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THE LALONDE–MCDUFF CONJECTURE AND THE FUNDAMENTAL GROUP

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Abstract We give a simple proof of the Lalonde–McDuff Conjecture for aspherical manifolds.

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1. Introduction

Let (M,ω) be a symplectic manifold. A locally trivial bundle

$$(M,\omega) \xrightarrow{i} E \xrightarrow{\pi} B$$

is called *Hamiltonian* if its structure group is a subgroup of the group $\mathrm{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of (M,ω). Lalonde and McDuff conjectured in [**5**] that the rational cohomology of the total space is isomorphic as a vector space to the tensor product of the cohomology of the base and the cohomology of the fibre. The conjecture has been proved in many particular cases using different methods. Examples include the following.

- Fibrations where the fibre is a closed Kähler manifold, proved by Blanchard [**2**]: the argument is purely cohomological and uses the Lefschetz property only.
- Fibrations for which the structure group is a compact Lie group [**1**].
- Four-dimensional manifolds [**5**, Lemma 4.8].
- Nilmanifolds [**8**]: the argument is purely homotopical and uses the Koszul–Sullivan models of a fibration.

If (M,ω) is aspherical, we get a stronger statement.

Theorem 1.1. *Let $(M,\omega) \to E \to B$ be a Hamiltonian fibration. If M is aspherical, then the fibration is homotopy trivial. In particular, the Lalonde–McDuff Conjecture holds for aspherical manifolds.*

The importance of the Lalonde–McDuff Conjecture comes from the fundamental question of understanding the equivariant cohomology. More precisely, let G be a topological group acting on a manifold M. Let

$$M \xrightarrow{i} M_G \xrightarrow{\pi} BG$$

be the universal fibration associated to the action. It is a fundamental question to determine the cohomology $H^*(M_G)$, also known as the equivariant cohomology of M associated with the action of G and denoted by $H^*_G(M)$. A cohomology class in $H^*(M_G)$ is called a G-equivariant class of M. Thus the Lalonde–McDuff Conjecture says that the $\mathrm{Ham}(M,\omega)$-equivariant cohomology of M has quite a simple form.

It follows from the Leray–Hirsch Theorem [**4**, Theorem 4D.1] that $H(M_G;\mathbb{Q})$ is isomorphic as an $H^*(BG;\mathbb{Q})$-module to the tensor product of the cohomology of the base and the cohomology of the fibre if and only if the homomorphism $i^*\colon H^*(M_G;\mathbb{Q}) \to H^*(BG;\mathbb{Q})$ induced by the inclusion of the fibre is surjective.

In our proof of Theorem 1.1 we shall show that the above homomorphism i^* is surjective under the hypothesis of the theorem. In other words, we shall prove that every cohomology class of a closed aspherical manifold M is an image of a $\mathrm{Ham}(M,\omega)$-equivariant class. This will follow from a simple and general observation made in Theorem 2.1 and a certain non-trivial fact about Hamiltonian actions (see § 3 for details).

Our result can be used in two ways. Firstly, it proves the conjecture for a very large class of manifolds. This yields some restrictions on a potential counterexample. Secondly, it can be used as a building block in a proof of the conjecture in full generality. More precisely, what we essentially prove is that π_1-classes (see § 2 for a definition) of $H^*(M;\mathbb{Q})$ are images of $\mathrm{Ham}(M,\omega)$-equivariant classes. Hence it remains to prove that all other classes are in the image as well. An example of this approach is presented in Theorem 3.1.

We approach the conjecture by trying to prove it for different classes of fibres. More precisely, we examine the properties of cohomology classes of M ensuring that they are images of $\mathrm{Ham}(M,\omega)$-equivariant classes. The approach presented by Lalonde and McDuff [**5**,**6**] is to prove the conjecture for different classes of bases. The most important step towards the proof of the conjecture has been made by McDuff in [**6**], where she proved the conjecture for all Hamiltonian fibrations over the two-dimensional sphere.

2. The main observation

Let G be a connected topological group acting on the symplectic manifold (M,ω). From now on, $H^*(X)$ will denote the rational cohomology of the space X. Let $c\colon M \to B\pi_1(M)$ be the map classifying the universal cover. A cohomology class $\alpha \in H^*(M)$ is called *a π_1-class* if it is in the image of the induced homomorphism

$$c^*\colon H^*(B\pi_1(M)) \to H^*(M).$$

Fix a point $p \in M$ and let $\mathrm{ev}\colon G \to M$ be the corresponding evaluation map defined by

$$\mathrm{ev}(f) := f(p).$$

Theorem 2.1. *Suppose that the evaluation map induces the trivial homomorphism on the fundamental group. Then every π_1-class is the image of some G-equivariant class.*

Proof. Consider the universal fibration $M \xrightarrow{i} M_G \xrightarrow{\pi} BG$ and the connecting homomorphism

$$\partial : \pi_2(BG) \to \pi_1(M).$$

In fact, it is the same homomorphism as the one induced by the evaluation, after the identification $\pi_1(G) \cong \pi_2(BG)$. This almost immediately follows from the definition of the connecting homomorphism. Thus the evaluation map induces the trivial homomorphism on the fundamental group if and only if the connecting homomorphism ∂ is trivial.

Since G is connected, BG is simply connected and, by the long exact sequence of homotopy groups of the universal fibration,

$$\pi_2(BG) \xrightarrow{\partial=0} \pi_1(M) \xrightarrow{i_*} \pi_1(M_G) \xrightarrow{\pi_*} \pi_1(BG) = 0,$$

we get the isomorphism $i_*\colon \pi_1(M) \to \pi_1(M_G)$. Hence the composition $M \xrightarrow{i} M_G \to B\pi_1(M)$ of the inclusion of the fibre followed by the map classifying the universal cover of M_G is homotopic to the classifying map $c\colon M \to B\pi_1(M)$. Thus every π_1-class of M is in the image of the map i^* induced by the inclusion of the fibre. □

Corollary 2.2. *Let M be an aspherical manifold. If the evaluation map induces the trivial homomorphism on the fundamental group, then the universal fibration $M \to M_G \to BG$ is homotopy trivial. In particular, the G-equivariant cohomology of M is isomorphic as a ring to the tensor product $H^*(BG) \otimes H^*(M)$.*

Proof. Since M is aspherical, we have that $M \simeq B\pi_1(M)$. The map $M_G \to B\pi_1(M)$ classifying the universal cover is a right homotopy inverse of the inclusion of the fibre $i\colon M \to M_G$. This implies that the fibration is homotopy trivial. □

Example 2.3. It is easy to see that the image of the homomorphism induced by the evaluation map is contained in the centre of the fundamental group of M. The above results therefore apply to manifolds whose fundamental group has a trivial centre. In particular, if M is such an aspherical manifold, then any bundle with fibre M over a simply connected base is homotopy trivial.

Remark 2.4. The last observation also follows from the classical result of Gottlieb [**3**, Theorem III.2] that states that the identity component of the space of homotopy equivalences of an aspherical manifold M is itself aspherical and has the fundamental group isomorphic to the centre of $\pi_1(M)$.

3. Applications to the Lalonde–McDuff Conjecture

Proof of Theorem 1.1. The group of Hamiltonian diffeomorphisms of a closed symplectic manifold has the property that the evaluation map induces the trivial homomorphism of the fundamental group. The proof of this fact is non-trivial and can be found in [**7**, Corollary 9.1.2]. Thus applying Theorem 2.1 in the Hamiltonian case and Corollary 2.2 yields the proof of Theorem 1.1. □

Combining our argument with some known results we obtain the following more general theorem.

Theorem 3.1. *Let (M, ω) be a symplectic manifold. If the cohomology ring $H^*(M; \mathbb{Q})$ is generated by $H^2(M)$ and by the Chern classes and the $\pi_1(M)$-classes, then the Lalonde–McDuff Conjecture holds for (M, ω).*

Proof. Let $(M, \omega) \xrightarrow{i} E \to B$ be a Hamiltonian bundle. The fact that the second cohomology is contained in the image of the map induced by the inclusion of the fibre is proven in [**5**, Theroem 1.1].

The Chern classes are in the image of $i^* \colon H^*(E) \to H^*(M)$ because the tangent bundle TM is the pullback of the bundle tangent to the fibres of $E \to B$. The statement now follows from Theorem 2.1 and the Leray–Hirsch Theorem. □

Example 3.2. Let (X, ω_X) be a closed symplectic manifold whose cohomology ring is generated by $H^2(X)$ and the Chern classes (e.g. the complex Grassmannian or a flag manifold). Let (M, ω) be a closed symplectic manifold that is the product of X and a symplectic aspherical manifold. It is then easy to see that the cohomology ring of (M, ω) satisfies the assumption of the above theorem and that the Lalonde–McDuff Conjecture holds.

On the other hand, it is not known in general if the Lalonde–McDuff Conjecture holds for products if it holds for the factors.

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