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Borel reducibility of equivalence relations on ω_1

Riccardo Camerlo

Abstract. The structure of Borel reducibility for equivalence relations on ω_1 is determined.

1 Introduction

Let X, Y be non-empty sets, and let E, F be equivalence relations on X, Y, respectively. A *reduction* of E to F is any function $f : X \to Y$ such that $\forall x, x' \in X\ (xEx' \Leftrightarrow f(x)Ff(x'))$. Thus the existence of a reduction of E to F simply amounts to the inequality $\operatorname{card}\left(X/E\right) \leqslant \operatorname{card}\left(Y/F\right)$.

Therefore, one is generally interested in restricting the classes of reductions to be considered. These restricted classes are often defined by exploiting some extra structure of the sets they are defined on. A typical example is *Borel reducibility*: if X, Y are topological spaces, then E *Borel reduces* to F, denoted $E \leqslant_B F$, if there exists a Borel reduction $f : X \to Y$ of E to F. If $E \leqslant_B F \leqslant_B E$, say that E, F are *Borel equivalent*, denoted $E \sim_B F$.

Borel reducibility has received much attention for equivalence relations defined on Polish spaces, and on such spaces, the structure of $\leqslant_B$ turns out to be very rich (see, for instance, [G2009] for a comprehensive introduction to the subject).

Much less is known concerning non-Polish spaces. This note focuses on one such space of interest in set theory: the ordinal ω_1 endowed with the order topology. The structure of $\leqslant_B$ on equivalence relations on ω_1 is much simpler than for Polish spaces, therefore allowing a complete description.

The paper is organized as follows: Section 2 recalls some basic set theoretic terminology and proves the facts on the Borel structure of ω_1 that are needed later. In particular, Theorem 2.7 identifies the Borel functions $\omega_1 \to \omega_1$ as those functions that are either constant on a club or the identity on a club, while Theorem 2.10 shows that the Cartesian product of Borel functions is Borel. This latter fact implies that Borel equivalence relations form an initial segment with respect to $\leqslant_B$ (Corollary 2.11). The analysis of $\leqslant_B$ on equivalence relations on ω_1 is carried out in Section 3, leading to the complete description of its structure in terms of the *characteristic triple* of an equivalence relation (taking into account, how many classes the relation has

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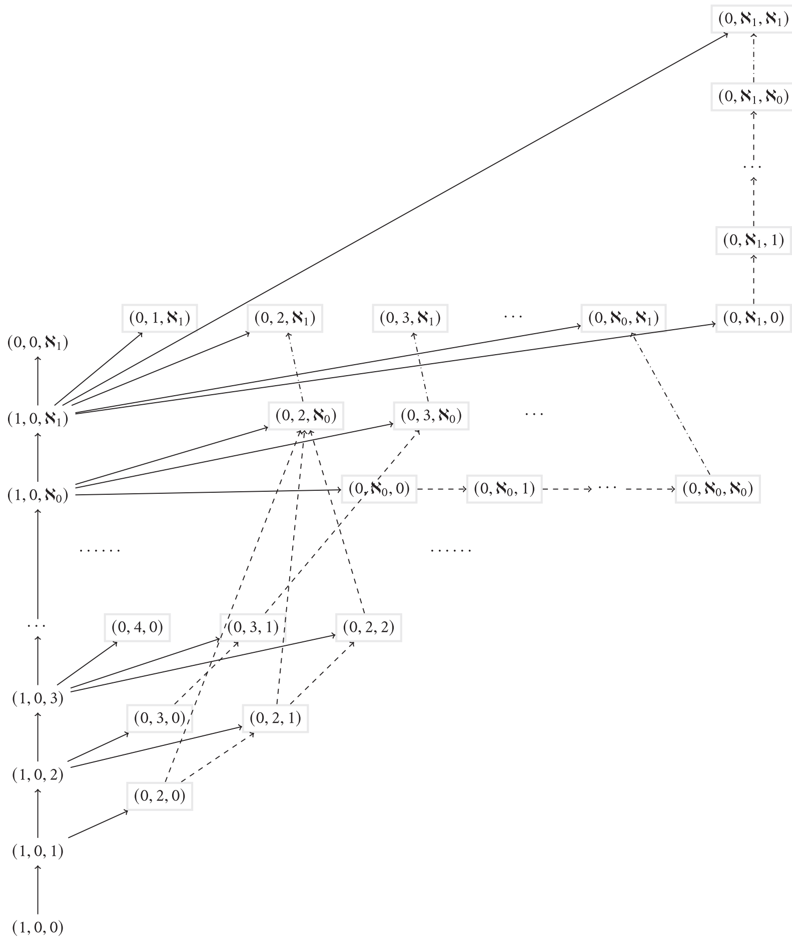

Figure 1: The structure of Borel reducibility on ω_1.

of a given size) and the compatibility of the bistationary classes. Theorem 3.14 and Figure 1 summarize the results. Finally, Section 4 connects equivalence relations on ω_1 with the better known realm of equivalence relations on Polish spaces.

Since by [GJK2008, Theorem 1.1], all ordinals α with $\omega_1 \leqslant \alpha < \omega_1 2$ are Borel isomorphic, the structure of $\leqslant_B$ is the same on all such ordinals. This raises the following question.

Question What is the structure of $\leqslant_B$ on equivalence relations on α when $\alpha \geq \omega_1 2$?

2 Generalities on Borel sets and functions

The simplicity of $\leqslant_B$ on ω_1 is due to the very rigid features of Borel subsets of ω_1 and Borel functions $\omega_1 \to \omega_1$.

Recall that a subset of a regular uncountable cardinal is a *club* if it is closed in the ordinal topology and unbounded, it is *stationary* if it intersects every club, it is *bistationary* if both it and its complement are stationary. A collection of the fundamental properties of these sets can be found, for instance, in [J2002], and in [BL2020] for the specific case of ω_1. The following is a convenient terminology.

Definition 2.1 Let $A \subseteq \omega_1$. Then:

- A is *thin* if A is not stationary.
- A is *thick* if its complement is thin, that is, if A contains a club.

Therefore, the subsets of ω_1 are partitioned into three classes: thin, bistationary, and thick subsets. The union of countably many thin sets is thin, the intersection of countably many thick sets is thick.

The following known fact is used repeatedly in the sequel.

Theorem 2.2 *A subset of ω_1 is Borel if and only if it is either thin or thick; more precisely, every thin set is of the form $\bigcup_{n\in\mathbb{N}} E_n$, where each E_n is the intersection of an open and a closed set. Consequently, a function $f : \omega_1 \to \omega_1$ is Borel if and only if the preimage under f of any thick set (equivalently, of any thin set) is either thin or thick.*

Proof See [BL2020, Proposition 4] and its proof. ∎

See theorem 2.7 for a characterization of Borel functions.

The proof of the existence of bistationary subsets of ω_1 uses the axiom of choice (see [Ru1957]). With the same idea, the following strengthening can be established.

Lemma 2.3 *Let $\mathcal{F}$ be a partition of a thick set C into non-thick sets. Then there exists $\mathcal{G} \subseteq \mathcal{F}$ such that $\bigcup \mathcal{G}$ is bistationary.*

Proof Notice that it can be assumed that $C = \omega_1$. Indeed, otherwise $\mathcal{F} \cup \{\omega_1 \setminus C\}$ is a partition of ω_1 into non-thick sets. Since $\omega_1 \setminus C$ is thin, if $\mathcal{G} \subseteq \mathcal{F} \cup \{\omega_1 \setminus C\}$ is such that $\bigcup \mathcal{G}$ is bistationary, then $\bigcup(\mathcal{G} \setminus \{\omega_1 \setminus C\})$ is bistationary as well.

It can also be assumed that all members of $\mathcal{F}$ are thin, otherwise $\mathcal{G} = \{A\}$ for A non-thin works because every element of $\mathcal{F}$ is assumed to be non-thick; thus such an A is bistationary. In particular, $\operatorname{card}(\mathcal{F}) = \aleph_1$, so let $\mathcal{F} = \{A_\alpha\}_{\alpha\in\omega_1}$.

Now, toward contradiction assume that for every $\mathcal{G} \subseteq \mathcal{F}$,

$$\text{either} \quad \bigcup \mathcal{G} \text{ is thick} \quad \text{or} \quad \omega_1 \setminus \bigcup \mathcal{G} = \bigcup(\mathcal{F} \setminus \mathcal{G}) \text{ is thick.} \tag{2.1}$$

Let $g : \omega_1 \to \mathbb{R}$ be an injection. For every $n \in \mathbb{N}$, let $\mathcal{J}(n) = \{B_{nm}\}_{m\in\mathbb{N}}$ be a partition of $\mathbb{R}$ into subsets of diameter less than $\frac{1}{n+1}$.

Claim 2.4 For every n, there exists m_n such that $D_n = \bigcup_{\alpha\in g^{-1}(B_{nm_n})} A_\alpha$ is thick.

Proof of the claim Otherwise, by (2.1), for every m, the set $\bigcup_{\alpha\in\omega_1\setminus g^{-1}(B_{nm})} A_\alpha$ would be thick, contradicting $\bigcap_{m\in\mathbb{N}} \bigcup_{\alpha\in\omega_1\setminus g^{-1}(B_{nm})} A_\alpha = \emptyset$. ∎

Let $D = \bigcap_{n\in\mathbb{N}} D_n = \bigcup_{\alpha\in\bigcap_{n\in\mathbb{N}} g^{-1}(B_{nm_n})} A_\alpha$, so that D is thick, implying that $\bigcap_{n\in\mathbb{N}} g^{-1}(B_{nm_n})$ has more than one element. Let $\alpha, \beta \in \bigcap_{n\in\mathbb{N}} g^{-1}(B_{nm_n})$ with $\alpha \neq \beta$, so that $g(\alpha) \neq g(\beta)$; on the other hand, $\alpha, \beta \in \bigcap_{n\in\mathbb{N}} g^{-1}(B_{nm_n})$ implies that $|g(\alpha) - g(\beta)| < \frac{1}{n+1}$ for every $n \in \mathbb{N}$, reaching a contradiction. ■

Lemma 2.5 *Let $\mathcal{F}$ be a family of pairwise disjoint thin sets. Then there exists a club C such that* $\forall A \in \mathcal{F}$ card$(C \cap A) \leqslant 1$.

Proof It can be assumed that no member of $\mathcal{F}$ is empty and that card$(\mathcal{F}) = \aleph_1$. Let $\mathcal{F} = \{A_\alpha\}_{\alpha\in\omega_1}$ such that $\alpha < \beta \Rightarrow \min A_\alpha < \min A_\beta$; in particular,

$$\forall \alpha \in \omega_1\ \alpha \leqslant \min A_\alpha. \tag{2.2}$$

For every $\alpha \in \omega_1$, let C_α be a club such that $C_\alpha \cap \bigcup_{\beta\leqslant\alpha} A_\beta = \emptyset$, and consider the diagonal intersection $C = \triangle_{\alpha\in\omega_1} C_\alpha = \{\beta \in \omega_1 \mid \beta \in \bigcap_{\alpha<\beta} C_\alpha\}$, which is a club by [J2002, Lemma 8.4]. If $\beta \in C$, then $\beta \notin \bigcup_{\alpha<\beta} A_\alpha$ by construction of C, and $\beta \notin \bigcup_{\alpha>\beta} A_\alpha$ by (2.2); in other words, any $\beta \in C$ cannot belong to any A_α but A_β. This establishes the lemma. ■

Lemma 2.6 *Let $f : \omega_1 \to \omega_1$ be Borel and assume that $f[C]$ is thin for some thick set C. Then there exists $\beta \in f[C]$ such that $f^{-1}(\{\beta\})$ is thick.*

Proof If the conclusion did not hold, $\{f^{-1}(\{\beta\}) \cap C\}_{\beta\in f[C]}$ would be a partition of C into thin sets, by Theorem 2.2. By Lemma 2.3, there exists $I \subseteq f[C]$ such that $\bigcup_{\beta\in I} f^{-1}(\{\beta\}) \cap C = f^{-1}(I) \cap C$ is bistationary. Since I is thin, this contradicts f being Borel, again by Theorem 2.2. ■

The following theorem imposes severe restrictions to Borel functions.

Theorem 2.7 *Let $f : \omega_1 \to \omega_1$. Then the following are equivalent:*

(1) *f is Borel.*
(2) *There exists a thick set C such that $f_{|_C}$ is continuous.*
(3) *There exists a thick set C such that either f is constant on C or $\forall \alpha \in C\ f(\alpha) = \alpha$.*

Proof (1) ⇒ (3). Assume that f is Borel. Let $S = \{\alpha \in \omega_1 \mid f(\alpha) < \alpha\}$.

If S is stationary, by Fodor's theorem, let β be such that $C = f^{-1}(\{\beta\})$ is stationary. Actually, C is thick: otherwise, it would be bistationary, contradicting the Borelness of f. So in this case, the conclusion is achieved.

Assume now that S is thin, and let F be a club disjoint from S. In particular, $\forall \alpha \in F\ \alpha \leqslant f(\alpha)$. Define an increasing, continuous at limits, sequence $\{\gamma_\rho\}_{\rho\in\omega_1}$ of elements of F as follows:

$$\gamma_0 = \min F,$$
$$\gamma_\rho = \min\{\alpha \in F \mid \forall \xi < \rho\ f(\gamma_\xi) < \alpha\}, \quad \text{for } \rho > 0.$$

Let $L = \{\gamma_\rho\}_{\rho\in\omega_1}$, so L is a club and $f_{|_L}$ is injective. If $C = \{\alpha \in L \mid f(\alpha) = \alpha\}$ is thick, then it has the desired properties, so toward contradiction assume otherwise: this means that either C is thin or it is bistationary. Let $G = \{\alpha \in L \mid \alpha < f(\alpha)\} = L \setminus C$.

Claim 2.8 $f[G]$ is thin.

Proof of the claim In fact, $f[G]$ does not contain any of its limit points. Let $\{\rho_n\}_{n\in\mathbb{N}}$ be any increasing sequence of elements of ω_1 such that $\forall n \in \mathbb{N}\ \gamma_{\rho_n} < f(\gamma_{\rho_n})$, and let $\lambda = \sup\{f(\gamma_{\rho_n})\}_{n\in\mathbb{N}} = \sup\{\gamma_{\rho_n}\}_{n\in\mathbb{N}} \in L$. Then $\lambda \leqslant f(\lambda)$. If $\lambda = f(\lambda)$, then $\lambda \notin G$, so $\lambda \notin f[G]$ by the injectivity of $f_{|_L}$; if $\lambda < f(\lambda)$, then $\lambda \notin f[G]$ by the construction of L. ■

If C were thin, then G would be thick. Thus, by the claim and Lemma 2.6, function f would be constant on a thick set, which is impossible as $f_{|_G}$ is injective.

If C were bistationary, G would be bistationary too. Now, note that $f^{-1}(f[G]) \cap L = G$ by the injectivity of $f_{|_L}$, but this is impossible, as $f[G]$ is Borel by the claim, while G is not.

(3) ⇒ (2) is immediate.

(2) ⇒ (1). Let C' be a club included in C. By [A2018, Theorem 4.1], either f is constant, say of value β, on a final segment of C', so on a thick set, or the set of fixed points of $f_{|_{C'}}$ is cofinal. In the former case, given any $B \subseteq \omega_1$, one has that $f^{-1}(B)$ is either thin or thick, according to whether $\beta \notin B$ or $\beta \in B$; so f is Borel by Theorem 2.2. In the latter case, by the continuity of $f_{|_{C'}}$, the set F of fixed points of $f_{|_{C'}}$ is also closed, therefore a thick set. Let then B be a thick set. Thus $B \cap F$ is thick as well. It follows that $f^{-1}(B) \supseteq f^{-1}(B \cap F) \supseteq B \cap F$, so $f^{-1}(B)$ is a thick set. Thus f is Borel by Theorem 2.2 again. ■

Therefore, even a very simple function like the successor function $f : \omega_1 \to \omega_1$, defined by $f(\alpha) = \alpha + 1$, is not Borel, since it does not satisfy the condition of Theorem 2.7(3).

For $S \subseteq \omega_1 \times \omega_1$ and $\xi \in \omega_1$ denote $S_\xi = \{\rho \in \omega_1 \mid (\xi, \rho) \in S\}$ the vertical section and $S^\xi = \{\rho \in \omega_1 \mid (\rho, \xi) \in S\}$ the horizontal section of S corresponding to ξ.

Lemma 2.9 *Let A be a thin set, and let $F \subseteq A \times \omega_1$ be a set all of whose vertical sections are Borel. Then F is Borel.*

Similarly, if $F \subseteq \omega_1 \times A$ is a set all of whose horizontal sections are Borel.

Proof The proof is an extension of the argument of [BL2020, Proposition 4]. Assume that $F \neq \emptyset$, and let $A' = \pi_1(F)$, where π_1 denotes the first projection.

Suppose first that all vertical sections of F are closed. Let H be a club such that $A' \cap H = \emptyset$, and let $\mathcal{D}$ be the partition of $\omega_1 \setminus H$ into maximal convex open sets, so that each member of $\mathcal{D}$ is countable. Let $\mathcal{D}' = \{D \in \mathcal{D} \mid A' \cap D \neq \emptyset\}$; then $A' = \bigcup_{D\in\mathcal{D}'}(A' \cap D)$. For every $D \in \mathcal{D}'$, let $A' \cap D = \{\xi_{Dn}\}_{n\in\mathbb{N}}$, allowing repetitions if $A' \cap D$ is finite. For every $n \in \mathbb{N}$, let $F(n) = \bigcup_{D\in\mathcal{D}'}(\{\xi_{Dn}\} \times F_{\xi_{Dn}})$ and note that $F(n) = \overline{F(n)} \cap ((\omega_1 \setminus H) \times \omega_1)$, so $F(n)$ is Borel. Finally, $F = \bigcup_{n\in\mathbb{N}} F(n)$ is Borel.

By taking in succession complements in $A \times \omega_1$, intersections, and countable unions, the statement holds for F having sections that are respectively open, intersections of an open and a closed set, countable unions of intersections of an open and a closed set, that is general thin sets. If the non-empty vertical sections of F are thick, apply the statement to $(A' \times \omega_1) \setminus F$, to get the statement for F. To conclude the proof, observe that every set with Borel vertical sections is the union of a set with thin vertical sections and a set with non-empty vertical sections that are thick. ■

Theorem 2.10 *Let $f, g : \omega_1 \to \omega_1$ be Borel. Then $f \times g : \omega_1 \times \omega_1 \to \omega_1 \times \omega_1$ is Borel.*

Proof By Theorem 2.7, there exists a club C such that one of the following four cases holds:

(i) $f_{|C}, g_{|C}$ are constant, say with values γ, δ, respectively.
(ii) $\forall \alpha \in C\ f(\alpha) = g(\alpha) = \alpha$.
(iii) $f_{|C}$ is constant, say with value γ, and $\forall \alpha \in C\ g(\alpha) = \alpha$.
(iv) $\forall \alpha \in C\ f(\alpha) = \alpha$ and $g_{|C}$ is constant.

Let B be a Borel subset of $\omega_1 \times \omega_1$, in order to prove that $(f \times g)^{-1}(B)$ is Borel. Note that

$$
\begin{aligned}
&(f \times g)^{-1}(B) = \\
&\qquad = ((f \times g)^{-1}(B) \cap C^2) \cup ((f \times g)^{-1}(B) \cap ((\omega_1 \setminus C) \times C)) \cup \\
&\qquad \cup ((f \times g)^{-1}(B) \cap (C \times (\omega_1 \setminus C))) \cup ((f \times g)^{-1}(B) \cap (\omega_1 \setminus C)^2).
\end{aligned}
$$

Every subset of $\omega_1 \setminus C$ is Borel, so every subset of $(\omega_1 \setminus C)^2$ is Borel by Lemma 2.9. Thus, it remains to prove that the other three terms in the union are Borel.

Case (i).

- $$(f \times g)^{-1}(B) \cap C^2 = \begin{cases} \emptyset, & \text{if } (\gamma, \delta) \notin B, \\ C^2, & \text{if } (\gamma, \delta) \in B, \end{cases}$$

  so $(f \times g)^{-1}(B) \cap C^2$ is Borel.

- $$
\begin{aligned}
&(f \times g)^{-1}(B) \cap ((\omega_1 \setminus C) \times C) = \\
&= \{(\alpha, \beta) \in (\omega_1 \setminus C) \times C \mid (f(\alpha), \delta) \in B\} = \\
&= \{(\alpha, \beta) \in (\omega_1 \setminus C) \times C \mid f(\alpha) \in B^\delta\} = (f^{-1}(B^\delta) \setminus C) \times C
\end{aligned}
$$

  is Borel. Similarly, for $(f \times g)^{-1}(B) \cap (C \times (\omega_1 \setminus C))$.

Case (ii).

- $(f \times g)^{-1}(B) \cap C^2 = B \cap C^2$ is Borel.
- $$
\begin{aligned}
&(f \times g)^{-1}(B) \cap ((\omega_1 \setminus C) \times C) = \\
&= \{(\alpha, \beta) \in (\omega_1 \setminus C) \times C \mid (f(\alpha), \beta) \in B\} = \\
&= \{(\alpha, \beta) \in (\omega_1 \setminus C) \times C \mid \beta \in B_{f(\alpha)}\} = \\
&= \textstyle\bigcup_{\alpha \in \omega_1 \setminus C} (\{\alpha\} \times (B_{f(\alpha)} \cap C))
\end{aligned}
\tag{2.3}
$$

  is Borel by Lemma 2.9. Similarly, for $(f \times g)^{-1}(B) \cap (C \times (\omega_1 \setminus C))$.

Case (iii).

-
$$(f \times g)^{-1}(B) \cap C^2 = \{(\alpha, \beta) \in C^2 \mid (\gamma, \beta) \in B\} =$$
$$= \{(\alpha, \beta) \in C^2 \mid \beta \in B_\gamma\} = C \times (B_\gamma \cap C)$$
is Borel.
- For $(f \times g)^{-1}(B) \cap ((\omega_1 \setminus C) \times C)$, the computation is the same as in (2.3).
-
$$(f \times g)^{-1}(B) \cap (C \times (\omega_1 \setminus C)) = \{(\alpha, \beta) \in C \times (\omega_1 \setminus C) \mid (\gamma, g(\beta)) \in B\} =$$
$$= \{(\alpha, \beta) \in C \times (\omega_1 \setminus C) \mid g(\beta) \in B^\gamma\} = C \times (g^{-1}(B^\gamma) \setminus C)$$
is Borel.

Case (iv). This is similar to case (iii). ■

Corollary 2.11 *If E, F are equivalence relations on ω_1 with F Borel and $E \leqslant_B F$, then E is Borel.*

Proof Let f be a Borel reduction of E to F. Then $E = (f \times f)^{-1}(F)$. Apply Theorem 2.10. ■

3 Equivalence relations on ω_1

A key tool to compare equivalence relations on ω_1 consists in counting their classes according to their size.

Definition 3.1 For E an equivalence relation on ω_1, let the *characteristic triple* of E be the triple of cardinal numbers $K = (\kappa_0^E, \kappa_1^E, \kappa_2^E)$ where:

κ_0^E is the number of thick equivalence classes of E.

κ_1^E is the number of bistationary equivalence classes of E.

κ_2^E is the number of thin equivalence classes of E.

Let $\mathcal{E}(\kappa_0, \kappa_1, \kappa_2)$ be the set of equivalence relations of characteristic triple $(\kappa_0, \kappa_1, \kappa_2)$.

Note that:

(3.1) $$\kappa_0^E \in \{0, 1\}, \quad 0 \leqslant \kappa_1^E \leqslant \aleph_1, \quad 0 \leqslant \kappa_2^E \leqslant \aleph_1,$$

moreover, by the definition of stationarity and the fact that the union of countably many thin sets is thin:

(i) If $\kappa_0^E = 1$ then $\kappa_1^E = 0$.
(ii) If $\kappa_0^E = \kappa_1^E = 0$, then $\kappa_2^E = \aleph_1$.
(iii) If $\kappa_1^E = 1$, then $\kappa_2^E = \aleph_1$.

Notice that (3.1) and the conditions (i–iii) above are the only constraints on the characteristic triple of an equivalence relation on ω_1: if $K = (\kappa_0, \kappa_1, \kappa_2)$ satisfies such conditions, there exists a partition of ω_1 with κ_0 thick elements, κ_1 bistationary elements, and κ_2 thin elements.

Definition 3.2 A triple $K = (\kappa_0, \kappa_1, \kappa_2)$ of cardinals satisfying (3.1) and the conditions (i–iii) above is a *legitimate triple*.

In other words, $(\kappa_0, \kappa_1, \kappa_2)$ is a legitimate triple if and only if $\mathcal{E}(\kappa_0, \kappa_1, \kappa_2) \neq \emptyset$. The legitimate triples are represented as labels of the nodes in Figure 1.

3.1 Equivalence relations with a thick class

This section deals with equivalence relations whose characteristic triple is $(1, 0, \kappa)$, where $0 \leqslant \kappa \leqslant \aleph_1$. It turns out that such relations form an initial segment with respect to $\leqslant_B$, and they Borel reduce to any equivalence relation with at least the same number of classes. Consequently, they are Borel by Corollary 2.11, since they Borel reduce to the equality relation which is closed.

Proposition 3.3 *Let E, F be equivalence relations on ω_1 such that E has a thick equivalence class. Then $E \leqslant_B F \Leftrightarrow \operatorname{card}(\omega_1/_E) \leqslant \operatorname{card}(\omega_1/_F)$.*

Proof The implication from left to right holds for any reduction.

As for the opposite implication, let $\{A_\gamma\}_\gamma$, $\{B_\gamma\}_\gamma$ be enumerations of the equivalence classes of E, F, respectively. Let $f : \omega_1 \to \omega_1$ be defined by letting $f(\alpha) = \min B_\gamma$ if $\alpha \in A_\gamma$. Being constant on a thick equivalence class, f is Borel by Theorem 2.7; moreover, it reduces E to F. ∎

Proposition 3.4 *Let E, F be equivalence relations on ω_1. Assume that E has a thick equivalence class and $F \leqslant_B E$. Then F has a thick equivalence class.*

Proof Deny. Let $f : \omega_1 \to \omega_1$ be a Borel reduction of F to E. Then f is not constant on a thick set, since F does not have thick equivalence classes. Thus, by Theorem 2.7, there exists a thick set C such that $\forall \alpha \in C\ f(\alpha) = \alpha$. Let A be the thick equivalence class of E, so that $C \cap A$ is thick; therefore, there exist F-inequivalent elements in $C \cap A$ that are sent by f to E-equivalent elements (namely, themselves), which is a contradiction. ∎

Corollary 3.5 *Let E, F be equivalence relations on ω_1. Assume that E has a thick equivalence class and no equivalence class of F is thick. Then*

$$E \leqslant_B F \Leftrightarrow E <_B F \Leftrightarrow \operatorname{card}(\omega_1/_E) \leqslant \operatorname{card}(\omega_1/_F).$$

3.2 Equivalence relations of all whose classes are thin

Corollary 3.5 implies that all equivalence relations with a thick class Borel reduce to the equivalence relations whose classes are thin, like the equality relation. Actually, up to $\sim_B$, there is just one such equivalence relation. Thus, by Corollary 2.11, all such relations are Borel.

Proposition 3.6 *Let E, F be equivalence relations on ω_1 all of whose equivalence classes are thin. Then $E \sim_B F$.*

Proof Using Lemma 2.5, let C_E be a club intersecting every equivalence class of E in at most one point, and let C_F be a club intersecting every equivalence class of F in at most one point. Let $C = C_E \cap C_F$, and let $C' \subseteq C$ be a club such that $\operatorname{card}(C \setminus C') = \aleph_1$ – for instance, let C' be the set of limit points of C. In particular, C' misses $\aleph_1$ equivalence classes of E and $\aleph_1$ equivalence classes of F: let $\{A_\gamma\}_{\gamma\in\omega_1}, \{B_\gamma\}_{\gamma\in\omega_1}$ be enumerations of such classes.

Let $f : \omega_1 \to \omega_1$ be defined by

$$f(\alpha) = \begin{cases} \beta, & \text{if } \alpha E \beta \in C', \\ \min B_\gamma, & \text{if } \alpha \in A_\gamma. \end{cases}$$

Then f is Borel by Theorem 2.7 and witnesses $E \leqslant_B F$.

Similarly, $F \leqslant_B E$. ■

3.3 Equivalence relations with some bistationary classes

This section focuses on equivalence relations having some bistationary classes; due to the presence of non-Borel classes, such equivalence relations are not Borel.

Definition 3.7 Let E, F be equivalence relations on ω_1 such that $\kappa_1^E = \kappa_1^F$. Let $\{A_\alpha\}_{\alpha<\kappa_1^E}, \{B_\alpha\}_{\alpha<\kappa_1^E}$ be enumerations of the bistationary classes of E, F, respectively. Say that such enumerations are *compatible* if there exists a club C such that

$$\forall \alpha < \kappa_1^E \; A_\alpha \cap C = B_\alpha \cap C. \tag{3.2}$$

The club C is a *witness of compatibility* for E, F.

Note that when (3.2) holds, every symmetric difference $A_\alpha \triangle B_\alpha$ is a thin set.

Proposition 3.8 *Let E, F be equivalence relations on ω_1 such that $E \leqslant_B F$. Assume that $\kappa_1^E > 0$. Then E, F admit compatible enumerations of their bistationary equivalence classes. In particular, $\kappa_1^E = \kappa_1^F$. Moreover, $\kappa_2^E \leqslant \kappa_2^F$.*

Proof Let f be a Borel reduction of E to F. Since $\kappa_0^E = 0$, function f cannot be constant on a thick set, so by Theorem 2.7, let C be a club such that $\forall \alpha \in C\; f(\alpha) = \alpha$. Then, for every bistationary equivalence class A of E, the set $A \cap C$ is bistationary and the elements of $f[A \cap C] = A \cap C$ are pairwise F-equivalent, so that $A \cap C$ is included in a single stationary F-class. Similarly, for every stationary equivalence class B of F, it holds that the stationary set $B \cap C$ is included in a single E-class, so $B \cap C$ is actually bistationary and B is bistationary as well. This shows that the bistationary equivalence classes of E and the bistationary equivalence classes of F can be enumerated compatibly, say as $\{A_\alpha\}_{\alpha<\kappa_1^E}, \{B_\alpha\}_{\alpha<\kappa_1^E}$, respectively, with C as witness of compatibility.

The last assertion follows from $\kappa_0^E = \kappa_0^F = 0$ and the fact that f sends the elements of each A_α to elements in the corresponding B_α; therefore, f sends elements belonging to thin classes of E to elements belonging to thin classes of F. ∎

The converse of Proposition 3.8 holds as well.

Proposition 3.9 *Let E, F be equivalence relations such that $\kappa_1^E = \kappa_1^F > 0$ and $\kappa_2^E \leqslant \kappa_2^F$. Suppose that E, F admit compatible enumerations of their bistationary equivalence classes. Then $E \leqslant_B F$.*

Proof Let $\{A_\alpha\}_{\alpha<\kappa_1^E}, \{B_\alpha\}_{\alpha<\kappa_1^E}$ be compatible enumerations of the bistationary equivalence classes of E, F, respectively. Let C be a witness of compatibility.

Assume first that $\bigcup_{\alpha<\kappa_1^E} A_\alpha$ is thick, and let C' be a club with $C' \subseteq \bigcup_{\alpha<\kappa_1^E} A_\alpha$. Define $f : \omega_1 \to \omega_1$ such that:

- If $\xi \in C \cap C'$, then $f(\xi) = \xi$.
- If $\xi \in A_\alpha \setminus (C \cap C')$, then $f(\xi) = \min B_\alpha$.
- f induces a reduction of the restriction of E to $\omega_1 \setminus \bigcup_{\alpha<\kappa_1^E} A_\alpha$ to the restriction of F to $\omega_1 \setminus \bigcup_{\alpha<\kappa_1^E} B_\alpha$. This can be done as $\kappa_2^E \leqslant \kappa_2^F$.

Then f is Borel by Theorem 2.7 and reduces E to F.

Assume now that $\bigcup_{\alpha<\kappa_1^E} A_\alpha$ is bistationary.

Claim 3.10 $\bigcup_{\alpha<\kappa_1^E} B_\alpha$ is bistationary.

Proof of the claim If $\bigcup_{\alpha<\kappa_1^E} B_\alpha$ were thick, let C_0 be a club with $C_0 \subseteq \bigcup_{\alpha<\kappa_1^E} B_\alpha$. Then $C \cap C_0$ is also a witness of compatibility for E, F and $C \cap C_0 \subseteq \bigcup_{\alpha<\kappa_1^E} A_\alpha$, so $\bigcup_{\alpha<\kappa_1^E} A_\alpha$ would be thick. ∎

From the assumption and the claim, it follows also that $\kappa_2^E = \kappa_2^F = \aleph_1$. By Lemma 2.5, let C' be a club intersecting in at most one point every thin equivalence class of E. Similarly, let C'' be a club intersecting in at most one point every thin equivalence class of F. By [KT2006, Chapter 20, Problem 25], let $\mathcal{F}$ be a family of cardinality $\aleph_1$ of thin equivalence classes of F such that $\bigcup \mathcal{F}$ is thin, and let C''' be a club disjoint from $\bigcup \mathcal{F}$. Set $D = C \cap C' \cap C'' \cap C'''$.

Define $f : \omega_1 \to \omega_1$ such that:

- If $\xi \in D$, then $f(\xi) = \xi$.
- If $\xi \notin D$ but there exists a least ζ such that $\xi E \zeta \in D$ then $f(\xi) = \zeta$.
- f induces a reduction of the restriction of E to $\{\xi \in \omega_1 \mid [\xi]_E \cap D = \emptyset\}$ to the restriction of F to $\{\xi \in \omega_1 \mid [\xi]_F \cap D = \emptyset\}$. This is possible by the choice of C'''.

Then f is Borel by Theorem 2.7 and witnesses $E \leqslant_B F$. ∎

Corollary 3.11 *If E, F are equivalence relations on ω_1 having the same characteristic triple $(0, \kappa_1, \kappa_2)$ with $\kappa_1 > 0$, then $E \leqslant_B F \Leftrightarrow E \sim_B F$.*

Proof Assume $E \leqslant_B F$. By Proposition 3.8, E, F admit compatible enumerations of their bistationary equivalence classes. Therefore, by Proposition 3.9, with the role of E, F switched, $F \leqslant_B E$. ∎

Thus, up to $\sim_B$, the equivalence relations with at least one bistationary class having a given characteristic triple form an antichain. This antichain is as big as it can be.

Proposition 3.12 *Let $K = (0, \kappa_1, \kappa_2)$ be a legitimate triple with $\kappa_1 > 0$. Then there is a $\leqslant_B$-antichain of cardinality $2^{\aleph_1}$ of equivalence relations having K as characteristic triple.*

Proof Let $\{S_\alpha\}_{\alpha\in\omega_1}$ be a partition of ω_1 into bistationary sets. This exists by [BL2020, Corollary 8].

To deal with the case $\kappa_1 = 1, \kappa_2 = \aleph_1$, for every non-empty, proper subset I of ω_1, let E_I be the equivalence relation having $\bigcup_{\alpha\in I} S_\alpha$ as one equivalence class, all other equivalence classes being singletons. Then characteristic triple of E_I is $(0, 1, \aleph_1)$ and $I \neq J \Rightarrow E_I \nleqslant_B E_J$ by Proposition 3.8.

For the case $\kappa_1 > 1$, let

$$S'_\alpha = \begin{cases} S_\alpha \setminus \{\min S_\alpha\}, & \text{if } \alpha < \kappa_2, \\ S_\alpha, & \text{if } \kappa_2 \leq \alpha. \end{cases}$$

Then each $S'\alpha$ is bistationary as well. For every partition $\mathcal{P}$ of ω_1 into κ_1 subsets, let $E_{\mathcal{P}}$ be the equivalence relations whose classes are all bistationary subsets $\bigcup_{\alpha\in P} S'_\alpha$ for $P \in \mathcal{P}$ and all singletons $\{\min S_\alpha\}$ for $\alpha < \kappa_2$. Then the characteristic triple of each $E_{\mathcal{P}}$ is $(0, \kappa_1, \kappa_2)$ and $\mathcal{P} \neq \mathcal{Q} \Rightarrow E_{\mathcal{P}} \nleqslant_B E_{\mathcal{Q}}$ by Proposition 3.8 again. ■

If E is an equivalence relation on ω_1 denote by $[E]_\sim$ the $\sim_B$-class of E.

Proposition 3.13 *Let $(0, \kappa_1, \kappa_2), (0, \kappa_1, \kappa'_2)$ be legitimate triples, with $\kappa_1 > 0$ and $\kappa_2 < \kappa'_2$. Setting $\varphi([E]_\sim) = [F]_\sim \Leftrightarrow E \leqslant_B F$ defines an injection $\varphi : \mathcal{E}(0, \kappa_1, \kappa_2)/_{\sim_B} \to \mathcal{E}(0, \kappa_1, \kappa'_2)/_{\sim_B}$. The range of φ is the set of all $[F]_\sim \in \mathcal{E}(0, \kappa_1, \kappa'_2)/_{\sim_B}$ such that the union of the thin equivalence classes of F is thin. In particular, φ is surjective if and only if $\kappa'_2 \leqslant \aleph_0$.*

Proof First, notice that for every $E \in \mathcal{E}(0, \kappa_1, \kappa_2)$, there exists $F \in \mathcal{E}(0, \kappa_1, \kappa'_2)$ such that $E \leqslant_B F$. Indeed, let A be a bistationary equivalence class of E, and let κ be such that $\kappa_2 + \kappa = \kappa'_2$. By [KT2006, Chapter 20, Problem 25], let $B \subseteq A$ such that B is thin and $\operatorname{card}(B) = \kappa$. Let F be the equivalence relation whose equivalence classes are $A \setminus B$, all equivalence classes of E distinct from A, and all singleton subsets of B. Then the characteristic triple of F is $(0, \kappa_1, \kappa'_2)$ and $E \leqslant_B F$ by Proposition 3.9, since any club disjoint from B is a witness of compatibility for E, F.

Function φ is well defined and injective by Propositions 3.8 and 3.9.

Assume that the union of the thin equivalence classes of $F \in \mathcal{E}(0, \kappa_1, \kappa'_2)$ is thin, and let C be a club disjoint from such a union. Let E be the equivalence relation having the same bistationary equivalence classes of F and κ_2 thin equivalence classes. Then $E \in \mathcal{E}(0, \kappa_1, \kappa_2)$ and $E \leqslant_B F$ by Proposition 3.9. Thus $[F]_\sim$ belongs to the range of φ.

Assume instead that the union of the thin equivalence classes of $F \in \mathcal{E}(0, \kappa_1, \kappa'_2)$ is stationary, so that in particular $\kappa'_2 = \aleph_1$. Toward contradiction, suppose that $E \in \mathcal{E}(0, \kappa_1, \kappa_2)$ and that there exists a Borel reduction f of E to F. By Theorem 2.7, let C be a club such that $\forall \alpha \in C\ f(\alpha) = \alpha$ and note that C intersects $\aleph_1$ thin equivalence classes of F, each of such intersections being a thin set. Therefore, E must have $\aleph_1$

equivalence classes intersecting C in a thin set, therefore, E has $\aleph_1$ thin equivalence classes, contradicting $\kappa_2 < \kappa_2' = \aleph_1$. So in this case, $[F]_\sim$ is not in the range of φ. ■

3.4 Summarizing the results

The following theorem collects the results of the preceding sections characterizing Borel reducibility of equivalence relations on ω_1.

Theorem 3.14 *Let E, F be equivalence relations on ω_1. Then $E \leqslant_B F$ if and only if one of the following conditions holds:*

(i) $\kappa_0^E = 1$ *and* $\mathrm{card}\left(\omega_1/E\right) \leqslant \mathrm{card}\left(\omega_1/F\right)$.
(ii) $\kappa_0^E = \kappa_0^F = \kappa_1^E = \kappa_1^F = 0$ *and* $\kappa_2^E = \kappa_2^F = \aleph_1$.
(iii) $\kappa_0^E = \kappa_0^F = 0, \kappa_1^E = \kappa_1^F > 0$, *relations E, F admit compatible enumerations of their bistationary classes, and* $\kappa_2^E \leqslant \kappa_2^F$.

This structure is also represented in Figure 1, where the triples labeling each node are the characteristic triples of the equivalence relations in that node.

- Unboxed nodes contain just one equivalence relation up to $\sim_B$. These have characteristic triple of the form $(1, 0, \kappa_2)$ or $(0, 0, \aleph_1)$ and are the Borel equivalence relations on ω_1.
- Boxed nodes, corresponding to characteristic triples of the form $(0, \kappa_1, \kappa_2)$ with $\kappa_1 \geqslant 1$, represent antichains of size $2^{\aleph_1}$.
- Solid arrows mean that every equivalence relation at the tail of the arrow Borel reduces to every equivalence relation at the head of the arrow.
- Dashed arrows, between nodes labeled $(0, \kappa_1, \kappa_2)$ and $(0, \kappa_1, \kappa' 2)$ with $\kappa_2 < \kappa_2' \leq \aleph_0$, mean that $\leqslant_B$ is a bijective correspondence between the $\sim_B$-classes of equivalence relations with such characteristic triples.
- Dashed and dotted arrows, between nodes labeled $(0, \kappa_1, \aleph_0)$ and $(0, \kappa_1, \aleph_1)$, mean that $\leqslant_B$ is an injective but not surjective correspondence between the $\sim_B$-classes of equivalence relations with such characteristic triples.

4 Comparison with equivalence relations on Polish spaces

This section determines all comparabilities between equivalence relations on Polish spaces and equivalence relations on ω_1.

Lemma 4.1 *Let X be a Polish space and $f : \omega_1 \to X$. Then f is Borel if and only if there exists a thick set $C \subseteq \omega_1$ such that $f_{|C}$ is constant.*

Proof Suppose that f is Borel. If X is finite, then it follows immediately that f is constant on a thick set, so assume that X is infinite. For every $n \in \mathbb{N}$, let $\mathcal{B}_n = \{B_{nm}\}_{m\in\mathbb{N}}$ be a partition of X into Borel subsets of diameter less than $\frac{1}{n+1}$, and such that $\mathcal{B}_{n+1}$ refines $\mathcal{B}_n$. Then there is a sequence $\{m_n\}_{n\in\mathbb{N}}$ of natural numbers such that $B_{n+1,m_{n+1}} \subseteq B_{nm_n}$ and $C_n = f^{-1}(B_{nm_n})$ is thick, for every $n \in \mathbb{N}$. Therefore, $C = \bigcap_{n\in\mathbb{N}} C_n$ is thick and $f_{|C}$ is constant.

Conversely, if C is a thick set such that f is constant on C, let x be the value taken by f on C. Then given $B \subseteq X$ it turns out that $f^{-1}(B)$ is either thin or thick, so Borel, according to whether $x \notin B$ or $x \in B$. ■

Proposition 4.2 *Let X be a Polish space, let E be an equivalence relation on ω_1, and let F be an equivalence relation on X. Then $E \leqslant_B F$ if and only if E has a thick equivalence class and* $\operatorname{card}(\omega_1/E) \leqslant \operatorname{card}(X/F)$.

Proof Let $f : \omega_1 \to X$ be a Borel reduction of E to F. By Lemma 4.1, let C be a thick set on which f is constant. Then C is contained in a thick equivalence class of E. Moreover, $\operatorname{card}(\omega_1/E) \leqslant \operatorname{card}(X/F)$ as f is a reduction.

Conversely, if E has a thick equivalence class and $\operatorname{card}(\omega_1/E) \leqslant \operatorname{card}(X/F)$, let $f : \omega_1 \to X$ be a reduction of E to F constant on every equivalence class of E. Then f is Borel by Lemma 4.1. ■

Lemma 4.3 *Let X be a Polish space, and let $\{B_\alpha\}_{\alpha\in\omega_1}$ be a family of pairwise disjoint, non-empty Borel subsets of X. Then there exists $I \subseteq \omega_1$ such that $\bigcup_{\alpha\in I} B_\alpha$ is not Borel.*

Proof If $\bigcup_{\alpha\in\omega_1} B_\alpha$ is not Borel, let $I = \omega_1$. Otherwise, toward contradiction, deny the conclusion. Let $Y = \{y_\alpha\}_{\alpha\in\omega_1}$ be a non-analytic subset of X of cardinality $\aleph_1$, and let $g : \bigcup_{\alpha\in\omega_1} B_\alpha \to X$ be defined by letting $g(x) = y_\alpha$ for the unique α such that $x \in B_\alpha$. Then g is Borel but has non-analytic range, a contradiction. ■

Lemma 4.4 *Let X be a Polish space, and let $f : X \to \omega_1$. Then f is Borel if and only if the range of f is countable and $f^{-1}(\{\beta\})$ is Borel for every $\beta \in \omega_1$.*

Proof Assume that f is Borel. Then the preimage of every singleton is Borel. Toward contradiction, assume that the range of f is uncountable. By [KT2006, Chapter 20, Problem 25], there exists a thin subset B of the range of f such that $\operatorname{card}(B) = \aleph_1$. Then the preimages of singleton subsets of B form a family $\{A_\alpha\}_{\alpha\in\omega_1}$ of pairwise disjoint Borel subsets of X. By Lemma 4.3, let I be such that $\bigcup_{\alpha\in I} A_\alpha$ is not Borel. Then $f[\bigcup_{\alpha\in I} A_\alpha]$ is thin, so Borel, while $f^{-1}(f[\bigcup_{\alpha\in I} A_\alpha]) = \bigcup_{\alpha\in I} A_\alpha$ is not Borel, a contradiction.

Conversely, if the range R of f is countable and the preimage of every singleton is Borel, given any $B \subseteq \omega_1$ one has that $f^{-1}(B) = \bigcup_{\beta\in B\cap R} f^{-1}(\{\beta\})$ is a countable union of Borel sets, so it is Borel. ■

Proposition 4.5 *Let X be a Polish space, let E be an equivalence relation on X, and let F be an equivalence relation on ω_1. Then $E \leqslant_B F$ if and only if all equivalence classes of E are Borel and* $\operatorname{card}(X/E) \leqslant \min(\aleph_0, \operatorname{card}(\omega_1/F))$.

Proof Let $f : X \to \omega_1$ be a Borel reduction of E to F. By Lemma 4.4, the preimage under f of any subset of ω_1 is Borel, in particular, all E-classes are Borel. The inequality $\operatorname{card}(X/E) \leqslant \aleph_0$ holds as the equivalence classes of E are preimages of pairwise disjoint subsets of the range of f, while the inequality $\operatorname{card}(X/E) \leqslant \operatorname{card}(\omega_1/F)$ holds as f is a reduction.

Conversely, assume that all equivalence classes of E are Borel and $\operatorname{card}\left({}^{X}/_{E}\right) \leqslant \min\left(\aleph_0, \operatorname{card}\left({}^{\omega_1}/_{F}\right)\right)$. Let $\{A_n \mid n < N\}$ be an enumeration of the equivalence classes of E, and let $\{\beta_n \mid n < N\}$ be a set of pairwise F-inequivalent elements of ω_1, for some $N \leqslant \aleph_0$. Let $f : X \to \omega_1$ be defined by letting $f(x) = \beta_n \Leftrightarrow x \in A_n$. Then f is Borel by Lemma 4.4 and reduces E to F by construction. ∎

References

[A2018] M. Alami, *Functions on stationary sets*. Palest. J. Math. **7**(2018), 222–226.

[BL2020] H. Bennett and D. Lutzer, *Topology inside* ω_1. Rocky Mountain J. Math. **50**(2020), 1989–2000.

[G2009] S. Gao, *Invariant descriptive set theory*, CRC Press, Boca Raton, FL, 2009.

[GJK2008] S. Gao, S. Jackson, and V. Kieftenbeld, *A classification of ordinals up to Borel isomorphism*. Fund. Math. **198**(2008), 61–76.

[J2002] T. Jech, *Set theory*, Springer, Berlin, 2002.

[KT2006] P. Komjáth and V. Totik, *Problems and theorems in classical set theory*, Springer, New York, 2006.

[Ru1957] M. E. Rudin, *A subset of the countable ordinals*. Amer. Math. Monthly **64**(1957), 351.

Department of mathematics, Università di Genova, Via Dodecaneso 35, 16146 Genova, Italy
e-mail: camerlo@dima.unige.it