polynomials with Galois groups that are not S_p. As most chromatic polynomials of strongly non-clique-separable graphs of order at most 10 have the symmetric Galois group, infinite families of graphs that have chromatic polynomials with other Galois groups are especially interesting. In particular we have found a family of graphs that have Galois group $C(3) \cong A_3$, a group that does not occur in our tables in Appendix A.

5.1. *Galois groups S_1 and S_2*

When $p = 2$, the chromatic polynomial of $P(G_{2,q,r}, \lambda)$ is

$$\begin{aligned} P(G_{2,q,r}, \lambda) &= P(K_r, \lambda)((\lambda - r)^2 + (r - q - 1)(\lambda - r + 1)) \\ &= P(K_r, \lambda)(\lambda^2 - (r + q + 1)\lambda + (2r + qr - q - 1)). \end{aligned} \tag{2}$$

Now the Galois group of (2) is the Galois group of its quadratic factor which is S_1 if the quadratic is reducible, that is when $r = q + 1$ or $r = q + 5$, and is S_2 otherwise. The quasi-chordal graphs are precisely the graphs that have chromatic polynomials with only integer roots and so have the trivial Galois group.

COROLLARY 1. *The graphs in the family $\{G_{2,r-1,r} : r \geqslant 2\} \cup \{G_{2,r-5,r} : r \geqslant 5\}$ are quasi-chordal graphs.*

It is clear that $G_{p,q,r}$, $q \neq r - 1$, is not chordal. So Corollary 1 gives an infinite family of non-chordal graphs, $\{G_{2,r-5,r} : r \geqslant 5\}$, that are chromatically equivalent to chordal graphs. The smallest graph belonging to this family is the graph $G_{2,0,5}$, the smallest (in terms of order) non-chordal quasi-chordal graph (see Figure 3).

5.2. *Galois group $C(3) \cong A_3$*

When $p = 3$ we have

$$\begin{aligned} P(G_{3,q,r}, \lambda) &= P(K_r, \lambda)((\lambda - r)^3 + (r - q - 1)(\lambda - r + 1)^2) \\ &= P(K_r, \lambda)(\lambda^3 - (2r + q + 1)\lambda^2 + (r^2 + (2q + 4)r - 2(q + 1))\lambda \\ &\quad - ((q + 3)r^2 - (2q + 3)r + q + 1). \end{aligned} \tag{3}$$

Now the Galois group of (3) is the Galois group of its cubic factor that we will call $f(\lambda)$. We show that when $r = c^2 + c + 8 + q$, $c \in \mathbb{N} \cup \{0\}$, this factor has Galois group A_3 and thus $P(G_{3,q,c^2+c+8+q}, \lambda)$ has Galois group A_3.

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When $r = c^2 + c + 8 + q$, the cubic factor in (3) becomes

$$
\begin{aligned}
f(\lambda) = \lambda^3 &- (3q + 1 + 2(c^2 + c + 8))\lambda^2 \\
&+ ((c^2 + c + 8)^2 + 4(q + 1)(c^2 + c + 8) + 3q^2 + 2(q - 1))\lambda \\
&- ((q + 3)(c^2 + c + 8)^2 + (2q^2 + 4q - 3)(c^2 + c + 8) + (q^3 + q^2 - 2q + 1)).
\end{aligned} \tag{4}
$$

We will use the following theorem from [**16**, Theorem 17.3, p. 259] to show that this cubic is irreducible for all $c \geqslant 0$.

THEOREM 3. *Let p be a prime and $f(x) \in \mathbb{Z}[x]$ with degree at least 1. Let $\overline{f}(x)$ be the polynomial in $\mathbb{Z}_p[x]$ obtained by reducing all the coefficients of $f(x)$ modulo p. If $\overline{f}(x)$ is irreducible $\mathbb{Z}_p$ and has the same degree as $f(x)$, then $f(x)$ is irreducible over $\mathbb{Q}$.*

LEMMA 1. *The cubic $f(\lambda)$ in (4) is irreducible for $c \in \mathbb{Z}$.*

Proof. The cubic $f(\lambda) \equiv \lambda^3 - (q + 1)\lambda^2 + q^2\lambda - 1 \pmod 2$. There are two cases: $q \equiv 0 \pmod 2$ and $q \equiv 1 \pmod 2$.

Case 1: ($q \equiv 0 \pmod 2$). If $q \equiv 0 \pmod 2$, then $f(\lambda) \equiv \lambda^3 - \lambda^2 - 1 \pmod 2$ which is irreducible mod 2.

Case 2: ($q \equiv 1 \pmod 2$). If $q \equiv 1 \pmod 2$, then $f(\lambda) \equiv \lambda^3 + \lambda - 1 \pmod 2$ which is irreducible mod 2.

By Theorem 3, as $f(\lambda)$ is irreducible modulo $p = 2$, the cubic $f(\lambda)$ is irreducible. □

THEOREM 4. *The family of graphs $\{G_{3,q,c^2+c+8+q} : c \geqslant 0, q \geqslant 0\}$ is a family of Galois equivalent graphs with each $P(G_{3,q,c^2+c+8+q}, \lambda)$ having Galois group $C(3) \cong A_3$.*

Proof. The chromatic polynomial of $G_{3,q,c^2+c+8+q}$ is

$$P(G_{3,q,c^2+c+8+q}, \lambda) = P(K_{c^2+c+8+q}, \lambda) f(\lambda)$$

where $f(\lambda)$ is the cubic given in (4). The Galois group of $P(G_{3,q,c^2+c+8+q}, \lambda)$ is determined by the Galois group of this cubic which by Lemma 1 is irreducible.

The discriminant of $f(\lambda)$ is

$$\Delta(f) = (c^2 + c + 7)^2(2c + 1)^2.$$

As $\sqrt{\Delta(f)} \in \mathbb{Q}$ and $f(\lambda)$ is a monic irreducible polynomial in $\mathbb{Q}[\lambda]$, the Galois group of $f(\lambda)$ is A_3 (see [**13**, Proposition 7.4.2, p. 169]). Thus $P(G_{3,q,c^2+c+8+q}, \lambda)$ has Galois group $C(3) \cong A_3$ and $\{G_{3,q,c^2+c+8+q} : c \geqslant 0, q \geqslant 0\}$ is a family of Galois equivalent graphs. □

The smallest graph in this family is $G_{3,0,8}$ which has order 11. Our comprehensive list of Galois groups of all chromatic polynomials of strongly non-clique-separable graphs of order $\leqslant 10$ in Appendix A has no chromatic polynomial with Galois group $C(3) \cong A_3$. Thus the graph $G_{3,0,8}$ is the smallest graph, in terms of order, that has a chromatic polynomial with Galois group $C(3) \cong A_3$.

The graph $G_{3,q,r}$, $q \neq r - 1$, is not clique-separable. If G is a graph, a Galois equivalent graph of larger order can always be obtained by gluing a quasi-chordal graph to G. In this case the Galois-equivalent graph is a clique-separable graph. However, the non-linear factors of the chromatic polynomial of this graph are the same as those of $P(G, \lambda)$. In contrast each $P(G_{3,q,c^2+c+8+q}, \lambda)$ has a different cubic factor. The graph $G_{3,q,c^2+c+8+q}$ is non-clique-separable and has order $n = c^2 + c + 11 + q$, $c \geqslant 0$, $q \geqslant 0$. By Theorem 4 it has a chromatic polynomial with Galois group $C(3) \cong A_3$. We therefore have the following corollary.

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COROLLARY 2. *For all $n \geqslant 11$, there exists a non-clique-separable graph G of order n whose chromatic polynomial has Galois group $C(3) \cong A_3$.*

5.3. *Galois groups $C(4)$ and $D(4)$*

We now consider $G_{p,q,r}$ when $p=4$. From Theorem 2 we have

$$\begin{aligned} P(G_{4,q,r},\lambda) &= P(K_r,\lambda)((\lambda-r)^4+(r-q-1)(\lambda-r+1)^3) \\ &= P(K_r,\lambda)(\lambda^4-(3r+q+1)\lambda^3+3(r^2+(q+2)r-q-1)\lambda^2 \\ &\quad -(r^3+3(q+3)r^2-3(2q+3)r+3(q+1))\lambda \\ &\quad +(q+4)r^3-3(q+2)r^2+(3q+4)r-(q+1)). \end{aligned} \tag{5}$$

Now the Galois group of (5) is the Galois group of its quartic factor. We consider the case when $r=c^3+4c^2+1+q$, $c\in\mathbb{N}$. When $c=1$, we show that this factor has Galois group $C(4)$ and thus $P(G_{4,q,6+q},\lambda)$ has Galois group $C(4)$. When $c\geqslant 2$, we show that this factor has Galois group $D(4)$ and thus $P(G_{4,q,c^3+4c^2+1+q},\lambda)$, $c\geqslant 2$, has Galois group $D(4)$.

In determining the Galois groups of quartics we will use the following theorem.

THEOREM 5 [**13**, Theorem 13.1.1]. *If $f(\lambda)=\lambda^4-a_1\lambda^3+a_2\lambda^2-a_3\lambda+a_4$ is an irreducible quartic in $\mathbb{Q}[\lambda]$ with discriminant $\Delta(f)$ and Ferrari resolvent $\theta_f(y)=y^3-a_2y^2+(a_1a_3-4a_4)y-a_3^2-a_1^2a_4+4a_2a_4$, then the Galois group G can be determined as follows.*

(i) *If $\theta_f(y)$ is irreducible over $\mathbb{Q}$, then*

$$G\cong\begin{cases} S_4 & \text{if } \sqrt{\Delta(f)}\notin\mathbb{Q}, \\ A_4 & \text{if } \sqrt{\Delta(f)}\in\mathbb{Q}. \end{cases}$$

(ii) *If $\theta_f(y)$ splits completely in $\mathbb{Q}$, then $G\cong E(4)$, the Klein group. Furthermore $\theta_f(y)$ splits completely in $\mathbb{Q}$ if and only if it is reducible over $\mathbb{Q}$ and $\sqrt{\Delta(f)}\in\mathbb{Q}$.*

(iii) *If $\theta_f(y)$ has a unique root $\beta\in\mathbb{Q}$, then*

$$G\cong\begin{cases} D(4) & \text{if } 4\beta+a_1^2-4a_2\neq 0 \text{ and } \Delta(f)(4\beta+a_1^2-4a_2) \text{ is not a square in } \mathbb{Q}\backslash\{0\}, \\ & \text{or } 4\beta+a_1^2-4a_2=0 \text{ and } \Delta(f)(\beta^2-4a_4) \text{ is not a square in } \mathbb{Q}\backslash\{0\}, \\ C(4) & \text{otherwise.} \end{cases}$$

When $r=c^3+4c^2+1+q$, $c\in\mathbb{N}$, the quartic factor of (5) becomes

$$\begin{aligned} g(\lambda) &= \lambda^4-(3(c^3+4c^2+1)+4q+1)\lambda^3 \\ &\quad +3((c^3+4c^2+1)^2+3(c^3+4c^2+1)q+2(c^3+4c^2+1)+2q^2+q-1)\lambda^2 \\ &\quad -((c^3+4c^2+1)^3+3(2q+3)(c^3+4c^2+1)^2+3(3q^2+4q-3)(c^3+4c^2+1) \\ &\qquad +(4q^3+3q^2-6q+3))\lambda \\ &\quad +(q+4)(c^3+4c^2+1)^3+3(q^2+3q-2)(c^3+4c^2+1)^2 \\ &\quad +(3q^3+6q^2-9q+4)(c^3+4c^2+1)+(q^4+q^3-3q^2+3q-1). \end{aligned} \tag{6}$$

We will use the following lemma in our proof of the irreducibility of $g(\lambda)$.

LEMMA 2. *There are no positive integer solutions (c,d) of the equation $c^2+4c=d^2$.*

Proof. If $d-c\leqslant 1$, then c is not a positive integer. Thus, $d-c\geqslant 2$ and so

$$\begin{gathered} 2(d-c)c\geqslant 4c \\ (d-c)^2+2(d-c)c>4c \\ d^2-c^2>4c. \end{gathered}$$

But this implies $c^2+4c<d^2$, and thus $c^2+4c\neq d^2$. □

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LEMMA 3. *The quartic $g(\lambda)$ in (6) is irreducible for $c \in \mathbb{N}$ and $q \geqslant 0$.*

Proof. Suppose, in order to obtain a contradiction, $g(\lambda)$ is reducible. The proof considers two cases: $g(\lambda)$ has an integer root, or $g(\lambda)$ can be expressed as a product of two quadratics in $\mathbb{Z}[\lambda]$.

Case 1: *$g(\lambda)$ has an integer root.* Suppose $g(\lambda)$ has an integer root. As $g(\lambda)$ is a factor of a chromatic polynomial, any integer root is a natural number $<r$. Let $r-k$, $1 \leqslant k \leqslant r-1$, be a root of $g(\lambda)$. Now

$$g(\lambda) = (\lambda - r)^4 + (r-q-1)(\lambda - r + 1)^3$$

where $r = c^3 + 4c^2 + q + 1$. If $r-k$ is a root, we have

$$k^4 - (r-q-1)(k-1)^3 = 0. \tag{7}$$

Now $k \neq 1$, as $k=1$ does not satisfy (7). So $r-1$ is not a root of $g(\lambda)$.

If $2 \leqslant k \leqslant r-1$, then (7) becomes

$$r = \frac{k^4}{(k-1)^3} + q + 1. \tag{8}$$

Now, $r \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{0\}$. So, if $r-k$ is a root of $g(\lambda)$, then $k^4/(k-1)^3$ must be in $\mathbb{N}$.

When $k=2$, $k^4/(k-1)^3 = 16 \in \mathbb{N}$. As $r = c^3 + 4c^2 + q + 1$, (8) becomes

$$c^3 + 4c^2 = 16.$$

When $c=1$, $c^3 + 4c^2 = 5 \neq 16$ and when $c \geqslant 2$ we have $c^3 + 4c^2 > 16$. So $r-2$ is not a root of $g(\lambda)$.

Now $k^4/(k-1)^3 = 81/8, 256/27, 625/64, 1296/125, 2401/216 \notin \mathbb{N}$ when $k = 3, 4, 5, 6, 7$, respectively, so $r-k$, $3 \leqslant k \leqslant 7$, is not a root of $g(\lambda)$.

Now

$$\frac{k^4}{(k-1)^3} = k + 3 + \frac{6k^2 - 8k + 3}{(k-1)^3}. \tag{9}$$

When $k \geqslant 8$, then

$$\begin{aligned} 0 < 6k^2 - 8k + 3 &< (k-1)^3, \\ 0 < \frac{6k^2 - 8k + 3}{(k-1)^3} &< 1, \\ k + 3 < k + 3 + \frac{6k^2 - 8k + 3}{(k-1)^3} &< k + 4, \\ k + 3 < \frac{k^4}{(k-1)^3} &< k + 4. \end{aligned}$$

So $k^4/(k-1)^3 \notin \mathbb{N}$, and so $r-k$ is not a root of $g(\lambda)$.

Case 2: *$g(\lambda)$ factors into quadratic polynomials.* Suppose $g(\lambda)$ factors into quadratic polynomials in $\mathbb{Q}[\lambda]$.

The reduced quartic of $g(\lambda)$ is

$$\begin{aligned} &g(y + q + \tfrac{3}{4}c^3 + 3c^2 + 1) \\ &\quad = y^4 - (\tfrac{3}{8}c^6 + 3c^5 + 6c^4 - 3c^3 - 12c^2)y^2 \\ &\qquad + (\tfrac{1}{8}c^9 + \tfrac{3}{2}c^8 + 6c^7 + \tfrac{13}{2}c^6 - 12c^5 - 24c^4 + 3c^3 + 12c^2)y \\ &\qquad - \tfrac{3}{256}c^{12} - \tfrac{3}{16}c^{11} - \tfrac{9}{8}c^{10} - \tfrac{45}{16}c^9 - \tfrac{3}{4}c^8 + 9c^7 + \tfrac{45}{4}c^6 - 6c^5 - 12c^4 + c^3 + 4c^2. \end{aligned}$$

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The quartic factors into quadratic factors if and only if at least one of the following holds:

- $D = 0$ and $C^2 - 4E$ is a square in $\mathbb{Q}$;
- the resolvent of g, $R(z) = z^3 + 2Cz^2 + (C^2 - 4E)z - D^2$, has a nonzero root that is a square in $\mathbb{Q}$,

where C and D are the the coefficients of y^2 and y respectively and E is the constant term in the reduced quartic [9, Theorem 1].

Now

$$\begin{aligned} D &= \tfrac{1}{8}c^9 + \tfrac{3}{2}c^8 - 12c^5 + \tfrac{13}{2}c^6 + 3c^3 - 24c^4 + 6c^7 + 12c^2 \\ &= \tfrac{1}{8}c^2(c+4)(c^2+2c-2)(c^4+6c^3+6c^2-12c-12) > 0 \end{aligned}$$

for $c > 1$. (When $c = 1$, $D = -55/8 \neq 0$.) So D is nonzero for all $c \in \mathbb{N}$.

Suppose the resolvent of g has a nonzero root that is a square in $\mathbb{Q}$. The resolvent of g is

$$\begin{aligned} R(z) = {} & \tfrac{1}{64}(4z - 16c + 28c^2 + 8c^3 - 16c^4 - 8c^5 - c^6) \\ & \times (16z^2 - 8zc^6 - 64zc^5 - 128zc^4 + 64zc^3 + 272zc^2 + 64zc - 816c^7 \\ & \quad - 1152c^6 + 288c^5 + 60c^8 + 240c^9 + 576c^3 + 1296c^4 + c^{12} + 16c^{11} + 96c^{10}). \end{aligned}$$

When $c = 1$, the resolvent becomes

$$\begin{aligned} R(z) &= \tfrac{1}{64}(64z^3 + 720z^2 + 1420z - 3025) \\ &= \tfrac{1}{64}(4z-5)(16z^2 + 200z + 605) \end{aligned}$$

which has a single rational root, $5/4$, which is not a square in $\mathbb{Q}$.

When $c \geqslant 2$, the resolvent has discriminant

$$-c^6(c+4)^3(3c-4)(3c+8)^2 < 0.$$

So $R(z)$ has one real root,

$$\frac{c(c+4)(c^2+2c-2)^2}{4}.$$

If this root is a square in $\mathbb{Q}$, then there exists $d \in \mathbb{N}$ such that

$$d^2 = c(c+4).$$

But this contradicts Lemma 2, and thus the quartic $g(\lambda)$ is irreducible. $\square$

THEOREM 6. *The family of graphs $\{G_{4,q,6+q} : q \geqslant 0\}$ is a family of Galois equivalent graphs with each $P(G_{4,q,6+q}, \lambda)$ having Galois group $C(4)$.*

Proof. The chromatic polynomial of $G_{4,q,6+q}$ is

$$\begin{aligned} P(G_{4,q,6+q}, \lambda) &= P(K_r, \lambda)((\lambda - 6 - q)^4 + 5(\lambda - 5 - q)^3) \\ &= P(K_r, \lambda)(\lambda^4 - (4q+19)\lambda^3 + 3(2q^2 + 19q + 47)\lambda^2 \\ &\quad - (4q^3 + 57q^2 + 282q + 489)\lambda + (q^4 + 19q^3 + 141q^2 + 489q + 671)). \end{aligned}$$

The Galois group of this polynomial is determined by the Galois group of the quartic

$$\begin{aligned} g(\lambda) = {} & \lambda^4 - (4q+19)\lambda^3 + 3(2q^2+19q+47)\lambda^2 \\ & - (4q^3 + 57q^2 + 282q + 489)\lambda + (q^4 + 19q^3 + 141q^2 + 489q + 671) \end{aligned} \tag{10}$$

which by Lemma 3 is irreducible.

The discriminant of $g(\lambda)$ is

$$\Delta(g) = 15125.$$

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The Ferrari resolvent is

$$\begin{aligned}\theta_g(y) &= y^3 - 3(2q^2+19q+47)y^2 + (12q^4+228q^3+5358q+1647q^2+6607)y \\ &\quad - (2q^2+19q+52)(4q^4+76q^3+539q^2+1691q+1979) \\ &= (y-52-19q-2q^2)(y^2-(4q^2+89+38q)y+(4q^4+76q^3+539q^2+1691q+1979)).\end{aligned}$$

Now the quadratic factor of $\theta_g(y)$ has discriminant 5 and thus this factor does not factor in $\mathbb{Q}[\lambda]$. Thus $\theta_g(y)$ has a unique rational root $\beta = 2q^2+19q+52$.

By Theorem 5 if $\theta_g(y)$ has a unique root in $\mathbb{Q}$, then the Galois group is $D(4)$ if:

- $4\beta + a_1^2 - 4a_2 \neq 0$ and $\Delta(g)(4\beta + a_1^2 - 4a_2)$ is not a non-zero square in $\mathbb{Q}$; or
- $4\beta + a_1^2 - 4a_2 = 0$ and $\Delta(g)(\beta^2 - 4a_4)$ is not a non-zero square in $\mathbb{Q}$,

and is $C(4)$ otherwise, where the a_i are the coefficients of λ^{4-i} in $g(\lambda)$.

Now

$$4\beta + a_1^2 - 4a_2 = 4(2q^2+19q+52) + (4q+19)^2 - 12(2q^2+19q+47) = 5 \neq 0,$$

and

$$\Delta(g)(4\beta + a_1^2 - 4a_2) = 15\,125 \times 5 = 75\,625 = 275^2$$

which is a non-zero square in $\mathbb{Q}$. Therefore $g(\lambda)$ has Galois group $C(4)$. □

THEOREM 7. *The family of graphs* $\{G_{4,q,c^3+4c^2+1+q} : c \geqslant 2, q \geqslant 0\}$ *is a family of Galois equivalent graphs with each* $P(G_{4,q,c^3+4c^2+1+q}, \lambda)$ *having Galois group* $D(4)$.

Proof. The chromatic polynomial of G_{4,q,c^3+4c^2+1+q} is

$$\begin{aligned}P(G_{4,q,c^3+4c^2+1+q}, \lambda) &= P(K_r, \lambda)((\lambda - (c^3+4c^2+1+q))^4 + (c^3+4c^2)(\lambda - (c^3+4c^2+q))^3 \\ &= P(K_r, \lambda) g(\lambda)\end{aligned}$$

where $g(\lambda)$ is the quartic in (6). The Galois group of this polynomial is the Galois group of the quartic $g(\lambda)$ which by Lemma 3 is irreducible.

The discriminant of $g(\lambda)$ is

$$\Delta(g) = -c^6(c+4)^3(3c-4)(3c+8)^2,$$

and the Ferrari resolvent is

$$\begin{aligned}\theta_g(y) = {} & (y - 2 - 4c - 17c^2 - 4c^3 - 16c^4 - 8c^5 - c^6 - 4q - 12c^2q - 3c^3q - 2q^2) \\ & \times (y^2 - (2c^6 + 16c^5 + 32c^4 + (8+6q)c^3 + (31+24q)c^2 - 4c + 8q + 4 + 4q^2)y \\ & + c^{12} + 16c^{11} + 96c^{10} + (6q+264)c^9 + (351+72q)c^8 + (288q+372)c^7 \\ & + (13q^2+484+416q)c^6 + (92+253q+104q^2)c^5 + (488q+208q^2+289)c^4 \\ & + (12q^3+40q^2-44-4q)c^3 + (62+48q^3+158q^2+172q)c^2 \\ & - (8q^2+16q+8)c + 4 + 16q^3 + 24q^2 + 16q + 4q^4).\end{aligned}$$

Now the quadratic factor of $\theta_g(y)$ has discriminant $-c^2(c+4)(3c-4)$ which is negative for all $c \geqslant 2$, and thus the quadratic factor does not factor in $\mathbb{Q}[\lambda]$. Thus $\theta_g(y)$ has a unique rational root $\beta = c^6 + 8c^5 + 16c^4 + 4c^3 + 17c^2 + 4c + 2q^2 + (3c^3 + 12c^2 + 4)q + 2$.

Now as $\theta_g(y)$ has a unique root, by Theorem 5 if $4\beta + a_1^2 - 4a_2 \neq 0$ and $\Delta(g)(4\beta + a_1^2 - 4a_2)$ is not a non-zero square in $\mathbb{Q}$, where the a_i are the coefficients of λ^{4-i} in $g(\lambda)$, then the Galois group of $g(\lambda)$ is $D(4)$.

Now

$$4\beta + a_1^2 - 4a_2 = c(c+4)(c^2+2c-2)^2 > 0 \quad \text{for } c \geqslant 2$$

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and

$$\Delta(g)(4\beta + a_1^2 - 4a_2) = -c^7(c+4)^4(3c-4)(3c+8)^2(c^2+2c-2)^2 < 0 \quad \text{for } c \geqslant 2$$

and thus cannot be a square in $\mathbb{Q}$.

Therefore $g(\lambda)$ has Galois group $D(4)$. □

We calculated $P(G_{4,0,r}, \lambda)$ for $2 \leqslant r \leqslant 20\,000$ and found that it had Galois group S_3 when $r = 17$, Galois group $C(4)$ when $r = 6$, Galois group $D(4)$ when $r = c^3 + 4c^2 + 1$, $c \geqslant 2$ and $r = 4, 9, 10$ and Galois group S_4 for all other values of $r \leqslant 20\,000$. We propose the following conjecture.

CONJECTURE 1. The family of graphs $\{G_{4,0,r}\}$, where $r \neq c^3 + 4c^2 + 1 + q$ and $c \in \mathbb{N}$, excluding the graphs $G_{4,0,4}$, $G_{4,0,9}$, $G_{4,0,10}$ and $G_{4,0,17}$, is a family of Galois equivalent graphs with each $P(G_{4,0,r}, \lambda)$ having Galois group S_4.

6. *Conclusion*

In this paper we presented the first results about Galois groups of chromatic polynomials. The summaries of all Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order at most 10 and of Galois groups of chromatic polynomials of θ-graphs of order at most 19 may be a useful resource for future work.

We gave several families of Galois equivalent graphs. These include graphs with chromatic polynomials that have symmetric, cyclic and dihedral Galois groups. As most chromatic polynomials of strongly non-clique-separable graphs of order at most 10 have symmetric Galois groups, it is particularly interesting to find infinite families of graphs that have chromatic polynomials with these 'rarer' groups. In particular, one of these families was an infinite family of graphs that have chromatic polynomials with the Galois group $C(3)$. The graph of smallest order with a chromatic polynomial with an odd non-trivial cyclic Galois group belongs to this family. There are no known chromatic polynomials with Galois group $C(n)$ for odd $n > 3$. An open problem is the following.

PROBLEM 2. Does there exist a graph whose chromatic polynomial has Galois group $C(n)$, where $n \geqslant 5$ is odd?

Which leads to a more general question posed by Cameron.

PROBLEM 3 [10, Conjecture 4]. For every Galois group G does there exist a graph whose chromatic polynomial has Galois group G?

A natural extension of this work is to investigate the relationship between the structure of a graph and the Galois group of its chromatic polynomial. Chordal graphs are a family of Galois equivalent graphs that have common structure. It is easy to see the relationship between the structure of a chordal graph and the Galois group of its chromatic polynomial. A chordal graph can be obtained by 'gluing' together complete graphs, which corresponds to multiplying the chromatic polynomials of complete graphs (minus some linear factors). As the chromatic polynomial of a complete graph has the trivial Galois group, it follows that the product of several chromatic polynomials of complete graphs also has the trivial Galois group. However, even connections between the structure of the quasi-chordal graphs and the trivial Galois group are not well understood.

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Appendix A. *Galois groups of chromatic polynomials of graphs of order* $\leqslant 10$

In the following tables we use the notation for Galois groups given in [**12**, Appendix A]. A summary of the groups used in this article and their generators is given in Table A.1.

TABLE A.1. *List of groups and their generators used in this article (see* [**12**, Appendix A]*).*

Groups	Generators
Symmetric groups	
S_2	$(0,1)$
S_3	$(0,1)$, $(0,2)$
S_4	$(0,1)$, $(0,2)$, $(0,3)$
S_5	$(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$
S_6	$(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$
S_7	$(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$, $(0,6)$
S_8	$(0,1)$, $(0,2)$, $(0,3)$, $(0,4)$, $(0,5)$, $(0,6)$, $(0,7)$
Alternating groups	
A_3	$(0,1,2)$
A_4	$(0,1,2)$, $(0,2,3)$
A_5	$(0,1,2)$, $(0,2,3)$, $(0,3,4)$
A_6	$(0,1,2)$, $(0,2,3)$, $(0,3,4)$, $(0,4,5)$
Cyclic groups	
$C(4)=4$	$(0,1,2,3)$
$C(6)=6=3[\times]2$	$(0,1,2,3,4,5)$
Dihedral groups	
$D(4)$	$(0,1,2,3)$, $(1,3)$
$D(5)=5:2$	$(0,1,2,3,4)$, $(1,4)(2,3)$
$D(6)=S(3)[\times]2$	$(0,1,2,3,4,5)$, $(0,5)(1,4)(2,3)$
$D(7)=7:2$	$(0,1,2,3,4,5,6)$, $(1,6)(2,5)(3,4)$
Klein group	
$E(4)=2[\times]2$	$(0,1)(2,3)$, $(0,3)(1,2)$
Other groups	
$A_4(6)=[2^2]3$	$(1,4)(2,5)$, $(0,2,4)(1,3,5)$
$2A_4(6)=[2^3]3=2\wr 3$	$(0,3)$, $(0,2,4)(1,3,5)$
$D_6(6)=[3]2$	$(0,2,4)(1,3,5)$, $(0,5)(1,4)(2,3)$
$E(8):D_6=S(4)[\times]2$	$(0,1)(2,3)(4,5)(6,7)$, $(0,2)(1,3)(4,6)(5,7)$, $(0,4)(1,5)(2,6)(3,7)$, $(1,2,3)(4,6,5)$, $(2,3)(4,5)$
$F(5)=5:4$	$(0,1,2,3,4)$, $(1,2,4,3)$
$F_{18}(6)=[3^2]2=3\wr 2$	$(0,2,4)$, $(0,3)(1,4)(2,5)$
$F_{18}(6):2=[\frac{1}{2}S(3)^2]2$	$(0,2,4)$, $(1,5)(2,4)$, $(0,3)(1,4)(2,5)$
$F_{36}(6)=\frac{1}{2}[S(3)^2]2$	$(0,2,4)$, $(1,5)(2,4)$, $(0,3)(1,4,5,2)$
$F_{36}(6):2=[S(3)^2]2=S(3)\wr 2$	$(0,2,4)$, $(2,4)$, $(0,3)(1,4)(2,5)$
$F_{42}(7)=7:6$	$(0,1,2,3,4,5,6)$, $(1,3,2,6,4,5)$
$L(6)=\mathrm{PSL}(2,5)=A_5(6)$	$(0,1,2,3,4)$, $(0,5)(1,4)$
$L(6):2=\mathrm{PGL}(2,5)=S_5(6)$	$(0,1,2,3,4)$, $(0,5)(1,2)(3,4)$
$S_4(6c)=\frac{1}{2}[2^3]S(3)$	$(1,4)(2,5)$, $(0,2,4)(1,3,5)$, $(0,3)(1,5)(2,4)$
$S_4(6d)=[2^2]S(3)$	$(1,4)(2,5)$, $(0,2,4)(1,3,5)$, $(1,5)(2,4)$
$2S_4(6)=[2^3]S(3)=2\wr S(3)$	$(0,3)$, $(0,2,4)(1,3,5)$, $(1,5)(2,4)$
$[2^4]S(4)$	$(0,4)$, $(0,1)(4,5)$, $(0,1,2,3)(4,5,6,7)$
$[S(4)^2]2$	$(0,1,2,3)$, $(2,3)$, $(0,4)(1,5)(2,6)(3,7)$

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TABLE A.2. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 4.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	1	1

TABLE A.3. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 5.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	3	3
S_3	1	1

TABLE A.4. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 6.

Galois group	# of chromatic polynomials	# of graphs
Trivial group	1	1
S_2	6	7
S_3	9	10
$C(4) = 4$	1	1
S_4	3	3

TABLE A.5. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 7.

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		16	23
S_3	6		30	41
$C(4) = 4$	4		3	6
$E(4) = 2[\times]2$	4		3	3
$D(4)$	8		14	20
S_4	24		31	46
S_5	120		6	6
S_2, S_2	2	$(1, 2)(3, 4)$	1	1
S_2, S_2	4	$(1, 2)$; $(3, 4)$	1	1
S_2, S_3	12	$(1, 2)$; $(3, 4, 5)$; $(3, 4)$	1	1

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TABLE A.6. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 8.*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		33	65
S_3	6		114	233
$C(4) = 4$	4		4	12
$E(4) = 2[\times]2$	4		8	36
$D(4)$	8		58	184
A_4	12		6	11
S_4	24		302	755
$D(5) = 5:2$	10		3	16
$F(5) = 5:4$	20		1	1
S_5	120		360	740
$C(6) = 6 = 3[\times]2$	6		1	1
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		1	1
S_6	720		25	27
$S_2,\ S_2$	2	(1, 2)(3, 4)	3	5
$S_2,\ S_2$	4	(1, 2); (3, 4)	17	34
$S_2,\ S_3$	6	(3, 5, 4); (1, 2)(3, 4)	1	1
$S_2,\ S_3$	12	(1, 2); (3, 4, 5); (3, 4)	46	97
$S_2,\ D(4)$	16	(3, 4); (5, 7)(6, 8); (7, 8)	1	1
$S_2,\ S_4$	48	(1, 2); (3, 4, 5, 6); (3, 4)	2	2

TABLE A.7. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 9 (continued in Table A.8).*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		78	296
S_3	3		373	1069
$C(4) = 4$	4		10	49
$E(4) = 2[\times]2$	4		38	210
$D(4)$	8		319	1709
A_4	12		25	156
S_4	24		2152	10 527
$D(5) = 5:2$	10		22	122
$F_5 = 5:4$	20		7	51
A_5	60		15	81
S_5	120		6385	29 924
$C(6) = 6 = 3[\times]2$	6		3	19
$D_6(6) = [3]2$	6		2	7
$D(6) = S(3)[\times]2$	12		14	77
$F_{18}(6) = [3^2]2 = 3 \wr 2$	18		9	86
$2A_4(6) = [2^3]3 = 2 \wr 3$	24		2	7
$S_4(6d) = [2^2]S(3)$	24		2	2
$S_4(6c) = \frac{1}{2}[2^3]S(3)$	24		3	6
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		114	591
$L(6) = \mathrm{PSL}(2, 5) = A_5(6)$	60		2	6
$F_{36}(6) : 2 = [S(3)^2]2 = S(3) \wr 2$	72		174	830
$L(6) : 2 = \mathrm{PGL}(2, 5) = S_5(6)$	120		1	3
S_6	720		5197	15895
S_7	5040		90	108

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TABLE A.8. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order 9 (continued).*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_2, S_2	2	$(1,2)$	4	10
S_2, S_2	2	$(1,2)(3,4)$	8	126
S_2, S_2	4	$(1,2)$; $(3,4)$	71	604
S_2, S_3	6	$(3,5,4)$; $(1,2)(3,4)$	7	44
S_2, S_3	12	$(1,2)$; $(3,4,5)$; $(3,4)$	347	2157
S_2, $C(4)=4$	8	$(1,2)$; $(3,4,5,6)$	3	27
S_2, $E(4)=2[\times]2$	4	$(1,2)(3,4)(5,6)$; $(3,5)(4,6)$	5	34
S_2, $E(4)=2[\times]2$	8	$(1,2)$; $(3,4)(5,6)$; $(3,5)(4,6)$	5	25
S_2, $D(4)$	8	$(1,2)(3,4)(5,6)$; $(4,5)$	12	59
S_2, $D(4)$	8	$(3,4,5,6)$; $(1,2)(4,5)$	3	15
S_2, $D(4)$	8	$(3,5)(4,6)$	1	4
S_2, $D(4)$	16	$(1,2)$; $(3,4)(5,6)$; $(3,5)$	38	212
S_2, A_4	24	$(1,2)$; $(3,4)(5,6)$; $(3,4,5)$	2	16
S_2, S_4	48	$(1,2)$; $(3,4,5,6)$; $(3,4)$	218	951
S_2, S_5	240	$(1,2)$; $(3,4,5,6,7)$; $(3,4)$	2	2
S_3, S_3	6	$(1,2,3)$; $(1,2)$	2	2
S_3, S_3	36	$(1,2,3)$; $(1,2)$; $(4,5,6)$; $(4,5)$	31	81
S_3, $C(4)=4$	24	$(1,2,3)$; $(1,2)$; $(4,5,6,7)$	1	1
S_3, $D(4)$	48	$(1,2,3)$; $(1,2)$; $(4,5)(6,7)$; $(5,7)$	1	1
S_3, S_4	144	$(1,2,3)$; $(1,2)$; $(4,5,6,7)$; $(4,5)$	3	3

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TABLE A.9. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 10 *(continued in Table* A.10*).*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
Trivial group	1		1	1
S_2	2		136	712
S_3	6		1309	6607
$C(4) = 4$	4		27	372
$E(4) = 2[\times]2$	4		113	1257
$D(4)$	8		1218	13 076
A_4	12		75	932
S_4	24		12 519	107 635
$D(5) = 5:2$	10		129	1103
$F(5) = 5:4$	20		60	771
A_5	60		108	1273
S_5	120		79 331	685 931
$C(6) = 6 = 3[\times]2$	6		16	571
$D_6(6) = [3]2$	6		17	250
$D(6) = S(3)[\times]2$	12		143	2808
$A_4(6) = [2^2]3$	12		8	121
$F_{18}(6) = [3^2]2 = 3 \wr 2$	18		50	1171
$2A_4(6) = [2^3]3 = 2 \wr 3$	24		13	141
$S_4(6d) = [2^2]S(3)$	24		59	1041
$S_4(6c) = \frac{1}{2}[2^3]S(3)$	24		21	621
$F_{18}(6):2 = [\frac{1}{2}S(3)^2]2$	36		30	414
$F_{36}(6) = \frac{1}{2}[S(3)^2]2$	72		6	22
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	48		1384	26 642
$L(6) = \mathrm{PSL}(2,5) = A_5(6)$	60		34	665
$F_{36}(6):2 = [S(3)^2]2 = S(3) \wr 2$	72		2683	40 345
$L(6):2 = \mathrm{PGL}(2,5) = S_5(6)$	120		57	662
A_6	360		26	211
S_6	720		257 203	2 395 512
$D(7) = 7:2$	14		1	21
$F_{42}(7) = 7:6$	42		1	32
S_7	5040		138 773	606 609
$E(8):D_6 = S(4)[\times]2$	48		1	1
$[2^4]S(4)$	384		3	5
$[S(4)^2]2$	1152		4	6
S_8	40 320		554	697

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TABLE A.10. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 10 *(continued in Table A.11).*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_2, S_2	2	$(1,2)$	7	41
S_2, S_2	2	$(1,2)(3,4)$	32	525
S_2, S_2	4	$(1,2)$; $(3,4)$	368	4969
S_2, S_3	6	$(3,5,4)$; $(1,2)(3,4)$	84	1042
S_2, S_3	12	$(1,2)$; $(3,4,5)$; $(3,4)$	2274	24 878
S_2, $C(4)=4$	8	$(1,2)$; $(3,4,5,6)$	53	922
S_2, $E(4)=2[\times]2$	4	$(1,2)(3,5)(4,6)$; $(1,2)(3,4)(5,6)$	36	1048
S_2, $E(4)=2[\times]2$	8	$(1,2)$; $(3,4)(5,6)$; $(3,5)(4,6)$	88	950
S_2, $D(4)$	8	$(1,2)(3,4)(5,6)$; $(4,6)$	86	1698
S_2, $D(4)$	8	$(3,4)(5,6)$; $(1,2)(4,6)$	18	294
S_2, $D(4)$	8	$(3,5,4,6)$; $(1,2)(5,6)$	34	836
S_2, $D(4)$	16	$(1,2)$; $(3,4)(5,6)$; $(3,6)$	607	9189
S_2, A_4	24	$(1,2)$; $(3,4)(5,6)$; $(3,4,5)$	49	514
S_2, S_4	24	$(1,2)(3,4)$; $(1,2)(4,5)$; $(1,2)(5,6)$	13	52
S_2, S_4	48	$(1,2)$; $(3,4,5,6)$; $(3,4)$	4681	65 110
S_2, $D(5)=5:2$	20	$(1,2)$; $(3,4)(5,6)$; $(3,5,6,4,7)$	30	350
S_2, $F(5)=5:4$	40	$(1,2)$; $(3,5,7,6)$; $(3,7)(5,6)$; $(3,7,6,4,5)$	3	20
S_2, A_5	120	$(1,2)$; $(5,6,7)$; $(3,4,5)$	5	22
S_2, S_5	240	$(1,2)$; $(3,4,5,6,7)$; $(3,4)$	3283	18 664
S_2, $C(6)=6=3[\times]2$	6	$(1,2)(3,5,7,6,8,4)$	1	1
S_2, $D_6(6)=[3]2$	6	$(1,2)(3,4)(5,7)(6,8)$; $(1,2)(3,5)(4,8)(6,7)$	1	1
S_2, $F_{36}(6):2$ $=[S(3)^2]2=S(3)\wr 2$	144	$(1,2)$; $(3,4)(5,6)(7,8)$; $(3,5,7)$; $(3,5)$	1	2
S_2, $2S_4(6)$ $=[2^3]S(3)=2\wr S(3)$	96	$(1,2)$; $(3,6,4)(5,7,8)$; $(3,4)(5,7)$; $(4,5)$	1	1
S_2, S_6	1440	$(1,2)$; $(3,4,5,6,7,8)$; $(3,4)$	21	26
S_2, S_2, S_2	2	$(1,2)(3,4)$	1	3
S_2, S_2, S_2	4	$(1,2)$; $(3,4)$	13	91
S_2, S_2, S_2	4	$(1,2)(5,6)$; $(3,4)$	4	4
S_2, S_2, S_2	8	$(1,2)$; $(3,4)$; $(5,6)$	14	30
S_2, S_2, S_3	6	$(3,5,4)$; $(1,2)(3,4)$	2	2
S_2, S_2, S_3	12	$(1,2)$; $(3,4,5)$; $(3,4)$	37	161
S_2, S_2, S_3	12	$(1,2)(3,4)$; $(5,7)$; $(5,6)$	10	46
S_2, S_2, S_3	12	$(1,2)$; $(3,4)(5,6)$; $(5,7,6)$	1	3
S_2, S_2, S_3	24	$(1,2)$; $(3,4)$; $(5,6,7)$; $(5,6)$	98	486

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Table A.11. *Galois groups of chromatic polynomials of strongly non-clique-separable graphs of order* 10 *(continued).*

Galois group	Order	Generators	# of chromatic polynomials	# of graphs
S_3, S_3	6	$(1,2,3)$; $(1,2)$	4	33
S_3, S_3	6	$(2,3)(5,6)$; $(1,3,2)(4,6,5)$	12	76
S_3, S_3	36	$(1,2,3)$; $(1,2)$; $(4,5,6)$; $(4,5)$	839	6119
S_3, $C(4)=4$	24	$(1,2,3)$; $(1,2)$; $(4,5,6,7)$	39	423
S_3, $E(4)=2[\times 2]$	12	$(1,2,3)$; $(2,3)(4,6)(5,7)$; $(2,3)(4,5)(6,7)$	3	4
S_3, $E(4)=2[\times]2$	24	$(1,2,3)$; $(1,2)$; $(4,5)(6,7)$; $(4,6)(5,7)$	46	232
S_3, $D(4)$	24	$(2,3)(4,5)(6,7)$; $(1,2,3)$; $(5,7)$	4	7
S_3, $D(4)$	24	$(4,6,5,7)$; $(1,2)(6,7)$; $(2,3)(6,7)$	2	8
S_3, $D(4)$	48	$(1,2,3)$; $(1,2)$; $(4,6)(5,7)$; $(6,7)$	295	2508
S_3, A_4	72	$(1,2,3)$; $(1,2)$; $(4,5)(6,7)$; $(4,5,6)$	21	248
S_3, S_4	144	$(1,2,3)$; $(1,2)$; $(4,5,6,7)$; $(4,5)$	1080	7267
S_3, S_5	720	$(1,2,3)$; $(1,2)$; $(4,5,6,7,8)$; $(4,5)$	1	1
S_4, $E(4)=2[\times]2$	96	$(1,2,3,4)$; $(1,2)$; $(5,6)(7,8)$; $(5,7)(6,8)$	1	1

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Appendix B. *Galois groups of chromatic polynomials of θ-graphs*

TABLE B.1. *Galois groups of chromatic polynomials of θ-graphs of order at most 19, where there is exactly one non-linear factor.*

Galois groups	# of graphs ($\chi = 2$)	# of graphs ($\chi = 3$)	Total # of graphs	$\theta_{a,b,c}$
S_3	1	1	2	2, 2, 2; 2, 2, 3
S_4	0	0	0	
S_5	1	1	2	2, 2, 4; 2, 2, 5
S_6	1	0	1	3, 3, 3
S_7	2	2	4	2, 2, 6; 2, 4, 4; 2, 2, 7; 2, 3, 6
S_8	1	2	3	3, 3, 5; 2, 5, 5; 3, 3, 6
S_9	2	3	5	2, 2, 8; 4, 4, 4; 2, 2, 9; 2, 3, 8; 2, 4, 7
S_{10}	1	3	4	3, 5, 5; 2, 3, 9; 2, 5, 7; 3, 3, 8
S_{11}	3	3	6	2, 2, 10; 2, 4, 8; 2, 6, 6; 2, 2, 11; 3, 6, 6; 4, 4, 7
S_{12}	2	3	5	3, 3, 9; 5, 5, 5; 2, 3, 11; 2, 5, 9; 2, 7, 7
S_{13}	4	6	10	2, 2, 12; 2, 4, 10; 2, 6, 8; 4, 4, 8; 2, 2, 13; 2, 3, 12; 2, 4, 11; 2, 5, 10; 2, 6, 9; 3, 6, 8
S_{14}	3	3	6	3, 3, 11; 3, 5, 9; 5, 5, 7; 3, 3, 12; 3, 6, 9; 4, 7, 7
S_{15}	4	6	10	2, 2, 14; 2, 8, 8; 4, 4, 10; 6, 6, 6; 2, 2, 15; 2, 3, 14; 2, 6, 11; 2, 7, 10; 3, 8, 8; 4, 4, 11
S_{16}	2	7	9	5, 5, 9; 5, 7, 7; 2, 3, 15; 2, 5, 13; 2, 7, 11; 2, 9, 9; 3, 3, 14; 3, 6, 11; 5, 5, 10
S_{17}	6	0	6	2, 2, 16; 2, 4, 14; 2, 6, 12; 2, 8, 10; 4, 8, 8; 6, 6, 8
$D(4)$	0	1	1	2, 3, 3
$2S_4(6) = [2^3]S(3) = 2 \wr S(3)$	0	1	1	2, 3, 5

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TABLE B.2. *Galois groups of chromatic polynomials of θ-graphs of order at most* 19, *where there is more than one non-linear factor.*

Galois groups	# of graphs ($\chi = 2$)	# of graphs ($\chi = 3$)	Total # of graphs	$\theta_{a,b,c}$
S_2, S_3	0	1	1	2, 3, 4
S_2, S_4	0	1	1	3, 3, 4
S_2, S_5	0	2	2	2, 4, 5; 3, 4, 4
S_2, S_6	0	1	1	2, 3, 7
S_2, S_7	1	2	3	2, 4, 6; 3, 4, 6; 4, 4, 5
S_2, S_8	1	2	3	3, 3, 7; 3, 4, 7 4, 5, 5
S_2, S_9	1	3	4	4, 4, 6; 2, 3, 10; 2, 5, 8; 3, 4, 8
S_2, S_{10}	1	3	4	3, 5, 7; 3, 3, 10; 3, 5, 8; 4, 5, 7
S_2, S_{11}	1	2	3	4, 6, 6; 3, 4, 10; 4, 5, 8
S_2, S_{12}	1	5	6	3, 7, 7; 2, 3, 13; 2, 7, 9; 3, 4, 11; 3, 5, 10; 5, 5, 8
S_2, S_{13}	2	4	6	2, 4, 12; 4, 6, 8; 2, 4, 13; 2, 5, 12; 3, 4, 12; 4, 5, 10
S_2, S_{14}	2	2	4	3, 3, 13; 3, 7, 9; 3, 5, 12; 3, 7, 10
S_2, S_{15}	1	0	1	4, 4, 12
S_2, S_2, S_4	0	1	1	3, 4, 5
S_2, S_2, S_7	0	2	2	2, 4, 9; 2, 6, 7
S_2, S_2, S_8	0	2	2	3, 4, 9; 3, 6, 7
S_2, S_2, S_9	0	2	2	4, 4, 9; 4, 6, 7
S_2, S_2, S_{10}	0	1	1	4, 5, 9
S_2, S_2, S_{11}	0	2	2	4, 6, 9; 6, 6, 7
S_2, S_2, S_{12}	0	3	3	3, 4, 13; 4, 7, 9; 6, 7, 7
$C(4), S_5$	0	1	1	2, 5, 6
$C(4), S_6$	0	1	1	3, 5, 6
$C(4), S_8$	0	1	1	5, 5, 6
$C(4), S_9$	0	1	1	5, 6, 6
$C(4), S_{10}$	0	1	1	2, 5, 11
$C(4), S_{12}$	1	1	2	3, 5, 11; 5, 6, 9
$C(4), S_2, S_9$	1	3	4	2, 6, 10; 3, 6, 10; 5, 6, 8; 2, 8, 9
$C(4), S_2, S_{10}$	0	1	1	4, 5, 11
$C(4), S_2, S_{11}$	1	0	1	4, 6, 10
$C(4), S_2, S_2, S_3$	0	1	1	4, 5, 6
$C(4), S_2, S_2, S_6$	0	1	1	5, 6, 7
$C(6), S_7$	0	1	1	2, 7, 8
$C(6), S_9$	0	1	1	4, 7, 8
$C(6), S_2, S_6$	0	1	1	3, 7, 8
$C(6), S_2, S_8$	0	1	1	5, 7, 8
$E(4) = 2[\times]2, S_2, S_{10}$	0	1	1	3, 8, 9

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TABLE B.3. *θ-graphs of order at most 19, where $P(C_n, \lambda)$, $n \geqslant 4$, divides $P(\theta_{a,b,c}, \lambda)$.*

n	$P(C_n, \lambda)$	Galois group(s)	$\theta_{a,a+1,b}$	$\theta_{a,2^{k-1}a+1,b}$	$\theta_{a,2^{k-1}(a-1),b}$	$\theta_{a,3a+1,b}$	$\theta_{a,3(a-1),b}$
4	$\lambda(\lambda-1)$ $\times(\lambda^2-3\lambda+3)$	S_2	2, 3, 4; 3, 3, 4; 3, 4, 4 3, 4, 6; 3, 4, 7; 3, 4, 8 3, 4, 10; 3, 4, 11	$(k=2)$: 2, 3, 7; 3, 3, 7 3, 5, 7; 3, 7, 7; 3, 7, 9; 3, 7, 10 $(k=3)$: 2, 3, 13; 3, 3, 13	$(k=2)$: 2, 4, 6; 4, 4, 6 4, 6, 6; 4, 6, 8 $(k=3)$: 2, 4, 12; 3, 4, 12 4, 4, 12	2, 3, 10 3, 3, 10 2, 4, 13; 3, 5, 10	2, 4, 9
5	$\lambda(\lambda-1)(\lambda-2)$ $\times(\lambda^2-2\lambda+2)$	S_2	2, 4, 5; 4, 4, 5 4, 5, 5: 4, 5, 7 4, 5, 8; 4, 5, 10		$(k=2)$: 2, 5, 8; 3, 5, 8 5, 5, 8		2, 5, 12; 3, 5, 12
6	$\lambda(\lambda-1)(\lambda^4-5\lambda^3$ $+10\lambda^2-10\lambda+5)$	$C(4)$	2, 5, 6; 3, 5, 6; 5, 5, 6 5, 6, 6; 5, 6, 9	$(k=2)$: 2, 5, 11; 3, 5, 11			
7	$\lambda(\lambda-1)(\lambda-2)$ $\times(\lambda^2-3\lambda+3)$ $\times(\lambda^2-\lambda+1)$	S_2, S_2	2, 6, 7; 3, 6, 7; 4, 6, 7 6, 6, 7; 6, 7, 7;				
8	$\lambda(\lambda-1)(\lambda^6-7\lambda^5$ $+21\lambda^4-35\lambda^3+35\lambda^2$ $-21\lambda+7)$	$C(6)$	2, 7, 8; 4, 7, 8; 5, 7, 8				
9	$\lambda(\lambda-1)(\lambda-2)$ $\times(\lambda^2-2\lambda+2)$ $\times(\lambda^4-4\lambda^3+6\lambda^2$ $-4\lambda+2)$	S_2, $E(4)$	2, 8, 9; 3, 8, 9				

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