J. Austral. Math. Soc. (Series A) 67 (1999), 85–103

EXISTENCE AND UNIQUENESS OF A SOLUTION FOR A MINIMIZATION PROBLEM WITH A GENERIC INCREASING FUNCTION

A. M. RUBINOV and A. J. ZASLAVSKI

(Received 13 May 1998; revised 14 April 1999)

Communicated by M. Sniedovich

Abstract

In this paper we study the existence and uniqueness of a solution for minimization problems with generic increasing functions in an ordered Banach space X. The standard approaches are not suitable in such a setting. We propose a new type of perturbation adjusted for the problem under consideration, prove the existence and point out sufficient conditions providing the uniqueness of a solution. These results are proved by assuming that the space X enjoys the following property: each decreasing norm-bounded sequence has a limit. We supply a counterexample, which shows that this property is essential and give a modification of obtained results for the space $C(T)$, which does not possess this property.

1991 *Mathematics subject classification* (*Amer. Math. Soc.*): primary 49J27, 90C30.
Keywords and phrases: Increasing function, optimization, generic property, ordered Banach space.

1. Introduction

Consider the following optimization problem

$$(P_f) \qquad\qquad f(x) \longrightarrow \min \quad \text{subject to} \quad x \in K$$

where K is a closed subset of a Banach space X and f is a lower semicontinuous function defined on K. It is well known that this problem has a solution if K is a compact set. For noncompact sets K it is quite natural the 'generic' setting of the existence problem. Instead of an individual problem (P_f) we consider a set of such problems with functions f belonging to a metric functional space equipped with some

The research of the first author has been supported by Australian Research Council Grant A69701407.

natural metric and show that a solution exists for problems (P_f) with f belonging to a certain large set (for example a countable intersection of open everywhere dense sets). Such approach when some property is studied not for a single object but for a class of such objects is common in global analysis and in the theory of dynamical systems (see, for example, [2, 3] and [7]).

The generic approach has been also applied in the optimization theory. Beer and Lucchetti [1] considered minimization problems with objective functions belonging to the space A of lower semicontinuous convex functions defined on a Banach space, equipped with the completely metrizable topology of uniform convergence of distance functions on bounded sets. They proved that there exists an everywhere dense G_δ set $A_0 \subset A$ such that for each $f \in A_0$ the problem (P_f) has a unique solution. Deville, Godefroy and Zizler [4] obtained an analogous result for a space of bounded continuous functions on a Banach space with the topology which is not weaker then the topology of the uniform convergence. The second author [9] established the existence result for optimal control problems with a generic integrand without convexity assumptions. Recently Ioffe [5] discovered the connection between variational principles and generic existence results. In a joint paper with the second author [5] they obtained a general variational principle (an extension of the variational principle of Deville, Godefroy and Zizler [4]) and showed that generic existence results in optimization theory and calculus of variations are obtained as a realization of this principle.

We now describe the scheme of the generic approach in the study of optimization problems. Consider problems (P_f) with f belonging to a metric space M. For a given function $f \in M$ and a small $\epsilon > 0$ we choose a positive number δ which is essentially smaller than ϵ and a point $x_0 \in X$ such that $f(x_0) \leq \inf_{x \in K} f(x) + \delta$, such a point is called the (f, δ)-solution. Then we construct a new function $\bar{f} \in M$ which belongs to a given neighborhood of f and show that all $(\bar{f}, 2\delta)$-solutions belong to the closed ball $B(x_0, \epsilon)$ centered at the point x_0 with radius ϵ. Finally we show that the following holds:

Property (D): For g belonging to a small neighborhood of $\bar{f}$ all (g, δ)-solutions belong to $B(x_0, \epsilon)$.

By the variational principle in [5] property (D) implies the existence of a G_δ set $M_0 \subset M$ such that for each $f \in M_0$ the problem (P_f) has a unique solution.

In this procedure the crucial stage is constructing a proper perturbation $\bar{f}$ which belongs to the space M and allows to establish property (D). Usually the function $x \mapsto f(x) + \gamma \|x - x_0\|$ or its modifications were taken as a perturbation of $\bar{f}$.

This kind of perturbations is suitable for many spaces of convex functions, continuous and lower semicontinuous functions but classes of objective functions arising in applications are more various. For example maximization of either utility functions or production function is considered in many models of mathematical economics. As a rule both utility and production functions are increasing with respect to natural order relations. Very often it is assumed that utility functions are quasiconcave (or even concave), and production function either quasiconcave or quasiconvex.

In the present paper we study the existence of solutions of the problem (P_f) for some classes of increasing lower semicontinuous functions f defined on a closed set K in an ordered Banach space. The perturbations which are usually used to obtain a generic existence result are not suitable for such classes since they break the monotonicity. Thus the problem arises to find appropriate perturbations of increasing functions which are also increasing functions. We propose a new kind of perturbations, which allows us to establish the following property:

Property (D′): For g belonging to a small neighborhood of $\bar{f}$ all (g, δ)-solutions belong to $B(x_0, \epsilon) - X_+$, where X_+ is a cone of positive elements of X.

Since property (D′) is only a weakened version of property (D), a generic existence result for increasing functions cannot be obtained as a realization of the variational principle in [5]. Nevertheless using property (D′) we establish the existence, in the generic sense, and the following property (instead of uniqueness): the set of solutions has a greatest element. Under some additional assumptions the uniqueness follows from this property. These results are obtained only for ordered Banach spaces such that each decreasing norm-bounded sequence has a limit. We supply a counterexample which shows that this property plays the crucial role. We also give a modification of the main theorem for the space $C(T)$, which does not enjoy this property.

2. Preliminaries

Let X be a Banach ordered space and $X_+ = \{x \in X : x \geq 0\}$ be the cone of its positive elements. Assume that X_+ is a closed convex normal cone. (A cone of positive elements is called *normal* if there exists an equivalent monotone norm $\|\cdot\|$: inequalities $0 \leq x \leq y$ imply $\|x\| \leq \|y\|$.) We assume that the cone X_+ is *completely regular* ([6], see also [8]) that is it enjoys the following property (A):

Property (A): If $(x_i)_{i=1}^{+\infty}$ is a decreasing sequence ($x_{i+1} \leq x_i$ for all i) and $\sup_i \|x_i\| < +\infty$ then the sequence x_i converges.

Recall [6] that a normal closed cone in a reflexive Banach space is completely regular; the cone of nonnegative functions (with respect to usual order relation) in the space L_1 of all integrable on a measure space functions is also completely regular.

Let $\Phi : [0, +\infty) \to [0, +\infty)$ be an increasing function such that $\Phi(t) \to +\infty$ as $t \to +\infty$ and a be a real number. Let K be a closed subset of X. Denote by $\mathscr{A}$ the

https://doi.org/10.1017/S1446788700000884 Published online by Cambridge University Press

set of all lower semicontinuous functions $K \to \mathbb{R}$ such that

$$f(x) \geq \Phi(\|x\|) - a \quad \text{for all} \quad x \in K. \tag{1}$$

REMARK 2.1. If K is a bounded set then $\mathscr{A}$ can be taken to be the set of all lower semicontinuous functions f defined on K such that $f(x) \geq -a$ for all $x \in K$ for an appropriate choice of a.

It follows directly from (1) that the following assertion holds.

PROPOSITION 2.1. *For each real number c there exists a positive integer N such that for each $x \in K$ and $h \in \mathscr{A}$ we have $h(x) \leq c$ implies $\|x\| \leq N$.*

We consider $\mathscr{A}$ with the topology of uniform convergence on bounded subsets of K. Namely, we consider the uniformity on K, which has the following base:

$$\{(f, g) \in \mathscr{A} \times \mathscr{A} : |f(x) - g(x)| \leq 1/N, x \in K, \|x\| \leq N\}, \quad N = 1, 2, \dots. \tag{2}$$

Clearly the uniform space $\mathscr{A}$ is metrizable and complete.

We denote by B the set of all strictly increasing continuous functions $\beta : [0, +\infty) \to [0, +\infty)$ such that $\beta(0) = 0$. For $\beta \in B$ denote by Λ_β the set of all functions λ defined on the set $K - K$ and such that $\lambda(x) \geq \inf\{\beta(\|z\|) : z \geq x\}$ for all $x \in K - K$ and $\lambda(0) = 0$.

PROPOSITION 2.2. *Let $\beta \in B$ and $\lambda : K - K \to [0, \infty)$. The following assertions are equivalent*:

(a) $\lambda \in \Lambda_\beta$;

(b) $\lambda(0) = 0$ *and for each* $\varepsilon > 0$ *and each* $x \in K - K$ *with* $\lambda(x) < \varepsilon$ *there exists* $y \geq x$ *such that* $\beta(\|y\|) < \varepsilon$.

PROOF. (a) implies (b): It follows from the definition of the class Λ_β that $\lambda(0) = 0$. If $\lambda \geq \inf\{\beta(\|z\|) : z \geq x\}$ and $\lambda(x) < \varepsilon$, then $\inf\{\beta(\|z\|) : z \geq x\} < \varepsilon$ as well, hence a required y exists.

(b) implies (a): Let $x \in K - K$ and $\varepsilon' > 0$. Then there exists $y \geq x$ such that $\beta(\|y\|) < \lambda(x) + \varepsilon'$. Thus $\inf_{z \geq x} \beta(\|z\|) \leq \lambda(x) + \varepsilon'$. □

Consider the simplest case when $\beta(t) = t$. In this case $\Lambda_\beta = \{\lambda : \lambda(0) = 0, \lambda(x) \geq \lambda'(x) \text{ for all } x \in K - K\}$ where

$$\lambda'(x) = \inf_{y \geq x} \|y\|. \tag{3}$$

Assume now that X is a Banach space with a trivial order relation $x \geq y$ if and only if $x = y$. In such a case $X_+ = \{0\}$ and $\lambda'(x) = \|x\|$.

We will consider various perturbations of functions belong to $\mathscr{A}$ which are constructed by means of a function $\lambda \in \Lambda_\beta$ with $\beta \in B$.

https://doi.org/10.1017/S1446788700000884 Published online by Cambridge University Press

DEFINITION 2.1. Let $\beta \in B$ and $\lambda \in \Lambda_\beta$. Let $f \in \mathscr{A}$. Consider sufficiently small positive numbers γ and δ and a point $x_* \in K$ such that $f(x_*) \le \inf_{x \in K} f(x) + \delta$. Let μ be a nonnegative function defined on K_K and possibly depending on f and λ such that $\sup_{\|x\| \le r, x \in K-K} \mu(x) < +\infty$ for all $r > 0$.

A function $g : K \to \mathbb{R}$ is called a λ-*perturbation* of a function $f \in \mathscr{A}$ (determined by the set (x_*, γ, δ), with respect to the function μ) if:

(a) $g \ge f$; $g(x_*) = f(x_*)$;
(b) $(x \in K,\ g(x) \le \inf_{x \in K} g(x) + \delta)$ implies $\lambda(x - x_*) < \gamma$;
(c) $g(x) - f(x) \le \gamma\mu(x - x_*)$ for all $x \in K$.

REMARK 2.2. A λ-perturbation g is actually a one sided perturbation. It follows from (a) and (c) that

$$0 = g(x_*) - f(x_*) \le g(x) - f(x) \le \gamma\mu(x - x_*)$$

The simplest example of a λ-perturbation is the sum $g(x) = f(x) + \gamma\lambda(x - x_*)$ (see Proposition 5.1 for details); here $\lambda \in \Lambda_\beta$ and λ is bounded on the intersection of $K - K$ with the ball $\{x : \|x\| \le r\}$ for each r. It is assumed that the function μ which appears in Definition 2.1 is equal to λ. Unfortunately, sometimes this perturbation is not suitable and we are forced to consider different kinds of perturbations.

Let f be a real-valued function defined on K. We shall use the following notation: $\inf_K(f) = \inf\{f(x) : x \in K\}$.

A point $x_* \in K$ is called a (f, δ)-solution of a function $f \in \mathscr{A}$ with $\delta > 0$ if $f(x_*) \le \inf_K(f) + \delta$. Thus if x_* is a (f, δ)-solution and g is a λ-perturbation of a function f determined by (x_*, γ, δ) then each (g, δ)-solution is close to x_* in the following sense: $\lambda(x - x_*) < \gamma$.

Some examples of λ-perturbations are given in Section 5.

In order to prove a generic existence result we need to have a fairly rich set of functions which contains with each its element a certain λ-perturbation of this element. Actually we need to have only perturbations determined by sets (x_*, γ, δ) where γ is rather small and δ much smaller then γ. We express this requirement in the following form: we shall consider a closed subset $\mathscr{M}$ of the topological space $\mathscr{A}$ which possesses the following property:

Property (M): There exist $\beta \in B$ and $\lambda \in \Lambda_\beta$ such that for each $f \in \mathscr{M}$, for each fairly small $\gamma > 0$ and positive $\delta < \gamma^2$ and for each (f, δ)-solution x_* the set $\mathscr{M}$ contains also a λ -perturbation of the function f determined by (x_*, γ, δ) (with respect to a certain function μ depending on λ and f).

Some concrete examples of sets with the property (M) are given in Section 6. We now present only the simplest example.

EXAMPLE 2.1. Let $K = X$ and $\mathscr{M}_1$ be a cone of functions defined on a set X with the following properties:

(1) $\mathscr{M}_1$ contains a function $\lambda \in \Lambda_\beta$ with $\beta \in B$ such that λ is bounded on each ball;
(2) if $f \in \mathscr{M}_1$ and $x \in X$ then the function $f_{x_*}(x) = f(x - x_*)$ belongs to $\mathscr{M}_1$ as well.

Then the set $\mathscr{M} = \mathscr{M}_1 \cap \mathscr{A}$ enjoys property (M). Indeed if $f \in \mathscr{M}$, $\gamma > 0$ and $x_* \in K$ then the function $g(x) = f(x) + \gamma\lambda(x - x_*)$ belongs to $\mathscr{M}$. We can consider g as a λ-perturbation (see Remark 2.2).

3. The main result

We will establish the following result.

THEOREM 3.1. *If a set $\mathscr{M}_1 \cap \mathscr{A}$ enjoys the property* (M) *then this set contains a subset $\mathscr{H}$, which is a countable intersection of open everywhere dense (in $\mathscr{M}$) sets, such that for each $h \in \mathscr{H}$ there exists an element $y^h \in K$ with the following properties*:

(1) $h(y^h) = \inf_K(h)$;
(2) *if $x \in K$ and $\inf_K(h) = h(x)$ then $x \le y^h$*;
(3) *for each $\varepsilon > 0$ there exist $\delta > 0$ and a neighborhood U of h in $\mathscr{M}$ with the following property*: *for each $h' \in U$ and each (h', δ)-solution x there exists a vector u such that $\|u\| < \varepsilon$ and $x \le y^h + u$.*

The proof of Theorem 3.1 is based on the use of special λ- perturbations $g^{f,i}$, which we introduce below.

4. Proof of the theorem

Let the class $\mathscr{M}$ be defined by means of a function $\lambda \in \Lambda_\beta$ with $\beta \in B$. For each integer $i \ge 1$ choose numbers $\gamma(i)$ and $\delta(i)$ such that

$$0 < \gamma(i) < \min\left\{1, \beta\left(\frac{1}{2^{i+1}}\right)\right\}, \quad 0 < \delta(i) < \frac{\gamma(i)^2}{4}. \tag{4}$$

Let $\delta'(i) = 3\delta(i) < \gamma(i)^2$. Consider a function $f \in \mathscr{M}$ and a (f, δ'_i)-solution $x(f, i)$. By the definition

$$f(x(f, i)) \le \inf_K(f) + \delta'(i). \tag{5}$$

https://doi.org/10.1017/S1446788700000884 Published online by Cambridge University Press

Since the set $\mathcal{M}$ possesses the property (M) we can choose $\gamma(i), \delta(i)$ so small that the set $\mathcal{M}$ contains λ-perturbation of the function f determined by the collection $(x(f, i), \gamma(i), \delta'(i))$ for all natural numbers i. We will denote these perturbations by $g^{f,i}$. Since $g^{f,i}(x) \geq f(x)$ we have

$$\inf_K(g^{f,i}) \geq \inf_K(f). \tag{6}$$

It follows from the definition of λ-perturbation and (5) that

$$g^{f,i}(x(f, i)) = f(x(f, i)) \leq \inf_K(f) + \delta'(i). \tag{7}$$

The following assertion immediately follows from the definition of λ-perturbation.

PROPOSITION 4.1. *If $x \in K$ and $g^{f,i}(x) \leq \inf_K(g^{f,i}) + \delta'(i)$ then $\lambda(x - x(f, i)) < \gamma(i)$.*

For $f \in \mathcal{A}$ set

$$c(f) = \left(\inf_K(f)\right)_+ + 4, \tag{8}$$

where $(\inf_K(f))_+ = \max(\inf_K(f), 0)$. It follows from Proposition 2.1 that there exists a positive integer $N = N(f)$ such that for each $x \in K$ and $h \in \mathcal{A}$ we have

$$h(x) \leq c(f) \text{ implies } \|x\| \leq N. \tag{9}$$

Let $N = N(f)$ be a positive integer such that (9) holds and

$$U(f, i) = \{h \in \mathcal{M} : |h(x) - g^{f,i}(x)| < \delta(i), \quad \text{for all} \quad x \in K, \ \|x\| \leq N\}. \tag{10}$$

Clearly $U(f, i)$ is an open neighborhood of the function $g^{f,i}$ (in the topological space $\mathcal{M}$).

LEMMA 4.1. *For each $h \in U(f, i)$ we have* $\inf_K(h) = \inf\{h(x) : x \in K, \|x\| \leq N\}$.

PROOF. Applying (6) and (7) we can conclude that

$$\inf_K(f) \leq \inf_K(g^{f,i}) \leq g^{f,i}(x(f, i)) \leq \inf_K(f) + \delta'(i) \leq \inf_K(f) + 1. \tag{11}$$

Take $y \in K$ such that $g^{f,i}(y) \leq \inf_K(g^{f,i}) + 1$. We have from (11) that

$$g^{f,i}(y) \leq \inf_K(g^{f,i}) + 1 \leq \inf_K(f) + 2 \leq \inf_K(f)_+ + 2. \tag{12}$$

Thus $g^{f,i}(y) \le \inf_K(f)_+ + 2 < c(f)$. The formula (9) shows that $\|y\| \le N$. Let $h \in U(f,i)$ that is (10) holds for all y with $\|y\| \le N$. Then

$$\inf_K(h) \le \delta(i) + g^{f,i}(y) \le \inf_K(f)_+ + 3. \tag{13}$$

Take $x \in K$ such that

$$h(x) \le \inf_K(h) + 1. \tag{14}$$

It follows from (13) then $h(x) \le \inf_K(f)_+ + 4 = c(f)$. Applying again (9) we have $\|x\| \le N$. Thus the inequality $\|x\| > N$ implies $h(x) > \inf_K(h) + 1$. Thus the desired result follows. □

REMARK 4.1. It has been proved that (14) implies $\|x\| \le N$. In particular $\|x(f,i)\| \le N$.

COROLLARY 4.1. $|\inf_K(h) - \inf_K(g^{f,i})| \le \delta(i)$ *for each* $h \in U(f,i)$.

COROLLARY 4.2. *If* $h \in U(f,i)$ *and* $h(x) \le \inf_K(h) + 1$ *then* $|h(x) - g^{f,i}(x)| \le \delta(i)$.

Indeed it follows from the inequality $\|x\| \le N$ and the definition of the neighborhood $U(f,i)$.

PROPOSITION 4.2. *Let* $h \in U(f,i)$. *Then the following hold*:

(1) $h(x(f,i)) \le \inf_K(h) + 2\delta'(i)$.

(2) *For a* $(h, \delta(i))$*-solution* x *the inequality* $\lambda(x - x(f,i)) < \gamma(i)$ *holds.*

PROOF. (1) The inequality $\|x(f,i)\| \le N$ implies $h(x(f,i)) \le g^{f,i}(x(f,i)) + \delta(i)$. Applying (7), (6) and Corollary 4.1 we can deduce that

$$h(x(f,i)) \le \inf_K(f) + \delta(i) + \delta'(i) \le \inf_K(g^{f,i}) + \delta(i) + \delta'(i) \le \inf_K(h) + 2\delta'(i).$$

(2) It follows from Remark 4.1 that $\|x\| \le N$. Corollary 4.1 also shows that $\inf_K(h) \le \inf_K(g^{f,i}) + \delta(i)$. Hence

$$g^{f,i}(x) \le h(x) + \delta(i) \le \inf_K(h) + 2\delta(i) \le \inf_K(g^{f,i}) + 3\delta(i) = \inf_K(g^{f,i}) + \delta'(i).$$

The desired result now follows from Proposition 4.1. □

PROPOSITION 4.3. *If* $f \in \mathscr{M}$ *then* $g^{f,i} \to f$ *(in the topological space* $\mathscr{M}$*) as* $i \to +\infty$.

PROOF. It follows from the definition of the λ-perturbation then there exists a function $\mu(x)$ such that $\sup_{\|y\|\le r, y\in K-K}\mu(y) < +\infty$ for all $r > 0$ and $g^{f,i}(x) - f(x) \le \gamma(i)\mu(x - x(f,i))$ for all $x \in K$. Since $\|x(f,i)\| \le N$ for all i, $g^{f,i} \ge f$ and $\gamma(i) \to 0$ it follows that $g^{f,i} \to f$. □

For each $q = 1, 2, \dots$ consider the set

$$\mathscr{H}_q = \bigcup_{i=q}^{+\infty} \bigcup_{f\in\mathscr{M}} U(f,i),$$

where $U(f,i)$ is defined by (10). Clearly $\mathscr{H}_q$ is open in $\mathscr{M}$ and $g^{f,i} \in \mathscr{H}_q$ for all $f \in \mathscr{M}$ and all $i \ge q$, $q = 1, 2, \dots$. It follows from Proposition 4.3 that $\mathscr{H}_q$ is everywhere dense in $\mathscr{M}$.

Let

$$\mathscr{H} = \bigcap_{q=1}^{+\infty} \mathscr{H}_q. \tag{15}$$

Thus $\mathscr{H}$ is a countable intersection of open everywhere dense sets in $\mathscr{M}$.

PROOF (of Theorem 3.1). We will show that the set $\mathscr{H}$ enjoys the required properties. Let $h \in \mathscr{H}$. It follows from the definition of $\mathscr{H}$ that there exists a sequence $(f_q)_{q=1}^{+\infty}$ of elements of $\mathscr{M}$ and a strictly increasing sequence of natural numbers $(i_q)_{q=1}^{+\infty}$ such that

$$h \in U(f_q, i_q), \qquad q = 1, 2, \dots. \tag{16}$$

Extracting, if it is necessary, a subsequence, we can assume that

$$2\delta'(i_{q+1}) \le \delta(i_q), \quad q = 1, 2, \dots. \tag{17}$$

Consider the sequence $\{x(f_q, i_q)\}_{q=1}^{\infty} \in K$. It follows from Proposition 4.2 (1) and (16) that $x(f_q, i_q)$ is a $(h, 2\delta'(i_q))$-solution, that is

$$h(x(f_q, i_q)) \le \inf_K(h) + 2\delta'(i_q), \quad q = 1, 2, \dots. \tag{18}$$

By (18) and (17) we have for each $q = 1, 2, \dots$

$$h(x(f_{q+1}, i_{q+1})) \le \inf_K(h) + 2\delta'(i_{q+1}) \le \inf_K(h) + \delta(i_q). \tag{19}$$

Applying Proposition 4.2 (2) we can conclude that

$$\lambda(x(f_{q+1}, i_{q+1}) - x(f_q, i_q)) < \gamma(i_q). \tag{20}$$

It follows from Proposition 2.2 and (4) that for each positive integer q there exists a vector v_q such that

$$(21)\qquad v_q \geq x(f_{q+1}, i_{q+1}) - x(f_q, i_q)), \quad \beta(\|v_q\|) < \gamma(i_q) < \beta\left(\frac{1}{2^{i_q+1}}\right).$$

Since β is strictly increasing we have

$$(22)\qquad \|v_q\| < \frac{1}{2^{i_q+1}}.$$

Define

$$y_n = x(f_n, i_n) + \sum_{q=n}^{+\infty} v_q, \qquad n = 1, 2, \dots .$$

It follows from (22) that the sequence (y_n) is well-defined. The relations (1) and (18) show that there exists a positive integer $N = N(h)$ such that $\|x(f_q, i_q)\| \leq N$ for all q. Thus

$$(23)\qquad \sup_n \|y_n\| < +\infty.$$

We have also

$$y_{n+1} - y_n = x(f_{n+1}, i_{n+1}) - x(f_n, i_n) - v_n \leq 0.$$

Thus the sequence (y_n) is norm-bounded and decreasing. It follows from the property (A) of the cone X_+ that there exists $\lim_n y_n = y^h$. Clearly also $y^h = \lim_n x(f_n, i_n)$. Since h is lower semicontinuous it follows from (18) that $h(y^h) = \inf_K(h)$. Thus the first statement of Theorem 3.1 is proved. We now check the second statement.

Take a minimizer x of the function h that is $x \in K$ such that $h(x) = \inf_K(h)$. Since $h \in U(f_q, i_q)$ we have by applying Proposition 4.2 (2) that $\lambda(x - x(f_q, i_q)) < \gamma(i_q)$. It follows from Proposition 2.2 that there exists a vector v'_q such that

$$v'_q \geq x - x(f_q, i_q) \quad \text{and} \quad \beta(\|v'_q\|) < \gamma(i_q).$$

Thus $0 = \lim_q v'_q \geq x - \lim_q x(f_q, i_q) = x - y^h$. The second statement of Theorem 3.1 is proved.

We now prove the third statement. Let $0 < \varepsilon < 1$ and $q > 0$ be a positive integer such that $\gamma(i_q) \leq \beta(\varepsilon/2)$. Since $y^h = \lim_n x(f_n, i_n)$ we can assume without loss of generality that $\|y^h - x(f_q, i_q)\| \leq \varepsilon/2$. Let $h' \in U(f_q, i_q)$ and x be a $(h', \delta(i_q))$-solution. It follows from Proposition 4.2 (2) that $\lambda(x - x(f_q, i_q)) < \gamma(i_q)$. Hence, there exists a vector v'_q such that $v'_q \geq x - x(f_q, i_q)$ and $\beta(\|v'_q\|) < \gamma(i_q) < \beta(\varepsilon/2)$. We have $\|v'_q\| < \varepsilon/2$. Let $u_q = v'_q - (y^h - x(f_q, i_q))$. Then $\|u_q\| \leq \varepsilon$ and $x \leq y^h + u_q$. $\square$

5. Examples of λ-perturbations

We now present two examples of λ-perturbations. In this section we consider functions λ defined on the entire space X, generating perturbations which are suitable for all closed sets K. We will define a function μ which appears in the definition of λ-perturbation by means of the function λ itself. So we will consider only functions λ with the following property:

$$\sup_{\|x\|\le r} \lambda(x) < +\infty \quad \text{for all} \quad r > 0. \tag{24}$$

PROPOSITION 5.1. *Let $\lambda \in \Lambda_\beta$, $\gamma > 0$, $\delta < \frac{1}{2}\gamma^2$ and let x_* be a (f, δ)-solution. Assume that* (24) *holds. Then*

$$g(x) = f(x) + \gamma\lambda(x - x_*) \tag{25}$$

is a λ-perturbation of the function f, determined by (x_*, γ, δ) with respect to the function $\mu = \lambda$.

PROOF. It is clear that $g \ge f$ and $g(x_*) = f(x_*)$. We also have $g(x) - f(x) = \gamma\lambda(x - x_*) = \gamma\mu(x - x_*)$. Let $x \in K$ and $g(x) \le \inf_K(g) + \delta$. Then

$$\begin{aligned} f(x) + \gamma\lambda(x - x_*) = g(x) &\le \inf_K(g) + \delta \le g(x_*) + \delta \\ &= f(x_*) + \delta \le \inf_K(f) + 2\delta \le f(x) + 2\delta. \end{aligned}$$

So $\gamma\lambda(x - x_*) \le 2\delta < \gamma^2$. □

PROPOSITION 5.2. *Let $\lambda \in \Lambda_\beta$, $\gamma \in (0, 1)$, $\delta < \gamma^2$ and*

$$\delta < \eta < \gamma^2. \tag{26}$$

Assume that (24) *holds. Further $f \in \mathscr{A}$ and let x_* be a (f, δ)-solution. Then the function*

$$g(x) = \max(f(x), \min(f(x_*) + \gamma\lambda(x - x_*), f(x_*) + \eta)) \tag{27}$$

is a λ-perturbation of the function f determined by (x_*, γ, δ) with respect to $\mu = 1 + \lambda$.

PROOF. It is clear that $g(x) \ge f(x)$ for all $x \in K$ and $g(x_*) = f(x_*)$. We now check that the inequality $g(x) \le \inf_K(g) + \delta$ implies $\lambda(x - x_*) < \gamma$. Assume the last inequality does not hold. Then

$$f(x_*) + \gamma\lambda(x - x_*) \ge f(x_*) + \gamma^2 > f(x_*) + \eta.$$

Thus $g(x) = \max(f(x), f(x_*) + \eta)$. We have

$$f(x_*) + \eta \le \max(f(x), f(x_*) + \eta) = g(x) \le \inf_K(g) + \delta \le g(x_*) + \delta = f(x_*) + \delta.$$

So $\eta \le \delta$, which contradicts (26). Hence the inequality $\lambda(x - x_*) < \gamma$ holds for (g, δ)-solutions.

We now estimate the difference $g(x) - f(x)$. It is sufficient to consider only points x such that $f(x) \le f(x_*) + \min(\gamma\lambda(x - x_*), \eta)$. For such points we have:

$$\begin{aligned} g(x) &= f(x_*) + \min(\gamma\lambda(x - x_*), \eta) \le \inf_K(f) + \delta + \min(\gamma\lambda(x - x_*), \eta) \\ &\le f(x) + \delta + \min(\gamma\lambda(x - x_*), \eta). \end{aligned}$$

Thus

$$g(x) - f(x) \le \delta + \min(\gamma\lambda(x - x_*), \eta) \le \gamma(1 + \lambda(x - x_*)) = \gamma\mu(x - x_*).$$

□

REMARK 5.1. A perturbation close to (27) with $\lambda(x) = \|x\|$ has been used in [5].

6. Examples

We now give some examples. First we consider the simplest case when $\beta(t) = t$. In this case $\Lambda_\beta = \{\lambda : \lambda(0) = 0,\ \lambda \ge \lambda'\}$ where the function λ' is defined by (3): $\lambda'(x) = \inf\{\|y\| : y \ge x\}$. We need the following assertion.

PROPOSITION 6.1. *The function λ' defined on the space X by* (3) *is a sublinear increasing function such that* $0 \le \lambda'(x) \le \|x\|$ *for all x and* $\lambda'(x) = 0$ *for* $x \le 0$. *If $\|\cdot\|$ is monotone on the cone X_+ (that is $y \ge x \ge 0$ implies $\|y\| \ge \|x\|$) then* $\lambda'(x) = \|x\|$ *for* $x \ge 0$.

PROOF. We will check only that λ' is a sublinear function. It is clear that $\lambda'(cx) = c\lambda'(x)$ for $c \ge 0$. Let $x_1, x_2 \in X$ and vectors $y_i \ge x_i$ such that $\|y_i\| < \lambda'(x_i) + \varepsilon$, $(i = 1, 2)$ with $\varepsilon > 0$. Then $y_1 + y_2 \ge x_1 + x_2$ and

$$\lambda'(x_1 + x_2) \le \|y_1 + y_2\| \le \|y_1\| + \|y_2\| < \lambda'(x_1) + \lambda'(x_2) + 2\varepsilon.$$

Thus λ' is a sublinear function. □

EXAMPLE 6.1. Let the cone X_+ in the Banach space X enjoys the property (A) and let K be a closed subset of X. Let $\mathscr{M}_{incr}$ be the set of all lower semicontinuous increasing functions defined on K such that (1) holds and (if K is, in addition, convex) $\mathscr{M}^c_{incr}$ be the subset of $\mathscr{M}_i$, consisting of convex functions. It is easy to check that Theorem 3.1 holds for both of these classes. Indeed, the function λ is increasing and convex, so we can use the sum $f(x) + \gamma\lambda(x - x_*)$ as a perturbation (compare with Remark 2.2 and Example 2.1).

We can use also the perturbation (27) for the class of all increasing functions, but we can not use it for the class $\mathscr{M}^c_{incr}$, since it contains the operation of taking minima.

EXAMPLE 6.2. Consider now the set $\mathscr{M}^{qc}$ of all lower semicontinuous quasiconvex functions defined on a closed convex set K and the set $\mathscr{M}^{qc}_{incr}$ of all lower semicontinuous quasiconvex increasing functions defined on this set. Recall that a function f is called quasiconvex if its level sets $\{x : f(x) \leq c\}$ are convex for each c. Since $\max(f_1(x), f_2(x))$ and $\min(f_1(x), c)$ with f_1, f_2 quasiconvex and $c \in \mathbb{R}$ are again quasiconvex, it follows that the perturbation (27) can be used for study these classes. We can choose again the function λ' for construction of perturbations. Thus Theorem 3.1 holds for these classes. Note that the sum of quasiconvex functions is not necessarily quasiconvex, so we cannot use the perturbation (25) in the case.

REMARK 6.1. A generic existence result for the set $\mathscr{M}^{qc}$ was established in [5].

EXAMPLE 6.3. Consider the sets $\mathscr{M}^{cv}_{incr}$ of all upper semicontinuous concave increasing functions f defined on a bounded closed convex set K and such that $\sup_{x\in K} f(x) \leq a < +\infty$. It is easy to check that a function f defined on K is increasing and concave if and only if the function $f_*(x) = -f(-x)$ defined on the set $-K$ is increasing and convex. Consider the set $\mathscr{M}^c_{incr}$ of all lower semicontinuous convex increasing functions defined on $-K$ and such that $\inf_{x\in -K} f(x) \geq -a$. Condition (1) holds for this class (see Remark 2.1). Applying Example 6.1 we can conclude that Theorem 3.1 holds for $\mathscr{M}^c_{incr}$. Therefore this theorem holds for $\mathscr{M}^{cv}_{incr}$. In the same manner we can show that Theorem 3.1 holds for the set $\mathscr{M}^{qcv}_{incr}$ of all upper semicontinuous quasiconcave increasing functions f such that $\sup_{x\in K} f(x) \leq a < +\infty$.

Now we consider the function $\beta(t) = t^\alpha$ with $\alpha > 0$. It is easy to check that $\Lambda_\beta(x) = \{\lambda : \lambda(0) = 0,\ \lambda \geq (\lambda')^\alpha(x)$ where λ' is defined by (3). If $X_+ = \{0\}$ then $(\lambda')^\alpha(x) = \|x\|^\alpha$.

EXAMPLE 6.4. Let X be a Banach space with $X_+ = \{0\}$. Let $\mathscr{B}$ be a closed convex cone of lower semicontinuous functions, defined on a closed set $K \subset X$. Assume $\mathscr{B}$ contains the restriction functions $x \to \|x - x_*\|^\alpha$ to K for all $x_* \in K$. Let $\mathscr{A}$ be the set of all functions $f \in \mathscr{B}$ such that (1) holds. Then Theorem 3.1 holds for the class $\mathscr{A}$. Note that the assertions (2) and (3) of the theorem can be reformulated for $X_+ = \{0\}$ in the following way:

(2′) If $\inf_K(h) = h(x)$ then $x = y^h$ (uniqueness);

(3′) For each $\varepsilon > 0$ there exist $\delta > 0$ and a neighbourhood of h in $\mathscr{M}$ such that for each (h', δ)-solution with $h' \in U$ the inequality $\|x - y^h\| < \varepsilon$ holds (stability).

For the space of all lower semicontinuous functions satisfying (1) the generic existence result was established in [5].

EXAMPLE 6.5. Let X be the space L_1 of all integrable functions defined on a measurable space (T, Σ, μ) and let K be a closed subset of L_1. Let $\lambda^+(x) = \int_T x_+ d\mu = \|x_+\|$, where $x_+ = \sup(x_+, 0)$ is a positive part of the element x. Let $\beta(t) = t$. Clearly $\lambda^+ \in \Lambda_\beta$. Let $\mathscr{B}$ be the space of all functions f represented in the form $f(x) = \int_T Q(x) d\mu$ where $Q : L_1 \to L_1$ be a convex increasing operator. Since the operator $x \mapsto (x - x_*)_+$ is convex and increasing it follows that the function $x \mapsto \|(x - x_*)_+\|$ belongs to $\mathscr{B}$. Thus Theorem 3.1 holds for the class $\mathscr{A}$ of all functions $f \in \mathscr{B}$ such that (1) is valid.

7. The uniqueness of a solution

For classes of increasing functions (see Example 6.1, Example 6.2 and Example 6.5) it is possible to establish stronger versions of Theorem 3.1 (under some additional assumptions).

PROPOSITION 7.1. *Assume that the set K possesses the following properties:*

(i) *for each $y \in K$ there is a minimal element x of the set K such that $x \le y$;*

(ii) *the set of all minimal elements of the set K is closed in norm-topology.*

If $\mathscr{M}$ consist of increasing functions then it is possible to replace assertion (2) *of Theorem* 3.1 *by the following assertion:*

(2″) *the vector y^h is the unique minimizer of a function $h \in \mathscr{H}$.*

PROOF. We will do some changes in the proof of Theorem 3.1. Since functions $g^{f,i}$ are increasing for all i it follows from (i) that we can choose a vector $x(f, i)$ as a minimal element of K. Define y_n and y^h like in the proof of the theorem. Then $y^h = \lim_n x(f_n, i_n)$. It follows from (ii) that y^h is a minimal element of K. Together with assertion (2) of Theorem 3.1 this implies assertion (2″). □

PROPOSITION 7.2. *Assume that all conditions of Proposition* 7.1 *hold and moreover the set Q of all minimal points of the set K is compact. Then the following assertion holds:*

https://doi.org/10.1017/S1446788700000884 Published online by Cambridge University Press

(3″) *for each* $h \in \mathscr{H}$ *and each* $\varepsilon > 0$ *there exists* $\delta > 0$ *and a neighborhood* U *of* h *in* $\mathscr{M}$ *with the following properties: for each* $h' \in U$ *and* $x \in Q$ *with* $h'(x) \leq \inf_K(h') + \delta$ *the inequality* $\|x - y^h\| < \varepsilon$ *holds.*

A proof easily follows from assertion (3) of Theorem 3.1 and the following lemma.

LEMMA 7.1. *Assume the set of* Q *of minimal points of the set* K *is compact. Then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *with the following property: if* $x, y \in Q$ *and* $x \leq y + u$ *with* $\|u\| \leq \delta$ *then* $\|x - y\| \leq \varepsilon$.

PROOF. Assume the contrary. Then there is $\varepsilon > 0$ and sequences $(x_n) \in Q$, $(y_n) \in Q$ and (u_n) such that $y_n \leq x_n + u_n$, $\|u_n\| \leq 1/n$ and $\|x_n - y_n\| \geq \varepsilon$. Since Q is compact we can assume without loss of generality that there exist $\lim x_n = x$ and $\lim y_n = y$. We have $x \in Q$, $y \in Q$ and $x \leq y$. Since y is a minimal element it follows that $x = y$. We obtain a contradiction. □

8. A modification of the main theorem for the space $C(T)$

The complete regularity of the cone X_+ (in other words the property (A)) plays the crucial role in the proof of Theorem 3.1. The example below shows that there exists a class $\mathscr{M}$ with the property (M) in a Banach lattice without property (A) such that infimum of each function $f \in \mathscr{M}$ is not attained. Nevertheless we can consider modifications of this theorem for some special cases where the property (A) does not hold. We consider one of these cases in this section.

Let $X = C(T)$ be the space of all continuous functions defined on the compact topological space T, equipped with the uniform norm and X_+ be the cone of all nonnegative functions. It is clear that X_+ does not possess in general the property (A). First we show that Theorem 3.1 is not valid for this space.

EXAMPLE 8.1. Let T be a finite dimensional compact space and let $\hat{x}$ be a bounded upper semicontinuous discontinuous function defined on T. Let $K = \{x \in C(T) : x \geq \hat{x}, \|x\| \leq c\}$ where c is a sufficiently large number. Then

$$\hat{x}(t) = \inf\{x(t) : t \in K\} \quad \text{for all} \quad t \in T. \tag{28}$$

Clearly K is closed. Since K is bounded it follows that $\mathscr{A}$ is the set of all lower semicontinuous functions f defined on K with the property $f(x) \geq -a$ where a is a real number. The topology generated in the set $\mathscr{A}$ by uniformity with the base (2) is the usual normed topology of the uniform convergence on K. Take a strictly increasing function $h \in \mathscr{A}$ (that is $h(y) > h(x)$ if $y > x$); the simplest example of such a function is $h(x) = \int_T x \, d\mu$ where μ is the Lebesgue measure. Consider the set $\mathcal{M}$ of all functions f of the form $f = h + g$ where g is a nonnegative lower semicontinuous increasing function defined on K. This set is closed in the topological space $\mathcal{A}$. Since the function $\lambda(x) = \inf\{\|y\| : y \geq x\}$ is increasing it follows that the function $x \mapsto f(x) + \gamma\lambda(x - x_*)$ $(x \in K)$ belongs to $\mathcal{M}$ if $f \in \mathcal{M}$. Thus $\mathcal{M}$ enjoys the property (M). It is clear that each function $f \in \mathcal{M}$ is strictly increasing. Let $f \in \mathcal{M}$ and $x \in K$. Since $x \neq \hat{x}$ it follows from (28) that there exists $x' \in K$ such that $x' < x$. Since f strictly increases it follows that $f(x') < f(x)$. Thus $\inf_{x \in K} f(x)$ is not attained for each $f \in \mathcal{M}$.

We will show that a minimizer for a generic increasing function exists if we consider increasing functions defined on a certain upper semicontinuous hull of the given set K of continuous functions.

Let T be a compact topological space. We denote by $\hat{C}(T)$ the set of all bounded upper semicontinuous functions defined on the space T. Let f be an increasing function defined on $C(T)$. If (x_n) is a decreasing bounded sequence of continuous functions on T then the pointwise $\lim x_n(t) = \inf x_n(t)$ belongs to $\hat{C}(t)$ and there exists $\lim f(x_n)$. We will consider increasing functions f which possess the following continuous-like property (C):

Property (C): if $y_n \in C(T)$ is a decreasing bounded sequence and $x_n \in C(T)$ is a sequence such that either

$$x_n \text{ is decreasing, } x_n \leq y_n, \ n = 1, 2, \ldots, \ \lim_n x_n(t) = \lim_n y_n(t) \text{ for all } t \in T;$$

or

$$x_n = y_n - \varepsilon_n \mathbf{1}, \quad n = 1, 2, \ldots, \quad \text{with} \quad \varepsilon_n \to 0,$$

then $\lim f(y_n) = \lim f(x_n)$. (Here $\mathbf{1}$ is a constant function, equal to 1.)

Denote by $\hat{f}$ the natural extension of the function f to the set $\hat{C}(T) : \hat{f}(\hat{y}) = \inf\{f(y) : y \in C(T),\ y \geq \hat{y}\}$ for $\hat{y} \in \hat{C}(T)$. Clearly $\hat{f}(x) = f(x)$ for $x \in C(T)$ and $\hat{f}$ is an increasing function.

The property (C) allows us to show that the following assertion holds:

PROPOSITION 8.1. *Let $\hat{y} \in \hat{C}(T)$ and $\hat{y}(t) = \inf_n y_n(t)$ for all $t \in T$ with a decreasing sequence $y_n(t)$. Then $\hat{f}(\hat{y}) = \inf_n f(y_n)$.*

PROOF. Let $\hat{y}$ and y_n be as above. Further let $y' \in C(T)$, and $y' \geq \hat{y}$. The sequence $y_n' = \inf(y_n, y')$, $(n = 1, 2 \ldots)$ is decreasing. We have

$$\inf_n y_n'(t) = \inf_n \inf(y_n(t), y'(t)) = \inf(\inf_n y_n(t), y'(t)) = y(t).$$

Since y'_n is decreasing it follows that $y'_n(t) \to y(t)$. Since also $y_n(t) \to y(t)$, the sequence y_n is decreasing and $y_n \geq y'_n$. We can conclude, by applying the property (C), that $\lim f(y'_n) = \lim f(y_n)$. Since $y' \geq y'_n$ for all n we have that $f(y') \geq \lim_n f(y'_n) = \lim f(y_n)$. Hence $\hat{f}(\hat{y}) = \inf_{y' \in C(T), y' \geq \hat{y}} f(y') = \inf_n f(y_n)$. □

Let K be a closed subset of $C(T)$. Denote by $\mathscr{A}$ the set of all increasing functions defined on $C(T)$, which enjoy the property (C) and satisfy the inequality (1).

Consider the set $\hat{K} \subset \hat{C}(T)$ defined in the following way: An element $\hat{x} \in \hat{C}(T)$ belongs to $\hat{K}$ if there exists a sequence of elements $x_n \in K$ and a sequence of positive numbers $\varepsilon_n \to 0$ that the sequence $x_n + \varepsilon_n \mathbf{1}$ is decreasing and

$$\hat{x}(t) = \inf_n (x_n(t) + \varepsilon_n) = \lim_n x_n(t) \quad \text{for all} \quad t \in T. \tag{29}$$

If f possesses the property (C) then

$$\inf_K(f) \equiv \inf_{x \in K} f(x) = \inf_{\hat{x} \in \hat{K}} \hat{f}(\hat{x}). \tag{30}$$

Indeed let $\hat{x} \in \hat{K}$ and sequences x_n and ε_n are chosen so that (29) holds and $y_n = x_n + \varepsilon_n \mathbf{1}$ is a decreasing sequence. Since the property (C) holds we have $\lim f(x_n) = \lim f(y_n)$. Since $\hat{x}(t) = \inf_n y_n(t)$ and (y_n) is a decreasing sequence it follows from Proposition 8.1 that

$$\hat{f}(\hat{x}) = \inf f(y_n) = \lim f(y_n) = \lim f(x_n) \geq \inf_K(f).$$

Thus (30) holds.

For $x \in C(T)$ define $\lambda^+(x)$ by the formula:

$$\lambda^+(x) = \inf\{\lambda > 0 : \lambda \mathbf{1} \geq x\}. \tag{31}$$

Let x_+ be the positive part of an element x. It is easy to check that $\lambda^+(x) = \max_{t \in T} x_+(t) = \|x_+(t)\|$ and $\lambda^+ \geq \lambda_\beta$ with $\beta(t) = t$.

We use the definitions, notation and results from the Section 2, by assuming that $X = C(Q)$, X_+ is the cone of nonnegative functions, $\mathscr{A}$ is the defined above set and $\lambda \geq \lambda^+$. The following Proposition will be used instead of Proposition 2.2.

PROPOSITION 8.2. *Let λ_0 be a function defined on $C(T)$. Then $\lambda_0 \geq \lambda^+$ if and only if for each $\varepsilon > 0$ and each $x \in C(T)$ with $\lambda_0(x) < \varepsilon$ there exists a positive number λ such that $\lambda \mathbf{1} \geq x$.*

The proof is similar to the proof of Proposition 2.2.

THEOREM 8.1. *Let a set* $\mathcal{M} \in \mathcal{A}$ *enjoy the property* (M) (*with respect to a function* $\lambda \geq \lambda^+$). *Then this set contains a set* $\mathcal{H}$, *which is a countable intersection of open everywhere dense* (*in* $\mathcal{M}$) *sets, such that for each* $h \in \mathcal{H}$ *there exists an element* $y^h \in \hat{K}$ *with the following properties*:

(1) $h(y^h) = \inf_K(h)$;

(2) *if* $x \in \hat{K}$ *and* $\inf_K(h) = h(x)$ *then* $y \leq y^h$.

PROOF. For the proof we can use results from Section 3; then we can argue like in the first part of the proof of Theorem 3.1, up to (20), by assuming that $\beta(t) = t$. Using Proposition 8.2 we can find a positive number η_q for each positive integer q, such that

$$\eta_q \mathbf{1} \geq x(f_{q+1}, i_{q+1}) - x(f_q, i_q), \quad \eta_q < \gamma_{i_q} < \frac{1}{2^{i_q+1}}. \tag{32}$$

Define

$$y_n = x(f_n, i_n) + \varepsilon_n \mathbf{1}\ , \tag{33}$$

where $\varepsilon_n = \sum_{q=n}^{+\infty} \eta_q$, $n = 1, 2, \dots$. Clearly $\varepsilon_n \to 0$. The sequence (y_n) is well-defined and bounded. It follows from (32) and (33) that this sequence is decreasing. We have also $y_n \geq x(f_n, i_n)$ for all n. Let

$$y^h(t) = \inf_n y_n(t) = \lim_n y_n(t) = \lim x(f_n, i_n)(t) \quad \text{for all} \quad t \in T.$$

It follows from the definition of the set $\hat{K}$ that $y^h \in \hat{K}$. Since the function h enjoys the property (C) we have, by applying (17), that

$$\hat{h}(y^h) = \lim_n h(y_n) = \lim f_n(x(f_n, i_n))(t) = \inf_K(h).$$

Thus the first assertion of the theorem is proved.

Let $\hat{x}$ be a minimizer of the function $\hat{h}$ over the set $\hat{K}$. Then there exist sequences $x_n \in K$ and $\varepsilon_n > 0$, $\varepsilon_n \to 0$ such that $z_n = x_n + \varepsilon_n \mathbf{1}$ is a decreasing sequence and $\hat{x} = \inf_n z_n$. Proposition 8.1 shows that $\inf_K(h) = \inf_K(\hat{h}) = \inf h(z_n)$. Let i be a positive integer. Then $h(z_n) \leq \inf_K(h) + \delta(i)$ for sufficiently large n. Since $h \in U(f_n, i_n)$ we have by applying Proposition 4.2 (2) that $\lambda(z_n - x(f_n, i_n)) < \gamma(i_n)$. Since $\lambda \geq \lambda^+$ we can find a positive number $\eta'_n \to 0$ such that $\eta'_n \mathbf{1} \geq z_n - x(f_n, i_n)$ We have

$$0 \geq \lim_n z_n(t) - \lim_n x(f_n, i_n)(t) = \hat{x}(t) - y^h(t).$$

The second statement of the theorem is proved. □

References

[1] G. Beer and R. Lucchetti, 'Convex optimization and the epi-distance topology', *Trans. Amer. Math. Soc.* **327** (1991), 795–813.

[2] F. S. De Blasi and J. Myjak, 'Sur la convergence des approximations successives pour les contractions non linéaires dans un espace de Banach', *C. R. Acad. Sc. Paris* **283** (1976), 185–187.

[3] ——, 'Generic flows generated by continuous vector fields in Banach spaces', *Adv. Math.* **50** (1983), 266–280.

[4] R. Deville, G. Godefroy and V. Zizler, 'A smooth variational principle with applications to Hamilton-Jacobi equations in infinite dimensions', *J. Funct. Anal.* **111** (1993), 197–212.

[5] A. D. Ioffe and A. J. Zaslavski, 'Variational principles and well-posedness in optimization and calculus of variations', *SIAM J. Control Optimiz.*, to appear.

[6] M. A. Krasnosel'skii, *Positive solutions of operator equations* (Noordhof, Leyden, 1964).

[7] S. Reich and A. J. Zaslavski, 'Convergence of generic infinite products of nonexpansive and uniformly continuous operators', *Nonlinear Anal.*, to appear.

[8] H. H. Schaefer, *Banach lattices and positive operators* (Springer, Berlin, 1974).

[9] A. J. Zaslavski, 'Existence of solutions of optimal control problems for a generic integrand without convexity assumptions', *Nonlinear Anal.*, to appear.

School of Information Technology
and Mathematical Sciences
University of Ballarat
Ballarat VIC 3353
Australia
e-mail: amr@ballarat.edu.au

Department of Mathematics
The Technion-Israel Institute of Technology
32 000 Haifa
Israel
e-mail: ajzasl@techunix.technion.ac.il

https://doi.org/10.1017/S1446788700000884 Published online by Cambridge University Press