LEFT ORDERS IN REGULAR $\mathscr{H}$-SEMIGROUPS II

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1. Introduction. We make the convention that if a is an element of a semigroup Q then by writing a^{-1} it is implicit that a lies in a subgroup of Q and has inverse a^{-1} in this subgroup; equivalently, $a\,\mathscr{H}\,a^2$ and a^{-1} is the inverse of a in H_a.

A subsemigroup S of a semigroup Q is a *left order* in Q and Q is a *semigroup of left quotients* of S if every element of Q can be written as $a^{-1}b$ where $a, b \in S$ and, in addition, every element of S satisfying a weak cancellation condition which we call *square-cancellable* lies in a subgroup of Q. The notions of *right order* and *semigroup of right quotients* are defined dually; if S is both a left order and a right order in Q then S is an *order* in Q and Q is a *semigroup of quotients* of S.

This concept of order was introduced by Fountain and Petrich in **[4]** and has been studied in a number of subsequent papers. Those semigroups that are orders in the following classes have been characterized:

(i) completely 0-simple semigroups **[4]**,
(ii) normal band of groups **[6]**,

as have left orders in semigroups in the classes:

(iii) Clifford semigroups **[7]**,
(iv) absolutely flat completely 0-simple semigroups **[10]**,
(v) Brandt semigroups **[5]**,
(vi) simple inverse ω-semigroups **[9]**.

It is easy to see that a regular semigroup is an order in itself. The classes (i), . . . , (vi) contain only regular semigroups and each has a satisfactory structure theorem. Thus a description of (left) orders in one of these classes yields a larger class of semigroups, for which no structure theorem is given, but such that each semigroup in the class may be embedded in a closely prescribed manner in a semigroup whose structure is well determined.

We define an $\mathscr{H}$-semigroup to be a semigroup on which Green's relation $\mathscr{H}$ is a congruence. The classes (i), . . . , (vi) have the common property that they contain only regular $\mathscr{H}$-semigroups. In a previous paper **[11]** we offer several characterizations of left orders in regular $\mathscr{H}$-semigroups. Here we show how the results of **[11]** can be applied to obtain descriptions of orders and left orders in the classes (i), . . . , (vi). In addition we give a characterization of left orders in bands of groups, a class not previously considered as semigroups of left quotients.

The generalisations $\mathscr{L}^*$ and $\mathscr{R}^*$ of Green's relations $\mathscr{L}$ and $\mathscr{R}$ play an important role in our theory. We recall that a semigroup S is *abundant* if every $\mathscr{L}^*$-class and every

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$\mathcal{R}^*$-class of S contains an idempotent. The last part of this paper concentrates on characterizing abundant semigroups that are left orders in regular $\mathcal{H}$-semigroups and in addition are full subsemigroups of their semigroup of left quotients. Here the cancellation conditions involved in the description of left orders in regular $\mathcal{H}$-semigroups may be dropped. If we specialise still further by insisting that the left orders be stratified, an even simpler characterization is obtained.

2. Preliminaries. We assume a basic knowledge of semigroup theory, such as may be found in **[2]** or **[12]**. This paper, although intended to be self-contained, is a sequel to **[11]**, where more details of some of the ideas set out in this preliminary section can be found. Where possible, we follow the notation and terminology of **[12]**. We deviate slightly in denoting by $\mathcal{H}_S$ (rather that $\mathcal{H}^S$) a relation $\mathcal{H}$ on a semigroup S, where to avoid ambiguity we wish to emphasize that S is the semigroup in question.

The relation $\mathcal{L}^*$ is defined on a semigroup S by the rule that if $a, b \in S$ then $a\,\mathcal{L}^*\,b$ if and only if $a\,\mathcal{L}\,b$ in some oversemigroup of S. The relation $\mathcal{R}^*$ is defined dually and we denote by $\mathcal{H}^*$ the intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$. The following alternative characterization of $\mathcal{L}^*$ and $\mathcal{R}^*$ is taken from **[3]**.

LEMMA 2.1. *Let S be a semigroup and let $a, b \in S$. Then $a\,\mathcal{L}^*\,b(a\,\mathcal{R}^*\,b)$ if and only if for all $x, y \in S^1$,*

$$ax = ay \quad \textit{if and only if} \quad bx = by$$

$$(xa = ya \quad \textit{if and only if} \quad xb = yb).$$

In view of Lemma 2.1 it is clear that $\mathcal{L}^*$ is a right congruence, $\mathcal{R}^*$ is a left congruence and $\mathcal{H}^*$ is an equivalence relation on any semigroup. An element a of a semigroup S is *square-cancellable* if $a\,\mathcal{H}^*\,a^2$. If S is a regular semigroup it is easily seen that $\mathcal{L} = \mathcal{L}^*$, $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{H} = \mathcal{H}^*$ on S. Thus, in this case, $a \in S$ is square-cancellable if and only if it lies in a subgroup of S. Square-cancellation is a necessary condition for an element of a semigroup S to lie in a subgroup of an oversemigroup—by definition we insist that all such elements must lie in subgroups of any semigroup of (left, right) quotients.

A semigroup T is *right* (*left*) *reversible* if given any elements a, b of T there are elements c, d of T with $ca = db$ ($ac = bd$). If T is both left and right reversible, then T is *reversible.* A classical result of Ore and Dubreil **[2]** states that a semigroup T is a left order in a group if and only if T is right reversible and cancellative.

Let S be a left order in Q. Then S is *straight* in Q if given any $q \in Q$, $q = a^{-1}b$ for some $a, b \in S$ with $a\,\mathcal{R}\,b$ in Q. We say that S is *local* in Q if $S \cap H$ is a left order in H, for all group $\mathcal{H}$-classes H of Q. Straight right orders and local right orders are defined dually.

PROPOSITION 2.2 **[8]**. *Let S be a left order in a regular $\mathcal{H}$-semigroup. Then S is a straight, local left order in Q. Further, if $q \in Q$ and $q = a^{-1}b$ where $a, b \in S$ and $a\,\mathcal{R}\,b$ in Q, then $q\,\mathcal{H}\,b$ in Q, so that S has non-empty intersection with every $\mathcal{H}$-class of Q.*

If S is a semigroup, then a *suitable pair* for S is an ordered pair $(\mathcal{L}', \mathcal{R}')$ of equivalence relations on S such that $\mathcal{L}'$ is a right congruence contained in $\mathcal{L}^*$, $\mathcal{R}'$ is a left congruence contained in $\mathscr{R}^*$ and for any $a \in S$

$$a\, \mathscr{H}^*\, a^2 \quad \text{if and only if} \quad a\, \mathscr{H}'\, a^2,$$

where $\mathscr{H}' = \mathscr{L}' \cap \mathscr{R}'$. Clearly $(\mathscr{L}^*, \mathscr{R}^*)$ is always a suitable pair for S. Moreover it is straightforward to show that if S is a left (right) order in a semigroup Q, then $(\mathscr{L}_Q \cap (S \times S), \mathscr{R}_Q \cap (S \times S))$ is a suitable pair for S; if S is a straight left (right) order in Q and $(\mathscr{L}_Q \cap (S \times S), \mathscr{R}_Q \cap (S \times S)) = (\mathscr{L}^*, \mathscr{R}^*)$ then S is said to be *stratified*.

Given a suitable pair $(\mathscr{L}', \mathscr{R}')$ for a semigroup S, we denote by L'_a, R'_a and H'_a respectively the $\mathscr{L}'$-class, $\mathscr{R}'$-class and $\mathscr{H}'$-class of an element a of S. We remark that throughout the paper we define and use conditions on a semigroup relating to a given suitable pair. For such a condition (X), we consistently omit the phrase "with respect to the given suitable pair $(\mathscr{L}', \mathscr{R}')$" when writing, for example, "S satisfies condition (X) with respect to the given suitable pair $(\mathscr{L}', \mathscr{R}')$".

We can now state the theorems from **[11]** on which this paper is based.

THEOREM 2.3 **[11]**. *Let S be a semigroup and let $(\mathscr{L}', \mathscr{R}')$ be a suitable pair for S. The following conditions are equivalent:*

(i) *S is a left order in a regular $\mathscr{H}$-semigroup Q such that $\mathscr{L}_Q \cap (S \times S) = \mathscr{L}'$ and $\mathscr{R}_Q \cap (S \times S) = \mathscr{R}'$;*

(ii) *S satisfies conditions* (A), (B), (C), (D), (E) *and the left-right duals* (C)′, (D)′ *and* (E)′ *of* (C), (D) *and* (E):

(A) *$\mathscr{H}'$ is a congruence on S and $S/\mathscr{H}'$ is regular,*

(B) *if $a \in S$ is square-cancellable, then H'_a is right reversible,*

(C) *if $a, b, c \in S$, a is square-cancellable, $a\, \mathscr{R}'\, b\, \mathscr{R}'\, c$ and $ab = ac$, then $b = c$,*

(D) *if $a, b \in S$ are square-cancellable and $a\, \mathscr{R}'\, b$, then $ab\, \mathscr{H}'\, b$,*

(E) *if $a, b, c \in S$ and $a\, \mathscr{R}'\, abc$, then $a\, \mathscr{R}'\, ab$;*

(iii) *S satisfies conditions* (A), (B), (C) *and* (C)′; *moreover, any onto homomorphism $\varphi: S \to S/\mathscr{H}' = T$ with* $\ker \varphi = \mathscr{H}'$ *(equivalently, the natural homomorphism) is such that for any $a, b \in S$,*

$$a\, \mathscr{L}'\, b \text{ in } S \quad \text{if and only if} \quad \varphi(a)\, \mathscr{L}\, \varphi(b) \text{ in } T$$

and

$$a\, \mathscr{R}'\, b \text{ in } S \quad \text{if and only if} \quad \varphi(a)\, \mathscr{R}\, \varphi(b) \text{ in } T.$$

THEOREM 2.4 **[11]**. *The following conditions are equivalent for a semigroup S:*

(i) *S is a stratified left order in a regular $\mathscr{H}$-semigroup;*

(ii) *S satisfies conditions* (A), (B), (C), (D), (C)′ *and* (D)′;

(iii) *S satisfies conditions* (A), (B), (C) *and* (C)′; *moreover any onto homomorphism $\varphi: S \to S/\mathscr{H}^* = T$ with* $\ker \varphi = \mathscr{H}^*$ *is such that for any $a, b \in S$,*

$$a\, \mathscr{L}^*\, b \text{ in } S \quad \text{if and only if} \quad \varphi(a)\, \mathscr{L}\, \varphi(b) \text{ in } T$$

and

$$a\, \mathscr{R}^*\, b \text{ in } S \quad \text{if and only if} \quad \varphi(a)\, \mathscr{R}\, \varphi(b) \text{ in } T;$$

(iv) *S satisfies conditions* (A), (B), (F) *and the left-right dual* (F)′ *of* (F):

(F) *if $a, b, c \in S$ where $ab = ac$, a is square-cancellable, $b = b'u$, $c = c'v$ for some $b', c' \in S$ with $a\, \mathscr{R}^*\, b'\, \mathscr{R}^*\, c'$ and $u, v \in S^1$, then $b = c$.*

In this paper we do not concern ourselves with the question of the uniqueness of a semigroup of (left) quotients of a given semigroup. However we comment that in view of Proposition 2.2 and Theorem 3.1 of **[8]**, if S is a left order in regular $\mathcal{H}$-semigroups P and Q, then P is isomorphic to Q under an isomorphism whose restriction to S is the identity mapping if and only if $\mathcal{L}_P \cap (S \times S) = \mathcal{L}_Q \cap (S \times S)$ and $\mathcal{R}_P \cap (S \times S) = \mathcal{R}_Q \cap (S \times S)$; that is, P and Q induce the same suitable pair for S.

We end this section with four straightforward results we shall call upon later.

Proposition 2.5. **[11]**. *Let Q be a regular $\mathcal{H}$-semigroup and let S be a left order in Q. Then $\mathcal{H}'' = \mathcal{H}_Q \cap (S \times S)$ is a congruence on S and $S/\mathcal{H}'' \cong Q/\mathcal{H}$.*

Proposition 2.6. *Let S be a semigroup on which the equivalence relation $\mathcal{H}^*$ is a congruence. Let $T \cong S/\mathcal{H}^*$ and let $\varphi: S \to T$ be any onto homomorphism with* ker $\varphi = \mathcal{H}^*$. *Then for any $a, b \in S$, if $\varphi(a) \mathcal{L} \varphi(b)$ then $a \mathcal{L}^* b$ and if $\varphi(a) \mathcal{R} \varphi(b)$ then $a \mathcal{R}^* b$. Further T is $\mathcal{H}$-trivial.*

Proof. Suppose that $\varphi(a) \mathcal{L} \varphi(b)$, where $a, b \in S$. Then either $\varphi(a) = \varphi(b)$, in which case certainly $a \mathcal{L}^* b$, or $\varphi(c)\varphi(a) = \varphi(b)$, $\varphi(d)\varphi(b) = \varphi(a)$ for some $c, d \in S$. This gives that $ca \mathcal{H}^* b$ and $db \mathcal{H}^* a$. Let $x, y \in S^1$. Then

$$ax = ay \Rightarrow cax = cay \Rightarrow bx = by \Rightarrow dbx = dby \Rightarrow ax = ay,$$

so that $a \mathcal{L}^* b$. The proof for the relations $\mathcal{R}$ and $\mathcal{R}^*$ is dual. Now if $\varphi(a) \mathcal{H} \varphi(b)$ in T then $a \mathcal{H}^* b$ in S, so that $\varphi(a) = \varphi(b)$ and T is $\mathcal{H}$-trivial.

Corollary 2.7. *Let Q be a regular $\mathcal{H}$-semigroup. Let $T \cong Q/\mathcal{H}$ and let $\varphi: Q \to T$ be an onto homomorphism with* ker $\varphi = \mathcal{H}$. *Then for all $a, b \in Q$,*

$$a \mathcal{L} b \quad \text{if and only if} \quad \varphi(a) \mathcal{L} \varphi(b),$$
$$a \mathcal{R} b \quad \text{if and only if} \quad \varphi(a) \mathcal{R} \varphi(b),$$

and T is $\mathcal{H}$-trivial.

Proof. Since Q is regular, $\mathcal{L}^* = \mathcal{L}$, $\mathcal{R}^* = \mathcal{R}$ and $\mathcal{H}^* = \mathcal{H}$ on Q. By Proposition 2.6 it is only necessary to show that if $a, b \in Q$ and $a \mathcal{L} b\,(a \mathcal{R} b)$ then $\varphi(a) \mathcal{L} \varphi(b)(\varphi(a) \mathcal{R} \varphi(b))$. But this is so since the relations $\mathcal{L}$ and $\mathcal{R}$ are always preserved by homomorphism.

Proposition 2.8. *Let S be a semigroup with a zero, 0_S. If S is a left order in Q then 0_S is the zero of Q.*

Proof. Let $a \in S$ be square-cancellable, so that a^{-1} exists in Q and $aa^{-1} = a^{-1}a$ is idempotent. Then

$$0_S Q^1 = a0_S Q^1 \subseteq aQ^1 = a^{-1}aQ^1,$$

so that $0_S = a^{-1}a0_S = a^{-1}0_S$. Dually, one obtains that $0_S = 0_S aa^{-1} = 0_S a^{-1}$. Thus for any element $h^{-1}k \in Q$ where $h, k \in S$, we have that $0_S h^{-1}k = 0_S k = 0_S$ and $h^{-1}k0_S = h^{-1}0_S = 0_S$, as required.

3. Clifford semigroups. A *Clifford semigroup* is a regular semigroup with central idempotents. We recall from Theorem IV 2.1 of **[12]** that a semigroup is a Clifford semigroup if and only if it is a semilattice of groups; further, $\mathcal{H}$ is always a congruence on a Clifford semigroup.

It is shown in **[7]** that if a semigroup S is a left order in a Clifford semigroup Q, then S is not necessarily stratified in Q, but there is some Clifford semigroup Q' in which S is a stratified left order. Thus we concentrate on stratified left orders in Clifford semigroups. Certainly Clifford semigroups are regular $\mathscr{H}$-semigroups and so we may apply Theorem 2.4 to obtain the following.

COROLLARY 3.1 [7]. *A semigroup S is a stratified left order in a Clifford semigroup if and only if $\mathscr{H}^*$ is a semilattice congruence on S and the $\mathscr{H}^*$-classes of S are right reversible and cancellative.*

Proof. Suppose first that S is a stratified left order in a Clifford semigroup Q. Since $\mathscr{H}$ is a congruence on Q, we can apply Proposition 2.5 to obtain that $\mathscr{H}_S^* = \mathscr{H}_Q \cap (S \times S)$ is a congruence on S and $S/\mathscr{H}^* \cong Q/\mathscr{H}$, so that $S/\mathscr{H}^*$ is a semilattice. It follows that every element of S is square-cancellable. From Theorem 2.4, S satisfies conditions (B), (C) and (C)′, so that for any $a \in S$, H_a^* is right reversible and cancellative.

Conversely, suppose that $\mathscr{H}^*$ is a semilattice congruence on S and the $\mathscr{H}^*$-classes of S are right reversible and cancellative. Let $a, b \in S$ and suppose that $a \mathscr{R}^* b$ and $ax = ay$ for some $x, y \in S^1$. Then $xaxa = xaya$ and since $\mathscr{H}^*$ is a semilattice congruence, $xa \mathscr{H}^* ax = ay \mathscr{H}^* ya$. But H_{xa}^* is cancellative, which gives that $xa = ya$, and so, as $a \mathscr{R}^* b$, $xb = yb$. But then $bxbx = bybx$ and $bx \mathscr{H}^* xb = yb \mathscr{H}^* by$, so that $bx = by$. Dually, if $bx = by$ then $ax = ay$ and so $a \mathscr{L}^* b$. Similarly one obtains that if $a, b \in S$ and $a \mathscr{L}^* b$ then $a \mathscr{R}^* b$, so that $\mathscr{R}^* = \mathscr{L}^* = \mathscr{H}^*$ on S. That S satisfies conditions (A), (B), (C), (C)′, (D) and (D)′ is now immediate. From Theorem 2.4, S is a stratified left order in a regular $\mathscr{H}$-semigroup Q. Proposition 2.5 gives that $Q/\mathscr{H} \cong S/\mathscr{H}^*$, so that $Q/\mathscr{H}$ is a semilattice. Hence if $a \in Q$, $a \mathscr{H} a^2$, so that H_a is a subgroup and Q is a semilattice of groups, as required.

4. Simple inverse ω-semigroups. Munn proves in [13] that $\mathscr{H}$ is a congruence on any inverse ω-semigroup. Thus inverse ω-semigroups are regular $\mathscr{H}$-semigroups and so we can use Theorems 2.3 and 2.4 to characterize left orders in inverse ω-semigroups. We concentrate here on simple inverse ω-semigroups. A similar situation holds to that of the previous section—it is shown in [9] that if S is a left order in a simple inverse ω-semigroup with d $\mathscr{D}$-classes, then it is a stratified left order in some simple inverse ω-semigroup having no more than d $\mathscr{D}$-classes. In particular, any left order in a bisimple inverse ω-semigroup must be stratified. Thus it is sufficient to characterize stratified left orders in simple inverse ω-semigroups.

We recall that for $d \in \mathbb{N}$, B_d is the subsemigroup of the bicyclic semigroup B given by $B_d = \{(m, n) \in B : m \equiv n (\mathrm{mod}\, d)\}$. It is shown in [13] that B_d is the unique simple inverse ω-semigroup with d $\mathscr{D}$-classes and trivial $\mathscr{H}$-classes.

COROLLARY 4.1 [9]. *A semigroup S is a stratified left order in a simple inverse ω-semigroup which has d $\mathscr{D}$-classes if and only if S satisfies conditions* (A), (B), (C) *and* (C)′ *and $S/\mathscr{H}^* \cong B_d$.*

Proof. Suppose first that S is a stratified left order in a simple inverse ω-semigroup Q, where Q has d $\mathscr{D}$-classes. By Proposition 2.5, $\mathscr{H}^*$ is a congruence on S and $S/\mathscr{H}^* \cong Q/\mathscr{H} \cong B_d$. From Theorem 2.4, certainly S satisfies conditions (A), (B), (C) and (C)′.

Conversely, suppose that S satisfies conditions (A), (B), (C) and (C)′ and $S/\mathscr{H}^* \cong B_d$. We must show that, in addition, (D) and (D)′ hold for S. Let $\varphi : S \to B_d$ be an onto homomorphism with ker $\varphi = \mathcal{H}^*$. Note that $a \in S$ is square-cancellable if and only if $\varphi(a) = (n, n)$ for some $n \in \mathbb{N}$.

Let $a, b \in S$ be square-cancellable and suppose that $a \mathcal{R}^* b$. Further, suppose that $\bar{x}, \bar{y} \in S^1$ and $a\bar{x} = a\bar{y}$. We have $\varphi(a) = (n, n)$ and $\varphi(b) = (m, m)$ for some $n, m \in \mathbb{N}$ and we may certainly choose $c \in S$ with $\varphi(c) = (0, 0)$ so that $ax = ay$ where $x = \bar{x}c$, $y = \bar{y}c \in S$. Denote $\varphi(x)$ by (p, q) and $\varphi(y)$ by (h, k). From $ax = ay$ we obtain that $\max\{n, p\} = \max\{n, h\}$ and $q - p = k - h$. Without loss of generality suppose that $q \geq k$. Pick $z \in S$ with $\varphi(z) = (k, h)$; this is possible since $h \equiv k (\text{mod}\, d)$. Then $\varphi(yz) = (h, h)$, $\varphi(xz) = (p, p)$ and $axz = ayz$. Now

$$\varphi(xza) = \varphi(axz) = \varphi(ayz) = \varphi(yza),$$

so from $xzaxza = xzayza$ and condition (C) we have that $xza = yza$. But $a \mathcal{R}^* b$, so that $xzb = yzb$. It follows that $\max\{p, m\} = \max\{h, m\}$ and $\varphi(bx) = \varphi(by)$. From $bxzbxz = byzbxz$ and (C)′ we have that $bxz = byz$, and so $bxzy = byzy$ and $\varphi(zy) = (k, k)$. Choose $w \in S$ with $\varphi(w) = (k - h + \max\{m, h\},\ k - h + \max\{m, h\})$ so that $\varphi(zyw) = \varphi(w)$ and $bx(zyw) = by(zyw)$. But $\varphi(bx) = \varphi(by) = (\max\{m, h\}, k - h + \max\{m, h\}) \mathcal{L} \varphi(zyw)$. From Proposition 2.6, $bx \mathcal{H}^* by \mathcal{L}^* zyw$ in S, and as zyw is square-cancellable, (C)′ gives that $bx = by$. Thus $b\bar{x}c = b\bar{y}c$ and one obtains that $\varphi(b\bar{x}) = \varphi(b\bar{y}) = (u, v)$ for some $u, v \in \mathbb{N}$ with $u \equiv v (\text{mod}\, d)$. Pick $d \in S$ with $\varphi(d) = (v, v)$. Then $b\bar{x}(cd) = b\bar{y}(cd)$ and $\varphi(b\bar{x}) = \varphi(b\bar{y})$, $\varphi(cd) = \varphi(d)$ so that in S, $b\bar{x} \mathcal{H}^* b\bar{y} \mathcal{L}^* cd$ and cd is square-cancellable. Another application of (C)′ gives that $b\bar{x} = b\bar{y}$. It follows that $a \mathcal{L}^* b$ and so $a \mathcal{H}^* b$. But H_a^* is a subsemigroup so that certainly $ab \mathcal{H}^* b$ and (D) holds. Dually, (D)′ holds for S. By Theorem 2.4, S is a stratified left order in a regular semigroup Q on which $\mathcal{H}$ is a congruence. But Proposition 2.5 gives that $Q/\mathcal{H} \cong S/\mathcal{H}^* \cong B_d$. It is then easy to deduce from Corollary 2.7 that Q is a simple inverse ω-semigroup with d $\mathcal{D}$-classes.

5. Completely 0-simple semigroups. Orders in completely 0-simple semigroups are characterized in **[4]**. The authors of that paper concentrate on the two-sided case, for, as they show, a semigroup can have non-isomorphic completely 0-simple semigroups of left quotients, whereas it is an order in at most one completely 0-simple semigroup. However, in view of Theorem 2.3 it is straightforward to deduce a characterization of left orders in completely 0-simple semigroups, from which the description given in **[4]** of two-sided orders follows.

It is useful at this point to recall some definitions given for any semigroup S with 0. The set of non-zero elements of S is denoted by S^*. One says that 0 *is a prime ideal* of S, or S is *prime*, if for any $a, b \in S^*$, $aSb \neq 0$. The semigroup S is *categorical at zero* if for any $a, b, c \in S$, if $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$. The relations ρ, λ on S are defined by the rule that for any $a, b \in S$,

$$a \lambda b \quad \text{if and only if} \quad a = b = 0 \quad \text{or} \quad Sa \cap Sb \neq 0$$

and

$$a \rho b \quad \text{if and only if} \quad a = b = 0 \quad \text{or} \quad aS \cap bS \neq 0.$$

If $a \in S$ then $l(a) = \{s \in S : sa = 0\}$ is a left ideal of S, the left *annihilator ideal* of a. The right annihilator ideal $r(a)$ of a is defined dually.

COROLLARY 5.1. *Let S be a semigroup with 0 and let $(\mathcal{L}', \mathcal{R}')$ be a suitable pair for S. Then the following are equivalent:*

(i) *S is a left order in a completely 0-simple semigroup Q such that $\mathcal{L}_Q \cap (S \times S) = \mathcal{L}'$ and $\mathcal{R}_Q \cap (S \times S) = \mathcal{R}'$;*

(ii) *S satisfies conditions* (A), (B), (C), (C)′, (E) *and* (E)′ *and $S/\mathcal{H}'$ is a rectangular 0-band;*

(iii) *S satisfies conditions* (G), (H), (I), (J) *and the left-right dual* (I)′ *of* (I):

(G) *0 is a prime ideal of S,*

(H) *$\lambda = \mathcal{L}'$,*

(I) *for any $a, b, c \in S$, if $a\,\mathcal{R}'\,b$ and $ca = cb \neq 0$, then $a = b$,*

(J) *if $a, b \in S$ and $ab \neq 0$, then $ab\,\mathcal{R}'\,a$.*

Proof. (ii)$\Rightarrow$(i). Let a, b be square-cancellable elements of S and suppose that $a\,\mathcal{R}'\,b$. Since $\mathcal{R}' \subseteq \mathcal{R}^*$, we have that $a = b = 0$, and so $ab\,\mathcal{H}'\,b$ or $a, b \in S^*$. In the latter case we use the fact that $\mathcal{R}'$ is a left congruence to obtain $ab\,\mathcal{R}'\,a^2\,\mathcal{H}'\,a\,\mathcal{R}'\,b$ so that $ab\,\mathcal{R}'\,b$; further, $bab\,\mathcal{R}'\,b^2\,\mathcal{H}'\,b$ and so $bab \neq 0$. It follows from the fact that $S/\mathcal{H}'$ is a rectangular 0-band that $bab\,\mathcal{H}'\,b$ and so by condition (E)′, $b\,\mathcal{L}'\,ab$, giving that $b\,\mathcal{H}'\,ab$. Thus (D), and dually (D)′, hold for S.

By Theorem 2.3, S is a left order in a regular semigroup Q such that $\mathcal{H}$ is a congruence on Q and $\mathcal{L}_Q \cap (S \times S) = \mathcal{L}'$, $\mathcal{R}_Q \cap (S \times S) = \mathcal{R}'$. Thus $\mathcal{H}' = \mathcal{H}_Q \cap (S \times S)$ and, by Proposition 2.5, $Q/\mathcal{H} \cong S/\mathcal{H}'$, so that $Q/\mathcal{H}$ is a rectangular 0-band.

In view of Proposition 2.8, Q is a semigroup with a 0. Let $\nu : Q \to Q/\mathcal{H}$ be the natural homomorphism. If $a, b \in Q^*$ then $\nu(a), \nu(b) \in (Q/\mathcal{H})^*$ so that $\nu(a)\nu(c)\nu(b) \neq 0$ for some $c \in Q^*$, which gives that $acb \neq 0$ and Q is prime. Further, if e and f are non-zero idempotents of Q and $e \leq f$, so that $e = ef = fe$, then $\nu(e)$ and $\nu(f)$ are non-zero idempotents in $Q/\mathcal{H}$ with $\nu(e) \leq \nu(f)$. But $Q/\mathcal{H}$ is primitive; thus $\nu(e) = \nu(f)$ and $e\mathcal{H}f$. Hence $e = f$ and Q is primitive. By Theorem III.3.5 of **[12]**, Q is a completely 0-simple semigroup.

(i)$\Rightarrow$(iii). Since S intersects every $\mathcal{H}$-class of Q it is easy to see that S is prime and categorical at zero. Let $a, b \in S$ and suppose that $a\,\lambda\,b$. If $a = b = 0$ then certainly $a\,\mathcal{L}'\,b$; otherwise, $Sa \cap Sb \neq 0$. In this case, $a\,\mathcal{L}\,b$ in Q and so by assumption $a\,\mathcal{L}'\,b$ in S. Thus $\lambda \subseteq \mathcal{L}'$. Conversely, if $a, b \in S$ and $a\,\mathcal{L}'\,b$, then $a\,\mathcal{L}\,b$ in Q so that $a = h^{-1}kb$, $b = u^{-1}va$ for some $h, k, u, v \in S$ with $h\,\mathcal{R}'\,k$, $u\,\mathcal{R}'\,v$ in S. Thus either $a = b = 0$ or $a, b \in S^*$ and $ha = kb \neq 0$ so that $a\,\lambda\,b$.

To see that condition (I) holds, let $a, b, c \in S$ where $a\,\mathcal{R}'\,b$ and $ca = cb \neq 0$. In Q, $a\,\mathcal{R}\,b$, and from $ca = cb \neq 0$ we also have that $a\,\mathcal{L}\,b$. It is then easy to see from the expression of Q as a Rees matrix semigroup that $a = b$.

It remains to show that (J) holds. Let $a, b \in S$ and suppose that $ab \neq 0$. Then in Q, $ab\,\mathcal{R}\,a$ so that $ab\,\mathcal{R}'\,a$ in S.

(iii)$\Rightarrow$(ii). We note first that the dual (J)′ of (J) is true. For if $ba \neq 0$ then, as S is prime, $tba \neq 0$ for some $t \in S$. Thus $a\,\lambda\,ba$ and by (H), $a\,\mathcal{L}'\,ba$ as required. We aim to show that $\mathcal{H}'$ is a congruence on S. Since $\mathcal{H}' \subseteq \mathcal{H}^*$, $\mathcal{H}'$ is 0-restricted, that is 0 is an $\mathcal{H}'$-class. Thus it is enough to show that if $a, b \in S^*$, $c \in S$ and $a\,\mathcal{H}'\,b$, then $ca\,\mathcal{H}'\,cb$ and $ac\,\mathcal{H}'\,bc$. So let a, b, c be as given. Then $ca\,\mathcal{R}'\,cb$ so that as $\mathcal{R}' \subseteq \mathcal{R}^*$ we have $ca = cb = 0$

or $ca, cb \in S^*$. Assume the latter is true. By the remark above, $a\,\mathscr{L}'\,ca$ and $b\,\mathscr{L}'\,cb$. This gives that $ca\,\mathscr{L}'\,a\,\mathscr{L}'\,b\,\mathscr{L}'\,cb$ and so $ca\,\mathscr{H}'\,cb$. Similarly, using the fact that $\mathscr{L}'$ is a right congruence contained in $\mathscr{L}^*$ we obtain that $ac\,\mathscr{L}'\,bc$ and so $ac = bc = 0$ or $ac, bc \in S^*$. But in the latter case, condition (J) gives that $ac\,\mathscr{R}'\,a\,\mathscr{R}'\,b\,\mathscr{R}'\,bc$ and so $ac\,\mathscr{H}'\,bc$ as required. Thus $\mathscr{H}'$ is a (0-restricted) congruence on S.

Let $\nu: S \to S/\mathscr{H}'$ be the natural homomorphism. Since S is prime, it is clear that so also is $S/\mathscr{H}'$. Suppose that $\nu(a) \neq 0$; now $aba \neq 0$ for some $b \in S^*$ and, as above, $a\,\mathscr{L}'\,aba\,\mathscr{R}'\,a$, so that $a\,\mathscr{H}'\,aba$ and $S/\mathscr{H}'$ is regular. If $\nu(a)$ and $\nu(b)$ are non-zero idempotents of $S/\mathscr{H}'$ with $\nu(a) \leq \nu(b)$, then $a\,\mathscr{H}'\,ab\,\mathscr{H}'\,ba$ so that $a\,\mathscr{H}'\,b$ and $\nu(a) = \nu(b)$. Again by Theorem III.3.5 of **[12]**, $S/\mathscr{H}'$ is completely 0-simple. Suppose that $\nu(a)\,\mathscr{H}\,\nu(b)$ where $a, b \in S^*$. Then there are elements c, d, u, v in S with $a\,\mathscr{H}'\,bc$, $b\,\mathscr{H}'\,ad$, $a\,\mathscr{H}'\,ub$ and $b\,\mathscr{H}'\,va$. Thus $a\,\mathscr{R}'\,bc\,\mathscr{R}'\,b$ and $a\,\mathscr{L}'\,ub\,\mathscr{L}'\,b$, so that $a\,\mathscr{H}'\,b$ and $\nu(a) = \nu(b)$. It follows that $S/\mathscr{H}'$ is a rectangular 0-band.

Since $H'_0 = \{0\}$, it is clear that H'_0 is right reversible. Suppose that $a \in S^*$ is square-cancellable. Then H'_a is a subsemigroup, so that if $b, c \in H'_a$ then so are ab and ac. But $ab\,\mathscr{L}'\,ac$ so that $pab = qac \neq 0$ for some $p, q \in S$ and, as S is prime, $arpab = arqac \neq 0$ for some $r \in S$. But $a\,\mathscr{H}'\,arpa\,\mathscr{H}'\,arqa$ so that H'_a is right reversible and (B) holds.

To see that Condition (C) is satisfied, let $a, b, c \in S$ where a is square-cancellable, $a\,\mathscr{R}'\,b\,\mathscr{R}'\,c$ and $ab = ac$. We may suppose that $a, b, c \in S^*$. Then $ab\,\mathscr{R}'\,a^2\,\mathscr{H}'\,a$ so that $ab \neq 0$. Directly from (I) we now have that $a = b$. Dually, (C)′ follows from (I)′.

Let $a, b, c \in S$ where $a\,\mathscr{R}'\,abc$. If $ab \neq 0$ then $ab\,\mathscr{R}'\,a$ by (J); if $ab = 0$ then, as $a\,\mathscr{R}'\,abc = 0$, $a = 0$ so that $a\,\mathscr{R}'\,ab$. Thus (E) and dually (E)′ are true of S.

We recall from **[1]** that a completely 0-simple semigroup Q is *absolutely flat* if for any elements a, b of Q,

$$a\,\mathscr{L}\,b \quad \text{if and if} \quad r(a) = r(b)$$

and

$$a\,\mathscr{R}\,b \quad \text{if and only if} \quad l(a) = l(b).$$

Corollary 5.2 **[10]**. *Let S be a semigroup with 0. Then S is a left order in an absolutely flat completely 0-simple semigroup if and only if S satisfies the following conditions:*

(i) *0 is a prime ideal of S;*

(ii) *for any $a, b \in S$,*

$$a\,\mathscr{R}^*\,b \quad \textit{if and only if} \quad l(a) = l(b)$$

and

$$a\,\lambda\,b \textit{ if and only if } r(a) = r(b);$$

(iii) *for any $a, b, c \in S$ where $a\,\mathscr{R}^*\,b$, $ca = cb \neq 0$ implies that $a = b$;*

(iii)′ *left-right dual of* (iii).

Proof. Suppose first that S is a left order in Q, where Q is an absolutely flat completely 0-simple semigroup. Let $a, b \in S$ and suppose that $a\,\mathscr{R}^*\,b$. Certainly $l_S(a) = l_S(b)$ where $l_S(a)(l_S(b))$ is the left annihilator ideal of a (b) in S. Let $q \in Q$ and suppose that $qa = 0$. Now $q\,\mathscr{H}\,c$ for some $c \in S$ so that $0 = qa\,\mathscr{H}\,ca$. Thus $ca = 0$ and, as $l_S(a) = l_S(b)$, we have that $cb = 0$. Then $0 = cb\,\mathscr{H}\,qb$, giving that $qb = 0$. It follows that with the obvious notation, $l_Q(a) = l_Q(b)$ and as by assumption Q is absolutely flat, we have that $a\,\mathscr{R}\,b$ in Q.

A dual argument shows that if $a, b \in S$ and $a\,\mathcal{L}^*\,b$, then $a\,\mathcal{L}\,b$ in Q. Thus S is a stratified left order in Q. By Corollary 5.1, (i), (iii) and (iii)$'$ hold for S. Further, $\lambda = \mathcal{L}^*$.

If $a, b \in S$ and $l_S(a) = l_S(b)$ then, as shown above, $a\,\mathcal{R}\,b$ in Q so that $a\,\mathcal{R}^*\,b$ in S. Thus $a\,\mathcal{R}^*\,b$ if and only if $l_S(a) = l_S(b)$ and dually, $c\,\mathcal{L}^*\,d$ if and only if $r_S(c) = r_S(d)$, for any $c, d \in S$. Since $\lambda = \mathcal{L}^*$ this gives that (ii) holds.

Conversely, suppose that S satisfies conditions (i), (ii), (iii) and (iii)$'$. If $a, b, c \in S$ and $ab, bc \in S^*$ then from (i) we have that $ab\,\lambda\,b$, so that $r(ab) = r(b)$, using (ii). Thus $abc \neq 0$, since $c \notin r(b)$. Hence S is categorical at 0. Thus if $a, b \in S$ and $ab \neq 0$, $l(ab) = l(a)$ and so by (ii) we have that $ab\,\mathcal{R}^*\,a$.

One needs to show that $\lambda = \mathcal{L}^*$. If $a, b \in S$ and $a\,\mathcal{L}^*\,b$ then $r(a) = r(b)$, so by (ii), $a\,\lambda\,b$. Conversely, if $a\,\lambda\,b$ then either $a = b = 0$, so that $a\,\mathcal{L}^*\,b$, or $ca = db \neq 0$ for some $c, d \in S$. Suppose the latter is true and $ax = ay$ for some $x, y \in S^1$. As S is categorical at zero we have that $r(a) = r(ca) = r(db) = r(b)$ and so $ax = ay = 0$ if and only if $bx = by = 0$. If $ax = ay \neq 0$ then $bx, by \in S^*$ and $dbx = dby \neq 0$ by categoricity at zero. But then $b\,\mathcal{R}^*\,bx\,\mathcal{R}^*\,by$ so that (iii) gives $bx = by$. It follows that $a\,\mathcal{L}^*\,b$ and so $\lambda = \mathcal{L}^*$.

We may now apply Corollary 5.1 to obtain that S is a stratified left order in a completely 0-simple semigroup Q. Let $p, q \in Q$, where $p\,\mathcal{H}_Q\,a$ and $q\,\mathcal{H}_Q\,b$ for some $a, b \in S$. Suppose that $l_Q(p) = l_Q(q)$; certainly then $l_Q(a) = l_Q(b)$ and $l_S(a) = l_S(b)$. Thus $a\,\mathcal{R}^*\,b$ in S and, as S is stratified in Q, $p\,\mathcal{H}_Q\,a\,\mathcal{R}_Q\,b\,\mathcal{H}_Q\,q$. Dually, if $r_Q(p) = r_Q(q)$ then $r_S(a) = r_S(b)$, so that $a\,\lambda\,b$, and hence $a\,\mathcal{L}^*\,b$ in S. Thus $p\,\mathcal{H}_Q\,a\,\mathcal{L}_Q\,b\,\mathcal{H}_Q\,q$ and Q is by definition absolutely flat.

It is shown in **[1]** that inverse semigroups are absolutely flat. Certainly then Brandt semigroups, that is, inverse completely 0-simple semigroups, are absolutely flat. On the other hand it is easy to see directly from the representation of a Brandt semigroup Q as a Rees matrix semigroup $B(G, I) = \mathcal{M}^0(I, G, I; P)$, where P is the $I \times I$ identity matrix, that for any $a, b \in Q$, $a\,\mathcal{L}\,b$ $(a\,\mathcal{R}\,b)$ if and only if $r(a) = r(b)(l(a) = l(b))$.

Left orders in Brandt semigroups were first characterized in **[5]**; some alternative descriptions are given in **[10]**. As noted in Corollary 5.2, any left order in an absolutely flat completely 0-simple semigroup is stratified, thus is any left order in a Brandt semigroup. Rather than specialising Corollary 5.2 to the case of Brandt semigroups we deduce the following result from Theorem 2.4.

First we recall that a semigroup S with 0 is *0-cancellative* if $xa = xb \neq 0 (ax = bx \neq 0)$ implies that $a = b$, for all $x, a, b \in S$.

Corollary 5.3 **[5]**. *Let S be a semigroup with 0. Then S is a left order in a Brandt semigroup if and only if S is 0-cancellative, the relation $\mathcal{H}^*$ is a congruence on S, $S/\mathcal{H}^*$ is a Brandt semigroup and the $\mathcal{H}^*$-class of any square-cancellable element of S is right reversible.*

Proof. If S is a left order in a Brandt semigroup $Q = B(G, I)$, then since S must be stratified in Q, Theorem 2.4 gives that $\mathcal{H}^*$ is a congruence on S and the $\mathcal{H}^*$-class of any square-cancellable element of S is right reversible. But from Proposition 2.5, $S/\mathcal{H}^* \cong Q/\mathcal{H}$, so that $S/\mathcal{H}^*$ is a Brandt semigroup. It is easy to check directly that $B(G, I)$ is 0-cancellative; certainly then so is S.

Conversely we suppose that $\mathcal{H}^*$ is a congruence on S, $S/\mathcal{H}^*$ is a Brandt semigroup, the $\mathcal{H}^*$-class of any square-cancellable element of S is right reversible and S is

0-cancellative. The latter property gives immediately that (C) and (C)′ hold for S. To see that (D) holds, suppose that $a, b \in S$ are non-zero square-cancellable elements where $a\,\mathcal{R}^*\,b$. Certainly $ab\,\mathcal{R}^*\,b$, so that $ab \neq 0$. But it follows from the fact that 0 is an $\mathcal{H}^*$-class and $S/\mathcal{H}^*$ is categorical at 0 that S is categorical at 0: thus $r(ab) = r(b)$. Let $x, y \in S^1$. Then $abx = aby = 0$ if and only if $bx = by = 0$. If $abx = aby \neq 0$ then $bx = by$ by 0-cancellation. Hence $ab\,\mathcal{H}^*\,b$ and (D) holds; dually, so does (D)′.

By Theorem 2.4, S is a stratified left order in a regular $\mathcal{H}$-semigroup Q. But it is easy to deduce from Proposition 2.5 and Corollary 2.7 that Q is a Brandt semigroup.

We now turn our attention to two-sided orders in completely 0-simple semigroups.

COROLLARY 5.4 [4]. *Let S be a semigroup with* 0. *Then S is an order in a completely* 0-*simple semigroup if and only if S satisfies conditions* (i), (ii), (iii), (iv) *and the left-right duals* (ii)′ *and* (iii)′ *of* (ii) *and* (iii):

(i) 0 *is a prime ideal of S;*
(ii) *if $a\,\rho\,b$, $au \neq 0$ and $bv \neq 0$, then $au\,\rho\,bv$;*
(iii) *if $a\,\rho\,b$ and $ca = cb \neq 0$, then $a = b$;*
(iv) *S is categorical at* 0.

Proof. Let S be an order in a completely 0-simple semigroup Q, that is, S is both a left order and a right order in Q. By Corollary 5.1 and its dual, 0 is a prime ideal of S, $\lambda = \mathcal{L}_Q \cap (S \times S)$ and $\rho = \mathcal{R}_Q \cap (S \times S)$. It is then immediate that (iii) and (iii)′ hold. Further, if $a\,\rho\,b$, $au \neq 0$ and $bv \neq 0$, then in Q, $au\,\mathcal{R}\,a\,\mathcal{R}\,b\,\mathcal{R}\,bv$, so that $au\,\rho\,bv$ in S. Dually, (ii)′ holds. Finally, it is clear that S is categorical at 0, since Q has this property.

Conversely, suppose that S satisfies conditions (i), (ii), (iii), (iv), (ii)′ and (iii)′. Let $a, b \in S$ and suppose that $a\,\rho\,b$. Thus $a = b = 0$ and $a\,\mathcal{R}^*\,b$, or $ac = bd \neq 0$ for some $c, d \in S$. Assume the latter is true and $xa = ya$ for some $x, y \in S^1$. Since S is categorical at 0, $l(a) = l(ac) = l(bd) = l(b)$ so that $xa = ya = 0$ if and only if $xb = yb = 0$. Suppose therefore that $xa = ya \neq 0$. Then $xbd = ybd \neq 0$, by categoricity at zero. Further, using (ii)′ and the fact that 0 is a prime ideal of S, $xb\,\lambda\,yb$ so that (iii)′ gives $xb = yb$. Thus $a\,\mathcal{R}^*\,b$ and we have shown that $\rho \subseteq \mathcal{R}^*$.

Since S is prime it is clear that the relation ρ is reflexive and symmetric. Further, if $a\,\rho\,b\,\rho\,c$ then either $a = b = c = 0$ or $au = bv \neq 0$, $bh = ck \neq 0$ for some $u, v, h, k \in S$. In the latter case one has by (ii) that $bv\,\rho\,bh$, so that $au\,\rho\,ck$. Thus $aus = ckt \neq 0$ for some $s, t \in S$ and so $a\,\rho\,c$ and ρ is an equivalence relation. To see that ρ is a left congruence, suppose that $a\,\rho\,b$ and $c \in S$. If $a = b = 0$ certainly $ca\,\rho\,cb$. If $a, b \in S^*$ then $ah = bk \neq 0$ for some $h, k \in S$ and it follows from (iv) that $ca = 0$ if and only if $cb = 0$. Suppose therefore that $ca, cb \in S^*$. Then again by (iv), $cah = cbk \neq 0$, so that $ca\,\rho\,cb$ and ρ is a left congruence.

The dual argument gives that λ is a right congruence contained in $\mathcal{L}^*$. To complete the proof that (λ, ρ) is a suitable pair, we must show that if $a\,\mathcal{H}^*\,a^2$ then $a\,\tau\,a^2$, where $\tau = \lambda \cap \rho$. If $a \in S$ and $a\,\mathcal{H}^*\,a^2$, then either $a = 0$, in which case $a\,\tau\,a^2$, or $a, a^2 \in S^*$. But then, as S is prime, $a\,\lambda\,a^2$ and $a\,\rho\,a^2$, so that $a\,\tau\,a^2$.

We may now apply Corollary 5.1 and its dual to obtain that S is a left (right) order in a completely 0-simple semigroup $P(Q)$ such that $\mathcal{L}_P \cap (S \times S) = \mathcal{L}_Q \cap (S \times S) = \lambda$ and $\mathcal{R}_Q \cap (S \times S) = \mathcal{R}_Q \cap (S \times S) = \rho$. By Proposition 2.2, S is straight in both P and Q. Then Proposition 2.5 of **[11]** says that P and Q are isomorphic semigroups of quotients of S.

6. Bands of groups. A semigroup is *completely regular* if it is a union of groups. We note that the relation $\mathscr{H}$ is not a congruence on an arbitrary completely regular semigroup. If Q is a completely regular semigroup and $\mathscr{H}$ is a congruence on Q, then clearly $Q/\mathscr{H}$ is a band. Conversely, if $\mathscr{H}$ is a band congruence on a semigroup Q, then given any $a \in Q$, $a\,\mathscr{H}\,a^2$, so that Q is completely regular. Thus the class of completely regular semigroups on which $\mathscr{H}$ is a congruence, which we denote $\mathscr{CRH}$, is the class of semigroups on which $\mathscr{H}$ is a band congruence. If $Q \in \mathscr{CRH}$ then certainly Q is a band of groups. On the other hand, suppose that a semigroup Q is a band of groups, that is, there is a band congruence θ on Q all of whose classes are groups. Let $\nu: Q \to Q/\theta$ be the natural homomorphism. Then if $a, b \in Q$ and $a\,\mathscr{H}\,b$, we have that $\nu(a)\,\mathscr{H}\,\nu(b)$ in Q/θ; but Q/θ is a band so that $\nu(a) = \nu(b)$ and $a\,\theta\,b$. Since it is clear that $\theta \subseteq \mathscr{H}$ it follows that $\theta = \mathscr{H}$, so that $\mathscr{H}$ is a band congruence on Q and $Q \in \mathscr{CRH}$.

Before considering left orders and orders in semigroups in $\mathscr{CRH}$ we prove the following technical lemma.

LEMMA 6.1. *Let S be a semigroup on which there is a band congruence θ all of whose classes are right reversible, cancellative semigroups. If $\varphi: S \to S/\theta$ is an onto homomorphism with kernel θ, then given $a, b \in S$, if $\varphi(a)\,\mathscr{L}\,\varphi(b)$ in S/θ, then $a\,\mathscr{L}^*\,b$.*

Proof. Let S, θ and φ be as given and denote S/θ by T. Suppose that $a, b \in S$ and $\varphi(a)\,\mathscr{L}\,\varphi(b)$ in T, and we are given $x, y \in S^1$ with $ax = ay$. Thus $\varphi(a)\varphi(x) = \varphi(a)\varphi(y)$ (where $\varphi(1) = 1$) and, as $\varphi(a)\,\mathscr{L}\,\varphi(b)$ in T, certainly $\varphi(a)\,\mathscr{L}^*\,\varphi(b)$ in T, so that $\varphi(b)\varphi(x) = \varphi(b)\varphi(y)$ and $bx\,\theta\,by$. Further, since T is a band, $\varphi(ba) = \varphi(b)$, so that $ba\,\theta\,b$ and, as $[b]_\theta$ is right reversible, there are elements $p, q \in [b]_\theta$ with $pba = qb$. This gives that $qbx = pbax = pbay = qby$ and so $bxqbxq = bxqbyq$. But $bxq\,\theta\,byq$, so we can cancel to obtain $bxq = byq$. Then $bxqx = byqx$ and $bx\,\theta\,by\,\theta\,qx$ so that $bx = by$. It follows that $a\,\mathscr{L}^*\,b$.

COROLLARY 6.2. *A semigroup S is a left order in a band B of groups if and only if S is a band B of right reversible, cancellative semigroups and, if $\varphi: S \to B$ is the natural homomorphism and $a, b \in S$, then $\varphi(a)\,\mathscr{R}\,\varphi(b)$ in B implies that $a\,\mathscr{R}^*\,b$ in S.*

Proof. Let S be a left order in Q, where Q is a band B of groups, that is, $Q/\mathscr{H} \cong B$. Put $\mathscr{L}' = \mathscr{L}_Q \cap (S \times S)$ and $\mathscr{R}' = \mathscr{R}_Q \cap (S \times S)$, so that $(\mathscr{L}', \mathscr{R}')$ is a suitable pair for S. By Theorem 2.3, $\mathscr{H}' = \mathscr{L}' \cap \mathscr{R}' = \mathscr{H}_Q \cap (S \times S)$ is a congruence on S and $S/\mathscr{H}'$ is regular; indeed, by Proposition 2.5, $S/\mathscr{H}' \cong Q/\mathscr{H} \cong B$. Since S satisfies conditions (B), (C) and (C)′, the $\mathscr{H}'$-class of any square-cancellable element is a right reversible, cancellative semigroup. But $S/\mathscr{H}'$ is a band, so every element of S is $\mathscr{H}'$-related to its square and so is square-cancellable. Thus S is a band B of right reversible, cancellative semigroups.

Let $\varphi: S \to B$ be the natural homomorphism and suppose that $\varphi(a)\,\mathscr{R}\,\varphi(b)$ in B where $a, b \in S$. Then there are elements c, d in S with $ac\,\mathscr{H}'\,b$ and $bd\,\mathscr{H}'\,a$. Thus $ac\,\mathscr{H}\,b$ and $bd\,\mathscr{H}\,a$ in Q, which gives that $a\,\mathscr{R}\,b$ in Q and so $a\,\mathscr{R}^*\,b$ in S.

Conversely, suppose that S is a band B of right reversible, cancellative semigroups and the natural homomorphism $\varphi: S \to B$ has the given property. We define relations $\mathscr{L}'$ and $\mathscr{R}'$ on S by

$$a\,\mathscr{L}'\,b \quad \text{if and only if} \quad \varphi(a)\,\mathscr{L}\,\varphi(b)$$

and

$$a\,\mathscr{R}'\,b \quad \text{if and only if} \quad \varphi(a)\,\mathscr{R}\,\varphi(b),$$

where $a, b \in S$. It is easy to see that $\mathscr{L}'$ is a right congruence and $\mathscr{R}'$ is a left congruence. We show that $(\mathscr{L}', \mathscr{R}')$ is a suitable pair for S.

If $a, b \in S$ and $a\,\mathscr{L}'\,b$ then $\varphi(a)\,\mathscr{L}\,\varphi(b)$ in B, so, by Lemma 6.1, $a\,\mathscr{L}^*\,b$ in S. On the other hand, if $a\,\mathscr{R}'\,b$ then $\varphi(a)\,\mathscr{R}\,\varphi(b)$ in B, so by assumption we have that $a\,\mathscr{R}^*\,b$ in S.

Since $\mathscr{H}$ is the trivial relation on B it is immediate that $\mathscr{H}' \equiv \mathscr{L}' \cap \mathscr{R}' = \ker \varphi$. Clearly $a\,\mathscr{H}'\,a^2$ for all $a \in S$, for $\ker \varphi$ is a band congruence. Suppose now that $a \in S$ and $a^2x = a^2y$ for some $x, y \in S^1$. Thus $ax\,\mathscr{H}'\,a^2x = a^2y\,\mathscr{H}'\,ay$ and so $axa\,\mathscr{H}'\,aya$. But $axaaxa = axaaya$; cancelling in H'_{axa} (the $\mathscr{H}'$-class of axa) yields $axa = aya$. But then $axax = ayax$, so one can cancel in H'_{ax} to obtain $ax = ay$. Together with its dual, this argument shows that every element of S is square-cancellable. Thus $(\mathscr{L}', \mathscr{R}')$ is a suitable pair for S.

Clearly S satisfies conditions (A) and (B) with respect to the suitable pair $(\mathscr{L}', \mathscr{R}')$. Let $a, b, c \in S$ where $a\,\mathscr{R}'\,b\,\mathscr{R}'\,c$ and suppose that $ab = ac$. In B, $\varphi(a)\,\mathscr{R}\,\varphi(b)\,\mathscr{R}\,\varphi(c)$, so that $b\,\mathscr{H}'\,ab = ac\,\mathscr{H}'\,c$. Now $baba = baca$ and $ba\,\mathscr{H}'\,ca$, so that $ba = ca$. But then $bab = cab$ and so cancelling in H'_b gives that $b = c$. Thus (C) and dually (C)′ hold for S. It is straightforward to show from the definition of $\mathscr{L}'$ and $\mathscr{R}'$ that conditions (D) and (E) and their duals hold.

We can now apply Theorem 2.3 to obtain that S is a left order in a regular $\mathscr{H}$-semigroup Q such that $\mathscr{L}_Q \cap (S \times S) = \mathscr{L}'$ and $\mathscr{R}_Q \cap (S \times S) = \mathscr{R}'$. Then $\mathscr{H}' = \mathscr{H}_Q \cap (S \times S)$ so that by Proposition 2.5, $Q/\mathscr{H} \cong S/\mathscr{H}' = S/\ker \varphi \cong B$. In view of the comments at the beginning of this section, Q is a band B of groups.

Orders in bands of groups have an even simpler characterization.

Corollary 6.3. *A semigroup S is an order in a band B of groups if and only if S is a band B of reversible, cancellative semigroups.*

Proof. If S is an order in a band B of groups then one obtains exactly as in Corollary 6.2 that S is a band B of reversible, cancellative semigroups.

Conversely, suppose that S is a band B of reversible, cancellative semigroups and let $\varphi: S \to B$ be the natural homomorphism. Using Lemma 6.1, Corollary 6.2 and their left-right duals, we obtain that S is a left (right) order in bands B of groups P (Q) such that $\mathscr{L}_P \cap (S \times S) = \mathscr{L}_Q \cap (S \times S) = \mathscr{L}'$ and $\mathscr{R}_P \cap (S \times S) = \mathscr{R}_Q \cap (S \times S) = \mathscr{R}'$, where for any $a, b \in S$,

$$a\,\mathscr{L}'\,b \quad \text{if and only if} \quad \varphi(a)\,\mathscr{L}\,\varphi(b)$$

and

$$a\,\mathscr{R}'\,b \quad \text{if and only if} \quad \varphi(a)\,\mathscr{R}\,\varphi(b).$$

By Proposition 2.5 of **[11]**, P and Q are isomorphic semigroups of quotients of S.

Orders in normal bands of groups are considered in **[6]**, where the authors make heavy use of the characterization of a normal band of groups as a strong semilattice of completely simple semigroups. They show in addition that if S is an order in a normal band of groups, then S is a 'canonical' order in some normal band of groups. Unlike the situation for Clifford semigroups or simple inverse ω-semigroups, the word 'canonical' cannot be replaced by 'stratified'. We refer the reader to **[6]** for its precise meaning in this context. However we may deduce here

Corollary 6.4 **[6]**. *A semigroup S is an order in a normal band of groups if and only if S is a normal band of reversible, cancellative semigroups.*

Proof. This follows immediately from Corollary 6.3.

7. Abundant orders. We recall that a semigroup S is *abundant* if every $\mathscr{L}^*$-class and every $\mathscr{R}^*$-class of S contains an idempotent. In view of the importance held by the relations $\mathscr{L}^*$ and $\mathscr{R}^*$ in the theory of semigroups of quotients, it seems interesting to consider abundant orders.

Abundant (left) orders in particular classes of semigroups have been considered in several papers, for example **[5]** and **[9]**. Here we offer two results as corollaries of Theorem 2.3.

A subsemigroup S of a semigroup T is *full* if S contains all the idempotents of T.

Corollary 7.1. *Let S be an abundant semigroup and let $(\mathscr{L}', \mathscr{R}')$ be a suitable pair for S. The following are equivalent:*

(i) *S is a full left order in a regular $\mathscr{H}$-semigroup Q, $\mathscr{L}_Q \cap (S \times S) = \mathscr{L}'$ and $\mathscr{R}_Q \cap (S \times S) = \mathscr{R}'$;*

(ii) *S satisfies conditions* (A), (B), (E) *and* (E)$'$ *and every square-cancellable element of S is $\mathscr{H}'$-related to an idempotent.*

Proof. (i)$\Rightarrow$(ii). In view of Theorem 2.3 it is only necessary to show that given $a \in S$ with $a\,\mathscr{H}^*\,a^2$, then $a\,\mathscr{H}'\,e$ for some idempotent e of S. Now $a\,\mathscr{H}_Q\,e$ for some idempotent e of Q, since S is a left order in Q. But S is full in Q so that $e \in S$ and $a\,\mathscr{H}'\,e$ as required.

(ii)$\Rightarrow$(i). Let $a, b, c \in S$ where a is square-cancellable, $a\,\mathscr{R}'\,b\,\mathscr{R}'\,c$ and $ab = ac$. Then $a\,\mathscr{H}'\,e$ for some $e = e^2 \in S$ so that $a\,\mathscr{H}^*\,e\,\mathscr{R}^*\,b\,\mathscr{R}^*\,c$. Since $a\,\mathscr{L}^*\,e$, $eb = ec$ and so as $e\,\mathscr{R}^*\,b\,\mathscr{R}^*\,c$ we have $b = c$. This shows that Condition (C) holds; dually, so does (C)$'$.

Suppose now that $a, b \in S$ are square-cancellable and $a\,\mathscr{R}'\,b$. Then $e\,\mathscr{H}'\,a\,\mathscr{R}'\,b$ for some idempotent $e \in S$ so that from $e\,\mathscr{R}^*\,b$ we have $b = eb$ and then using the fact that $\mathscr{H}'$ is a congruence, $b\,\mathscr{H}'\,ab$. Thus (D), and similarly (D)$'$, hold for S.

Theorem 2.3 may now be applied to obtain that S is a left order in a regular semigroup Q on which $\mathscr{H}$ is a congruence such that $\mathscr{L}_Q \cap (S \times S) = \mathscr{L}'$ and $\mathscr{R}_Q \cap (S \times S) = \mathscr{R}'$. Let $e \in Q$ be idempotent. Since S is straight in Q, S intersects every $\mathscr{H}$-class of Q so that $e = a^{-1}a$ for some $a \in S$. As a is square-cancellable we have by assumption that $a\,\mathscr{H}'\,f$ for some $f = f^2 \in S$. Thus in Q, $e\,\mathscr{H}\,a\,\mathscr{H}\,f$ so that $e = f \in S$ and S is a full subsemigroup of Q.

Corollary 7.2. *Let S be an abundant semigroup. Then S is a full stratified left order in a regular $\mathscr{H}$-semigroup Q if and only if $\mathscr{H}^*$ is a congruence on S, $S/\mathscr{H}^*$ is regular and the $\mathscr{H}^*$-class of any square-cancellable element is a right reversible monoid.*

Proof. As remarked in Section 2, $(\mathscr{L}^*, \mathscr{R}^*)$ is a suitable pair for S.

Suppose that S is a full stratified left order in a regular $\mathscr{H}$-semigroup Q. By Corollary 7.1, $\mathscr{H}^*$ is a congruence on S and $S/\mathscr{H}^*$ is regular. Certainly the $\mathscr{H}^*$-class of any square cancellable element must be a subsemigroup and, again by Corollary 7.1, it is right reversible and contains an idempotent, which must therefore be an identity for the $\mathscr{H}^*$-class.

Conversely, suppose that $\mathscr{H}^*$ is a congruence on S, $S/\mathscr{H}^*$ is regular and the $\mathscr{H}^*$-class of any square-cancellable element is a right reversible monoid. Certainly S satisfies (A), (B), (E) and (E)$'$. Further, if a is square-cancellable then $a\,\mathscr{H}^*\,e$ for some idempotent e. Thus, by Corollary 7.1, S is a full left order in a regular semigroup Q such that $\mathscr{H}$ is a congruence on Q, $\mathscr{L}_Q \cap (S \times S) = \mathscr{L}^*$ and $\mathscr{R}_Q \cap (S \times S) = \mathscr{R}^*$. Since S must be straight in Q, S is stratified in Q.

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