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# On Certain Classes of Unitary Representations for Split Classical Groups

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*Abstract.* In this paper we prove the unitarity of duals of tempered representations supported on minimal parabolic subgroups for split classical $p$-adic groups. We also construct a family of unitary spherical representations for real and complex classical groups

## Introduction

The strategy of classifying the unitary dual of a reductive group over a local field usually breaks into two parts: classification of tempered representations and then classification of non-tempered Langlands quotients that are unitary [T2]. It is especially pleasant to do the classification for some classes of representations that are fully induced from their standard modules, such as generic representations [LMT], since then the second part of the classification is only about the determination of complementary series. In general, we work with representations that are not obviously unitarizable like tempered representations nor do they appear in complementary series. They are the so-called (non-tempered) isolated unitary representations. There are not many techniques known for establishing the unitarity of such representations. One that is well known is an obvious one: one can use Langland's spectral theory [MW3] to construct a representation $\pi = \bigotimes_v \pi_v$ in a residual spectrum with a prescribed representation $\sigma$ in a fixed local place $v_0$ establishing unitarity of $\sigma \simeq \pi_{v_0}$. As far as the author knows, this method was used for the first time by Speh [Sp] establishing the unitarity of Speh representations (archimedean case) and a little later by Tadić [T1] establishing unitarity of analogues of Speh representations for $p$-adic general linear groups. The author himself also proved unitarity of some isolated representations for $p$-adic $G_2$ [M1] using the same technique. In any case, the proof is based on a construction of a relatively simple residual representation, the so-called regular case, that simplifies the local computation of the poles and images of normalized intertwining operators coming from the constant term of the Eisenstein series used. (We refer to [MW2, K1, Ža] for the full classification of residual spectrum for general linear groups and $G_2$, respectively.) The case of split classical groups is more complicated.

In this paper we study unitarizable representations of split classical groups $G_n = \mathrm{Sp}(n)$ (symplectic group of rank $n$), $G_n = \mathrm{SO}(2n+1)$ (split special odd-orthogonal group of rank $n$), and $G_n = \mathrm{O}(2n)$ (split special even-orthogonal group of rank $n$) over a non-archimedean local field $F$ of characteristic 0. The main result of the paper

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(see Theorem 3.7) proves that the dual of a tempered representation, induced from the minimal parabolic subgroup under the Aubert–Schneider–Stuhler involution [A, Jan1, SS] is a unitarizable representation. In fact, the only serious work is done for discrete series representations, since there is an obvious reduction to that case using the fact that any tempered representation is a subrepresentation of a representation unitarily induced from a discrete series representation (see Corollary 3.2). Earlier, Barbasch and Moy [BM1, BM2] proved the analogue of our results for the Iwahori–Matsumoto involution in the special case of Iwahori spherical representations, but for general split reductive groups with connected center, using the Kazhdan–Lusztig classification.

We should mention that there are now several papers treating various parts of the residual spectrum for the split classical groups that we consider [Mœ1, Mœ2, K2, JK], but none of them consider all the residual representations that we need here. Now, we describe our approach. We use the classification of discrete series established by Mœglin and Tadić [Mœ3, MT]. In fact, we use only the part that comes from the minimal parabolic subgroup of $G_n$, so that those discrete series are classified without any assumption. The classification of discrete series is inductive starting from strongly positive discrete series (see §3.1, §3.2) and adding two Jordan blocks in each step (see §3.3). Let $\sigma \in \operatorname{Irr} G_n(F)$ be a representation in the discrete series supported on the minimal parabolic subgroup. Then we choose a global field $K$ so that $K_{v_0} \simeq F$ and all archimedean places are complex. The proof is then done by the induction on the number of Jordan blocks of $\sigma$, carefully increasing (see Lemma 4.3) the set of finite places $S$ (containing $v_0$) such that all residual representations $\pi = \bigotimes_v \pi_v$ satisfy the following: $\pi_v$ is spherical for $v \notin S$; $\sigma \simeq \pi_{v_0}$; for $v \in S \setminus \{v_0\}$, $\pi_v$ is in the discrete series and we inductively build $\pi_v$ in the same way we build $\sigma$ (see Proposition 4.2). In doing so, the inductive construction of discrete series enables us to simplify global computations by choosing a degenerate Eisenstein series coming from some maximal parabolic subgroup instead of one coming from the minimal parabolic subgroup. Their poles are then computed considering the constant terms along the minimal parabolic subgroup of $G_n$. The preliminary computations and explanation of the method used is in Section 2 (Lemmas 2.2 and 2.3). Section 2 also recalls the normalization of intertwining operators proposed by Langlands and originally proved by Shahidi for $\mathrm{GL}(n)$ [Sh1] and then extended [Sh2, KSh] (see Theorem 2.1). Section 3 is the technical heart of the paper. There we prove that all normalized intertwining operators are holomorphic at particular points (Lemma 3.3 and Lemma 3.6). This is possible mainly because we carefully choose our degenerate Eisenstein series. In Section 4, we prove Theorem 3.7 by induction, constructing global representations $\pi$ (Propositions 4.1 and 4.2). We believe that the methods of Section 2 and Section 4 are of independent interest and they will have some other applications in the theory of automorphic forms.

In Section 5, we reprove a well-known result for real and complex groups about unitarity of unipotent spherical representations [Ba]. The discussion is a direct consequence of the results of the previous section. We feel that this might be interesting since it sheds some light on unitary duals of archimedean groups from a different perspective. We also get very precise information on the position of spherical representations in certain principal series. The main result is Theorem 5.1.

# 1 Preliminaries

Let $K$ be a global or local field (archimedean or not) of characteristic zero. If $K$ is local field we let $|\cdot|$ denote its absolute value. Let $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ be the ring of rational integers, the field of real numbers, and the field of complex numbers, respectively. In this paper we work with groups:

- $G_n = \mathrm{Sp}(n)$ (symplectic group of rank $n$ over $K$):

$$G_n = \left\{ g \in \mathrm{GL}(2n) \mid g \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & -J_n \\ J_n & 0 \end{pmatrix} \right\}.$$

- $G_n = \mathrm{SO}(2n+1)$ (split special odd-orthogonal group of rank $n$ over $K$):

$$G_n = \{ g \in \mathrm{SL}(2n+1) \mid g J_{2n+1} g^t = J_{2n+1} \}.$$

- $G_n = \mathrm{O}(2n)$ (split even-orthogonal group of rank $n$ over $K$):

$$G_n = \{ g \in GL(2n) \mid g J_{2n} g^t = J_{2n} \},$$

where $J_n$ is the $n \times n$ matrix defined by

$$J_n = \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \cdot & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & & & & \end{pmatrix}.$$

If $A$ is a $K$-algebra, then we denote the group of $A$-points of $G_n$ by $G_n(A)$. Let $B_n = T_n U_n$ be its fixed Borel subgroup over $K$, where $T_n$ is its maximal split torus and $U_n$ is its unipotent radical. We may and will realize them as the usual groups of upper triangular matrices. We let $W_n$ denote the corresponding Weyl group. Let $\mathrm{Rat}(T_n)$ denote the group of all (algebraic) characters of $T_n$. Since $T_n$ is split, it is precisely the group of all $K$-rational characters. It is a free $\mathbb{Z}$-module

$$\mathrm{Rat}(T_n) \simeq \mathbb{Z}\phi_1 \oplus \mathbb{Z}\phi_2 \oplus \cdots \oplus \mathbb{Z}\phi_n, \tag{1.1}$$

where $\phi_i$, $1 \le i \le n$, is defined by $\phi_i(t) = t_i$, where

$$t = \begin{pmatrix} t_1 & & & & & & & & \\ & t_2 & & & & & & & \\ & & \ddots & & & & & & \\ & & & t_n & & & & & \\ & & & & 1 & & & & \\ & & & & & t_n^{-1} & & & \\ & & & & & & \ddots & & \\ & & & & & & & t_2^{-1} & \\ & & & & & & & & t_1^{-1} \end{pmatrix}, \tag{1.2}$$

for $G_n = \mathrm{SO}(2n+1)$, and for the other two cases, we remove 1 from the middle. We write the group operation in $\mathrm{Rat}(T_n)$ additively.

Now, we follow [T3] rather closely. The Weyl group $W_n$ is isomorphic to $S_n \rtimes (\{\pm 1\})^n$, where $S_n$ is the group of permutations of $n$-letters. This semidirect product, as well as the isomorphism with $W_n$, are described in the standard action of the Weyl group $W_n$ on $\mathrm{Rat}(T_n)$. More precisely, for $p \in S_n$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in (\{\pm 1\})^n$, we have $(p\epsilon)(\phi_i) = \phi_{p(i)}^{\epsilon_i}$.

We write $\Sigma$ for the root system of $T_n$ with respect to $G_n$. Further, $\Sigma_+$ and $\Delta_+$ will denote positive roots and simple roots with respect to $U_n$. We have $\Sigma = \Sigma_+ \cup (-\Sigma_+)$, and in terms of (1.1) we have:

$$\Sigma_+ = \{\phi_i - \phi_j, \phi_i + \phi_j; 1 \le i < j \le n\} \cup \begin{cases} \{\phi_i \,;\, 1 \le i \le n\} & G_n = \mathrm{SO}(2n+1), \\ \{2\phi_i \,;\, 1 \le i \le n\} & G_n = \mathrm{Sp}(n). \end{cases}$$

Let $\alpha_i = \phi_i - \phi_{i+1}$, $1 \le i \le n-1$, and $\alpha_n = \phi_n$ $(G_n = \mathrm{SO}(2n+1))$, $\alpha_n = 2\phi_n$ $(G_n = \mathrm{Sp}(n))$ and $\alpha_n = \phi_{n-1} + \phi_n$ $(G_n = \mathrm{O}(2n))$. Then $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Now, we recall the computation of [T3, Lemma 4.4].

***Lemma 1.1*** *Fix $i$, $1 \le i \le n$. The set $\{w \in W_n \,;\, w(\Delta \setminus \{\alpha_i\}) > 0\}$ can be described as the disjoint union of the sets $\bigcup_{0 \le j \le i} W_j$, where $W_j$ is the set of all $p\epsilon$ (see the description above) such that the following conditions hold:*

$$\epsilon_k = \begin{cases} 1, & 1 \le k \le j,\ i+1 \le k \le n, \\ -1, & j+1 \le k \le i, \end{cases}$$

*and we have the following:*

- *$p$ restricted to $\{1, \ldots, j\}$ or $\{i+1, \ldots, n\}$ is increasing (but not necessarily when restricted to their union).*
- *$p$ restricted to $\{j+1, \ldots, i\}$ is decreasing.*

The next corollary follows directly from Lemma 1.1. We leave the details to the reader.

***Corollary 1.2*** *Fix $i$, $1 \le i \le n$. Let $w = p\epsilon \in W_j$. (This means $w(\phi_k) = \phi_{p(k)}$, $1 \le k \le j$ or $i+1 \le k \le n$, and $w(\phi_k) = \phi_{p(k)}^{-1}$, $j \le k \le i$ .) Let $u = 1$ if $G_n = \mathrm{SO}(2n+1)$, and $u = 2$ otherwise. Then the set of all roots $\alpha \in \Sigma_+$ such that $w(\alpha) < 0$ is given by the union of the following cases:*

- $\{u\phi_k \,;\, j+1 \le k \le i\}$ *(does not exist for $G_n = \mathrm{O}(2n)$),*
- $\{\phi_k + \phi_l \,;\, j+1 \le k < l \le i\}$,
- $\{\phi_k - \phi_l \,;\, j+1 \le k \le i,\ i+1 \le l \le n\}$,
- $\{\phi_k + \phi_l \,;\, 1 \le k \le j,\ j+1 \le l \le i,\ p(l) < p(k)\}$ ,
- $\{\phi_k - \phi_l \,;\, 1 \le k \le j,\ i+1 \le l \le n,\ p(l) < p(k)\}$,
- $\{\phi_k + \phi_l \,;\, j+1 \le k \le i,\ i+1 \le l \le n,\ p(l) > p(k)\}$.

We denote morphisms $\varphi_i\colon \mathrm{GL}(1) \to T_n$, $1 \le i \le n$, given by $\varphi_i(p) = t$, where $t$ is the matrix given by (1.2), where $t_j = 1$, $j \ne i$ and $t_i = p$. Put

$$\mathrm{Rat}_*(T_n) \simeq \mathbb{Z}\varphi_1 \oplus \mathbb{Z}\varphi_2 \oplus \cdots \oplus \mathbb{Z}\varphi_n. \tag{1.3}$$

We write the group operation in $\mathrm{Rat}_*(T_n)$ additively.

We have the usual pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon\ \mathrm{Rat}(T_n) \times \mathrm{Rat}_*(T_n) \to \mathbb{Z},$$

defined by

$$(a \circ b)(p) = p^{\langle a,b\rangle}, \quad p \in \mathrm{GL}(1).$$

We have $(\phi_i \circ \varphi_j)(p) = 1$ $(i \ne j)$ and $(\phi_i \circ \varphi_i)(p) = p$. Thus $\langle \phi_i, \varphi_j\rangle = \delta_{ij}$. Finally, we have the following list of roots and coroots ($u$ is defined in Corollary 1.2):

- $\alpha = \phi_i \pm \phi_j, \quad \alpha^\vee = \varphi_i \pm \varphi_j,$
- $\alpha = u\phi_i, \quad \alpha^\vee = (2/u)\varphi_i.$

We end this section recalling some representation theory of general linear groups [Ze]. We assume $K$ is non-archimedean here. First, if $\chi$ is a character of $K^\times$, then we can consider it as a character of $\mathrm{GL}(n, K)$: $g \rightsquigarrow \chi(\det g)$, that we usually write as $\chi\mathbf{1}_n$. In this way we get a one-to-one correspondence between characters of $\mathrm{GL}(n, K)$ and $K^\times$ that preserves the sets of unitary and unramified characters, respectively. Further, we have the following lemma [Ze]:

***Lemma 1.3*** *Assume that $\chi, \chi'$ are unitary characters of $K^\times$ and $\alpha, \beta, \alpha', \beta' \in \mathbb{R}$ such that $\alpha + \beta \in \mathbb{Z}_{\ge 0}$ and $\alpha' + \beta' \in \mathbb{Z}_{\ge 0}$ Then we have the following:*

$$\chi|\det|^{(\alpha-\beta)/2}\mathbf{1}_{\mathrm{GL}(\alpha+\beta+1,K)} \hookrightarrow |\cdot|^{-\beta}\chi \times |\cdot|^{-\beta+1}\chi \times \cdots \times |\cdot|^{\alpha}\chi$$
$$|\cdot|^{\alpha}\chi \times |\cdot|^{\alpha-1}\chi \times \cdots \times |\cdot|^{-\beta}\chi \twoheadrightarrow \chi|\det|^{(\alpha-\beta)/2}\mathbf{1}_{\mathrm{GL}(\alpha+\beta+1,K)}.$$

*We write*

$$\zeta(-\beta, \alpha, \chi) := \chi|\cdot|^{(\alpha-\beta)/2}\mathbf{1}_{\mathrm{GL}(\alpha+\beta+1,K)}.$$

*Moreover, $\zeta(-\beta, \alpha, \chi) \times \zeta(-\beta', \alpha', \chi')$ is reducible if and only if $\chi = \chi'$, $\alpha - \alpha' \in \mathbb{Z}$, and one of the following holds:*

$$-\beta \le -\beta' - 1 \le \alpha < \alpha',$$
$$-\beta' \le -\beta - 1 \le \alpha' < \alpha.$$

*Moreover, if $\zeta(-\beta, \alpha, \chi) \times \zeta(-\beta', \alpha', \chi')$ reduces, then it has length 2 and it contains $\zeta(-\beta', \alpha, \chi) \times \zeta(-\beta, \alpha', \chi)$ in its composition series with multiplicity one. More precisely,*

$$\zeta(-\beta', \alpha, \chi) \times \zeta(-\beta, \alpha', \chi) \hookrightarrow \begin{cases} \zeta(-\beta, \alpha, \chi) \times \zeta(-\beta', \alpha', \chi), & \alpha < \alpha' \\ \zeta(-\beta', \alpha', \chi) \times \zeta(-\beta, \alpha, \chi), & \alpha' < \alpha \end{cases}$$

*and*

$$\zeta(-\beta',\alpha,\chi)\times\zeta(-\beta,\alpha',\chi)\twoheadleftarrow\begin{cases}\zeta(-\beta,\alpha,\chi)\times\zeta(-\beta',\alpha',\chi), & \alpha'<\alpha\\ \zeta(-\beta',\alpha',\chi)\times\zeta(-\beta,\alpha,\chi), & \alpha<\alpha'.\end{cases}$$

*(We omit $\zeta(-\beta',\alpha,\chi)$ if $\alpha=-\beta'+1$, and $\zeta(-\beta',\alpha',\chi)$ if $\alpha'=-\beta+1$. In this paper, $\twoheadleftarrow$ and $\twoheadrightarrow$ stand for epimorphisms in the direction of the arrows.)*

## 2 Some Degenerate Eisenstein Series

In this section we give the setup for the computation of poles of some degenerate principal series given in Section 4. We follow Mœglin and Waldspurger [MW3] for the theory of automorphic forms, we use results of Keys and Shahidi [KSh] on normalization of intertwining operators and we follow the results of Tate on $L$-functions associated to grössencharacters [Ta].

We continue using the notation introduced in the previous section. Let $K$ be a global field, $\mathbb{A}$ its ring of adeles and $\mathbb{A}^\times$ its group of ideles. We write $\{v\}$ for the set of places of $K$. For each place $v$ of $K$, we let $K_v$ be its completion at that place. Let $|\cdot|_v$ be the normalized absolute value of $K_v$. Put $|\cdot|=\prod_v|\cdot|_v$. If $v$ is finite (denoted by $v<\infty$), then we let $\mathcal{O}_v$ be the ring of integers of $K_v$ and $\widetilde{\omega}_v$ a generator of the maximal ideal in $\mathcal{O}_v$. Then $\mathbb{A}\simeq\prod_v' K_v$ is a restricted tensor product over all places $\{v\}$. We choose a non-trivial additive character $\psi\colon K\setminus\mathbb{A}\to\mathbb{C}^\times$ and write it as a restricted product of local characters $\psi=\prod_v'\psi_v$. (Note that for all but a finite set of places $v$ we have $\psi_v|_{\mathcal{O}_v}=1$.) We fix a Haar measure $dx=\prod_v' dx_v$ such that $dx_v$ is self-dual with respect to $\psi_v$; this means for any $f\in C_c^\infty(K_v)$ the Fourier transform $\hat{f}(x)=\int_{K_v}f(y)\psi_v(xy)dy_v$ satisfies $f(x)=\int_{K_v}\hat{f}(y)\psi_v(-xy)\,dy_v$. The induced measure on $K\setminus\mathbb{A}$ has total measure equal to one.

Let $\mu\colon K^\times\setminus\mathbb{A}^\times\to\mathbb{C}^\times$ be a (unitary) grössencharacter. We can write it as a restricted tensor product $\mu=\bigotimes_v'\mu_v$ of local characters $\mu_v\colon K_v^\times\to\mathbb{C}^\times$. In the above setup, Tate associated local factors $L(s,\mu_v)$ and $\epsilon(s,\mu_v,\psi_v)$ such that the following holds. First, the product $\epsilon(s,\mu)=\prod_v\epsilon(s,\mu_v,\psi_v)$ is independent of $\psi$ and is a finite product since for all but finite places $\psi_v$ and $\mu_v$ are unramified, and, in that case, $\epsilon(s,\mu_v,\psi_v)=1$. Moreover, we have that the product $L(s,\mu)=\prod_v L(s,\mu_v)$ initially converges for $\mathrm{Re}(s)>1$ and extends to the whole complex plane as a meromorphic function satisfying the following functional equation:

$$L(s,\mu)=\epsilon(s,\mu)L(1-s,\mu^{-1}).$$

Moreover, if $\mu\neq\mathbf{1}$, then $L(s,\mu)$ is holomorphic. $L(s,\mathbf{1})$ has simple poles at $s=0$ and $s=1$ and no other poles.

Let $G_n$ be one of the groups described in Section 1 over $K$. (We remind the reader that $K$ is global here.) As usual, we let $G_n(K)$, $G_n(\mathbb{A})$, $G_n(K_v)$ be the group of $K$-points, $\mathbb{A}$-points, and $K_v$-points of $G_n$, respectively. We have $G_n(\mathbb{A})\simeq\prod_v' G_n(K_v)$ (where the restricted product is taken with respect to $\mathcal{K}_v:=G_n(\mathcal{O}_v)$, $v<\infty$). Analogous notation is used for the torus $T_n$, unipotent radical $U_n$ and the Borel group $B_n$ introduced in Section 1. We fix a maximal compact subgroup $\mathcal{K}=\prod_v\mathcal{K}_v\subset G_n(\mathbb{A})$

such that for any finite place $v$, $\mathcal{K}_v$ is defined as above, and for an archimedean place $v$ we fix some maximal compact subgroup $\mathcal{K}_v$. We have the usual Iwasawa decomposition $G_n(\mathbb{A}) = \mathcal{K}B_n(\mathbb{A})$.

In order to normalize intertwining operators, we need to fix Haar measures on unipotent radicals. We follow [KSh]. For any root subgroup $U_{n,\alpha} \subseteq U_n$, we have $U_{n,\alpha}(\mathbb{A}) \simeq \mathbb{A}$ and we transfer Haar measure from $\mathbb{A}$ to $U_{n,\alpha}(\mathbb{A})$. The Haar measure on $U_{n,\alpha}(\mathbb{A})$ will be denoted by $du_{n,\alpha}$. Similarly, for any place $v$, we have $U_{n,\alpha}(K_v) \simeq K_v$ and we transfer the Haar measure from $K_v$ to $U_{n,\alpha}(K_v)$. The Haar measure on $U_{n,\alpha}(K_v)$ will be denoted by $du_{n,\alpha,v}$. We have $U_{n,\alpha}(\mathbb{A}) \simeq \prod'_v U_{n,\alpha}(K_v)$ and $du_{n,\alpha} = \prod'_v du_{n,\alpha,v}$. Furthermore, this also fixes Haar measures $du_n$ and $du_{n,v}$ on both $\mathbb{A}$-points and $K_v$-points of $U_n \simeq \prod_{\alpha\in\Sigma_+} U_{n,\alpha}$ (the product map is an isomorphism of affine $K$-varieties). The induced measure on $U_n(K) \setminus U_n(\mathbb{A})$ has total volume equal to 1. Also, in this way we obtain a unique Haar measure on any subgroup of $\mathbb{A}$-points or $K_v$-points that can be written as a product of root subgroups. Typically, we use this in the following way. Let $w \in W := W_n$ be an element of the Weyl group of $G_n$ with respect to $T_n$. Then

$$U_n \cap w\overline{U}_n w^{-1} = \prod_{\substack{\alpha\in\Sigma_+ \\ w^{-1}(\alpha)<0}} U_{n,\alpha},$$

and our remark applies to this subgroup.

We also need to fix a representative $\widetilde{w}$ of $w \in W$ properly. We do this as follows. If $\alpha \in \Delta$, the simple reflection $r_\alpha$ has the representative

$$\widetilde{r}_\alpha = \varphi_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\varphi_\alpha \colon \mathrm{SL}(2) \to G_n$ is a $K$-morphism of algebraic groups attached to $\alpha$ [KSh]. In general, for $w \in W$ we obtain the representative in the following way. For any reduced expression $w = r_{\alpha_1} r_{\alpha_2} \cdots r_{\alpha_l}$, we let $\widetilde{w} = \widetilde{r}_{\alpha_1}\widetilde{r}_{\alpha_2} \cdots \widetilde{r}_{\alpha_l}$. The representative $\widetilde{w}$ is independent of the choice of the reduced expression used [KSh, §2]. We write $\widetilde{W}$ for the set of representatives. It is clear $\widetilde{W} \subseteq G_n(K)$.

Now, we discuss automorphic forms following [MW3]. Put

$$\mathbf{s} = (s_1, s_2, \dots, s_n) \in \mathbb{C}^n \simeq \mathrm{Rat}(T_n) \bigotimes_{\mathbb{Z}} \mathbb{C} =: \mathfrak{a}^*_{\mathbb{C}}. \tag{2.1}$$

Next, given a sequence of grössencharacters $K^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$:

$$\boldsymbol{\mu} := (\mu_1, \mu_2, \dots, \mu_n), \tag{2.2}$$

we consider a family of characters $\boldsymbol{\mu}_{\mathbf{s}} \colon T_n(K) \setminus T_n(\mathbb{A}) \to \mathbb{C}^\times$ defined as follows ($t$ is given by (1.2)):

$$\boldsymbol{\mu}_{\mathbf{s}}(t) = \prod_{i=1}^n \mu_i(t_i)|t_i|^{s_i}, \tag{2.3}$$

parametrized by $\mathbf{s} \in \mathfrak{a}^*_{\mathbb{C}}$. All characters given by (2.3) have the same restriction to $T_n(\mathbb{A}) \cap \mathcal{K}$. We write $\underline{\mu}_0$ for that restriction. We also write $\mathrm{Ind}^{\mathcal{K}}_{T_n(\mathbb{A})\cap\mathcal{K}}(\underline{\mu}_0)$ for the space of $\mathcal{K}$-finite functions spanned by the factorizable functions $f = \prod_v f_v \colon \mathcal{K} \to \mathbb{C}$ such that $f(tk) = \underline{\mu}_0(t)f(k)$, for all $t \in T_n(\mathbb{A}) \cap \mathcal{K}$ and $k \in \mathcal{K}$, and for all but a finite number of places $v$, $f_v$ is right $\mathcal{K}_v$-invariant. $\mathcal{K}$ acts by right translations there.

Now, if we take the character (2.3) and $f \in \mathrm{Ind}^{\mathcal{K}}_{T_n(\mathbb{A})\cap\mathcal{K}}(\underline{\mu}_0)$, we define a function $f_{\boldsymbol{\mu}_\mathbf{s}}$ on $G_n(\mathbb{A})$ using the Iwasawa decomposition:

$$f_{\boldsymbol{\mu}_\mathbf{s}}(tuk) := \delta^{1/2}_{B_n}(t)\boldsymbol{\mu}_\mathbf{s}(t)f(k), \quad t \in T_n(\mathbb{A}), u \in U_n(\mathbb{A}), k \in \mathcal{K}.$$

One checks that it is well defined. In the notation of [MW3], the space of such functions is denoted by $A(U_n(\mathbb{A})T_n(K) \setminus G_n(\mathbb{A}), \boldsymbol{\mu}_\mathbf{s})$. It is a space of $\mathcal{K}$-finite automorphic forms. It is a $\prod_{v<\infty} G_n(K_v) \times (\mathfrak{g}_\infty, \mathcal{K}_\infty)$ module, where $\mathfrak{g}_\infty$ is (real) Lie algebra of the group $\prod_{v \text{ arch}} G_n(K_v)$ considered as a real Lie group and $\mathcal{K}_\infty = \prod_{v \text{ arch}} \mathcal{K}_v$. Moreover, as an abstract representation of $\prod_{v<\infty} G_n(K_v) \times (\mathfrak{g}_\infty, \mathcal{K}_\infty)$, $A(U_n(\mathbb{A})T_n(K) \setminus G_n(\mathbb{A}), \boldsymbol{\mu}_\mathbf{s})$ is isomorphic to the global induced representation

$$|\cdot|^{s_1}\mu_1 \times |\cdot|^{s_2}\mu_2 \times \cdots \times |\cdot|^{s_n}\mu_n \rtimes \mathbf{1} = \mathrm{Ind}^{G_n(\mathbb{A})}_{B_n(\mathbb{A})}\big(|\cdot|^{s_1}\mu_1 \otimes |\cdot|^{s_2}\mu_2 \otimes \cdots \otimes |\cdot|^{s_n}\mu_n \otimes \mathbf{1}_{U_n}\big)$$

(normalized induction; $\mathcal{K}$-finite vectors) that factors into the restricted tensor product of local representations:

$$|\cdot|^{s_1}\mu_1 \times |\cdot|^{s_2}\mu_2 \times \cdots \times |\cdot|^{s_n}\mu_n \rtimes \mathbf{1} \simeq \bigotimes_v |\cdot|^{s_1}_v\mu_{1,v} \times |\cdot|^{s_2}_v\mu_{2,v} \times \cdots \times |\cdot|^{s_n}_v\mu_{n,v} \rtimes \mathbf{1}.$$

Let $f \in \mathrm{Ind}^{\mathcal{K}}_{T_n(\mathbb{A})\cap\mathcal{K}}(\underline{\mu}_0)$. Then we define Eisenstein series [MW3, II.1.5]

$$E(f_{\boldsymbol{\mu}_\mathbf{s}}, g) = \sum_{\gamma \in B_n(K)\setminus G_n(K)} f_{\boldsymbol{\mu}_\mathbf{s}}(\gamma \cdot g). \tag{2.4}$$

The constant term along $B_n$ of (2.4) determines the poles and analytic continuation of the Eisenstein series:

$$E_0(f_{\boldsymbol{\mu}_\mathbf{s}}, g) = \sum_{\widetilde{w}\in\widetilde{W}} \int_{U_n(\mathbb{A})\cap w\overline{U}_n(\mathbb{A})w^{-1}} f_{\boldsymbol{\mu}_\mathbf{s}}(\widetilde{w}^{-1}ug). \tag{2.5}$$

Next, the global intertwining operator

$$M(\boldsymbol{\mu}_\mathbf{s}, w)f = \int_{U_n(\mathbb{A})\cap w\overline{U}_n(\mathbb{A})w^{-1}} f_{\boldsymbol{\mu}_\mathbf{s}}(\widetilde{w}^{-1}ug) \tag{2.6}$$

does not depend on the choice of the representative for $w$ in $G_n(K)$. Thus, the constant term given by (2.5) does not depend on the set of representatives $\widetilde{W}$. The global intertwining operator factors into the product of local intertwining operators

$$M(\boldsymbol{\mu}_\mathbf{s}, w)f = \bigotimes_v A(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w})f_v,$$

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where $f = \bigotimes_v f_v \in \mathrm{Ind}_{T_n(\mathbb{A})\cap\mathcal{K}}^{\mathcal{K}}(\underline{\mu}_0)$. Away from poles, $A(\widetilde{w}, \boldsymbol{\mu}_{v,\mathbf{s}})$ intertwines representations:

$$\mathrm{Ind}_{B_n(K_v)}^{G_n(K_v)}(\boldsymbol{\mu}_{v,\mathbf{s}}) \to \mathrm{Ind}_{B_n(K_v)}^{G_n(K_v)}(w(\boldsymbol{\mu}_{v,\mathbf{s}})).$$

Its normalizing factor is defined by

$$r(\boldsymbol{\mu}_{v,\mathbf{s}}, w) = \prod_{\substack{\alpha\in\Sigma_+\\ w(\alpha)<0}} \frac{L(1, \boldsymbol{\mu}_{v,\mathbf{s}} \circ \alpha^\vee)\epsilon(1, \boldsymbol{\mu}_{v,\mathbf{s}} \circ \alpha^\vee, \psi_v)}{L(0, \boldsymbol{\mu}_{v,\mathbf{s}} \circ \alpha^\vee)}.$$

We put

$$R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w}) = r(\boldsymbol{\mu}_{v,\mathbf{s}}, w)A(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w}).$$

The following theorem summarizes the basic properties of the normalization.

***Theorem 2.1*** ([KSh]) *Under the above assumptions, we have the following:*

(i) $R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w}_1\widetilde{w}_2) = R(w_2(\boldsymbol{\mu}_{v,\mathbf{s}}), \widetilde{w}_1)R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w}_2)$.

(ii) $R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w})R(w(\boldsymbol{\mu}_{v,\mathbf{s}}), \widetilde{w}^{-1}) = id$.

(iii) *Assume $v < \infty$. Then if $\psi_v$ and $\boldsymbol{\mu}_{v,\mathbf{s}}$ are unramified and $f_{\boldsymbol{\mu}_{v,\mathbf{s}}} \in \mathrm{Ind}_{B_n(K_v)}^{G_n(K_v)}(\boldsymbol{\mu}_{v,\mathbf{s}})$ is $K_v$-invariant, $f_{\boldsymbol{\mu}_{v,\mathbf{s}}}(1) = 1$, then $R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w})f_{\boldsymbol{\mu}_{v,\mathbf{s}}} = f_{w(\boldsymbol{\mu}_{v,\mathbf{s}})}$.*

(iv) *Assume that $\boldsymbol{\mu}_{v,\mathbf{s}}$ is in the closure of positive Weyl chamber with respect to $\Sigma_+$. (This means that for any $\alpha \in \Sigma_+$, the character $\boldsymbol{\mu}_{v,\mathbf{s}} \circ \alpha^\vee$ can be written as a product of a unitary character and $|\cdot|_v^a$, for some $a \geq 0$.) If $w$ is the longest element of the Weyl group $W$, then $R(\boldsymbol{\mu}_{v,\mathbf{s}}, \widetilde{w})$ is holomorphic and non-zero.*

We will need the next extension of the above results on Eisenstein series. The reader may skip this part on the first reading and refer back only when a result is needed. Let $P = MN$ be a maximal ($K$-)parabolic subgroup of $G_n$. Assume $M \simeq \mathrm{GL}(m) \times G_{n'}$. We will use irreducible subspace $(V, \Pi)$ (for $\prod_{v<\infty} G_n(K_v) \times (\mathfrak{g}_\infty, \mathcal{K}_\infty)$) of the space of automorphic forms $A(G_{n'}(K) \setminus G_{n'}(\mathbb{A}))$. Let

$$\mu\colon K^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$$

be a (unitary) grössencharacter.

We now consider the global induced representation:

$$|\det|^s \mathbf{1}_{\mathrm{GL}(m,\mathbb{A})}\mu \rtimes \Pi. \tag{2.7}$$

The induced representation in (2.7) is realized on the space of ($\mathcal{K}$-finite) automorphic forms $A(N(\mathbb{A})M(K) \setminus G_n(\mathbb{A}), |\det|^s\mu \otimes \Pi)$ consisting of the functions

$$f_s\colon N(\mathbb{A})M(K) \setminus G_n(\mathbb{A}) \to \mathbb{C},$$

such that, for $k \in \mathcal{K}$, we have that the function

$$(m_1, m') \in \mathrm{GL}(m, \mathbb{A}) \times G_{n'}(\mathbb{A}) \rightsquigarrow |\det m_1|^{-s}\mu^{-1}(\det m_1)\delta_P^{-1/2}(m_1)f_s((m_1, m')k)$$

is independent of $s$ and, considered as a function of the second argument, belongs to $V$.

We consider a degenerate Eisenstein series as a meromorphic function of $s \in \mathbb{C}$:

$$E(f_s, g) = \sum_{\gamma \in P(K) \backslash G_n(K)} f_s(\gamma g), \quad f_s \in A\big(N(\mathbb{A})M(K) \setminus G_n(\mathbb{A}), |\det|^s \mu \otimes \Pi\big). \tag{2.8}$$

We write $\alpha_i$ for the simple root in $\Delta$, so that $P$ is attached to $\Delta \setminus \{\alpha_i\}$. Now, we are ready to state the main result in this section.

***Lemma 2.2*** *Assume that $\Pi$ is concentrated on the Borel subgroup (that is, in our fixed automorphic realization $V$ some constant term along $B_{n'}$ does not vanish). Then the Eisenstein series given by* (2.8) *is also concentrated on the Borel subgroup, and its constant term along $B_n$ is given by*

$$E_0(f_s, g) = \sum_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} \int_{U_n(\mathbb{A}) \cap w\overline{U}_n(\mathbb{A}) w^{-1}} (f_s)_0(\widetilde{w}^{-1} ug)\, du, \tag{2.9}$$

*where $(f_s)_0$ is defined by*

$$(f_s)_0(g) = \int_{U_{n'}(K) \backslash U_{n'}(\mathbb{A})} f_s(u'g)\, du'. \tag{2.10}$$

**Proof** This is elementary, but we were unable to find a convenient reference, so we present the proof. Using the Bruhat decomposition (disjoint union)

$$G_n = \bigcup_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} Pw^{-1}U_n = \bigcup_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} Pw^{-1}(U_n \cap w\overline{U}_n w^{-1}),$$

we compute, by definition:

$$\begin{aligned}
E_0(f_s, g) &= \int_{U_n(K) \backslash U_n(\mathbb{A})} E(f_s, ug)\, du = \int_{U_n(K) \backslash U_n(\mathbb{A})} \Big( \sum_{\gamma \in P(K) \backslash G_n(K)} f_s(\gamma ug) \Big)\, du \\
&= \int_{U_n(K) \backslash U_n(\mathbb{A})} \Big( \sum_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} \Big( \sum_{\gamma \in (U_n \cap w\overline{U}_n w^{-1})(K)} f_s(w^{-1}\gamma ug) \Big) \Big)\, du \\
&= \sum_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} \int_{U_n(K) \backslash U_n(\mathbb{A})} \Big( \sum_{\gamma \in (U_n \cap w\overline{U}_n w^{-1})(K)} f_s(w^{-1}\gamma ug) \Big)\, du \\
&= \sum_{\substack{w \in W \\ w(\Delta \backslash \{\alpha_i\}) > 0}} \int_{(U_n \cap wU_n w^{-1})(K) \backslash U_n(\mathbb{A})} f_s(w^{-1}ug)\, du,
\end{aligned}$$

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since for $\mathrm{Re}(s)$ large enough the Eisenstein series converges uniformly on compact sets. Finally, the inner integral is given by

$$\begin{aligned}\int_{(U_n\cap wU_nw^{-1})(K)\backslash U_n(\mathbb{A})} f_s(w^{-1}ug)\,du &= \iiint f_s(u_Nu'w^{-1}ug)\,du_N du' du\\ &= \iint f_s(u'w^{-1}ug)\,du' du\\ &= \int_{(U_n\cap w\overline{U}_nw^{-1})(\mathbb{A})} (f_s)_0(w^{-1}ug)\,du,\end{aligned}$$

where the triple integral is taken over the set

$$(N\cap wU_nw^{-1})(K)\setminus(N\cap wU_nw^{-1})(\mathbb{A})\times U_{n'}(K)\setminus U_{n'}(\mathbb{A})\times(U_n\cap w\overline{U}_nw^{-1})(\mathbb{A}),$$

while the double integral is taken over the set

$$U_{n'}(K)\setminus U_{n'}(\mathbb{A})\times(U_n\cap w\overline{U}_nw^{-1})(\mathbb{A}). \qquad \blacksquare$$

We will explain how we use the above results. First, we note that for $m'\in G_{n'}(\mathbb{A})$ and $k\in\mathcal{K}$, the function (see (2.10))

$$(f_s)_0(m'k)=\int_{U_{n'}(K)\backslash U_{n'}(\mathbb{A})} f_s(u'm'k)\,du'. \tag{2.11}$$

is independent of $s$. It represents a canonical map $V\longrightarrow V_0$ from the automorphic realization of $\Pi$ to its constant term along $B_{n'}$. We know that $V_0$ decomposes as the direct sum over generalized eigenspaces of $T_{n'}(\mathbb{A})$. According to [La, Lemma 6], for any such generalized eigenspace of $T_{n'}(\mathbb{A})$, we may find a character $\boldsymbol{\mu}'\colon T_n(\mathbb{A})\longrightarrow\mathbb{C}^\times$, necessarily trivial on $T_n(K)$, and $f_s\neq 0$, such that:

$$(f_s)_0(u't'm'k)=\delta_{B_{n'}}(t')^{1/2}\boldsymbol{\mu}'(t')(f_s)_0(m'k),$$

$t'\in T_{n'}(\mathbb{A}), u'\in U_{n'}(\mathbb{A}), m'\in G_{n'}(\mathbb{A}), k\in\mathcal{K}$.

This is, of course, independent of $s$, and this means that we have the following well-defined function:

$$(f_s)_0(t'm'm_1nk)=|\det m_1|^s\mu(\det m_1)\delta_P^{1/2}(m_1)\delta_{B_{n'}}(t')^{1/2}\boldsymbol{\mu}'(t')(f_s)_0(m'k), \tag{2.12}$$

where we recall that $m_1\in\mathrm{GL}(m,\mathbb{A})$, $n\in N(\mathbb{A})$. Defining $\delta$ to be the modular character of $T_{n'}(\mathbb{A})$ on $N(\mathbb{A})$, we put $\delta^{-1/2}\boldsymbol{\mu}'=\mu_1\otimes\cdots\otimes\mu_{n'}$. Then (2.11) defines an embedding

$$|\det|^s\mu\rtimes\Pi\hookrightarrow|\cdot|^{s-(m-1)/2}\mu\times\cdots\times|\cdot|^{s+(m-1)/2}\mu\times\mu_1\times\cdots\times\mu_{n'}\rtimes\mathbf{1}, \tag{2.13}$$

which is induced from the one given above,

$$\Pi\hookrightarrow\mu_1\times\cdots\times\mu_{n'}\rtimes\mathbf{1}, \tag{2.14}$$

that came from the computation of the constant term on $V$.

This discussion proves the following lemma:

***Lemma 2.3*** *Assume that $f_s$ is as above and that its constant term $(f_s)_0$ is in the induced representation in* (2.13)*. Put $\boldsymbol{\mu}_{\mathbf{s}} = |\cdot|^{s-(m-1)/2}\mu \otimes \cdots \otimes |\cdot|^{s+(m-1)/2}\mu \otimes \mu_1 \otimes \cdots \otimes \mu_{n'}$. Then the poles of the Eisenstein series in* (2.8) *are the same as the poles of the sum of intertwining operators:*

$$\sum_{w, w(\Delta \setminus \{\alpha_i\})>0} M(\boldsymbol{\mu}_{\mathbf{s}}, w)(f_s)_0.$$

The Eisenstein series (2.8) strongly depends on the choice of automorphic realization of $\Pi$. The same holds for the embedding (2.14). In Section 4 we will carefully choose both in order to compute poles of Eisenstein series given by (2.8).

## 3 Discrete Series, Their Duals, and Poles of Normalized Intertwining Operators

Let $F$ be a local non-archimedean field of characteristic zero. In this section we give a description of discrete series supported on the minimal parabolic subgroup given by Mœglin and Tadić [Mœ3, MT], and we also describe their duals [A]. We should mention that in that particular case the results of [Mœ3, MT] hold without further assumptions. We also show that certain normalized intertwining operators are holomorphic.

We denote by $D = D_{G_n}$ the involution introduced by Aubert for general connected groups in [A] and extended to $\mathrm{O}(2n, F)$ by Jantzen [Jan1].

We now give the description of discrete series in three steps, compute their duals and show that certain normalized intertwining operators are holomorphic.

### 3.1 Trivial Representation $\mathbf{1} \in \operatorname{Irr} G_0(F)$ as a Supercuspidal Representation

Let $(\mathrm{Jord}(\mathbf{1}), \mathbf{1}, \epsilon)$ be its admissible triple. We recall that $\mathrm{Jord}(\mathbf{1})$ consists of the pairs $(a, \rho)$ ($\rho \cong \widetilde{\rho}$ is a supercuspidal representation of some $\mathrm{GL}(m_\rho, F)$, $a > 0$ is an integer) such that $a$ is even if and only if $L(s, \rho, r)$ has a pole at $s = 0$, and

$$\delta\big(\big[\,|\det|^{-(a-1)/2}\rho,\ |\det|^{(a-1)/2}\rho\big]\big) \rtimes \mathbf{1}$$

is irreducible. Also, the local $L$-function $L(s, \rho, r)$ is the one defined by Shahidi [Sh2, Sh3], where $r = \wedge^2 \mathbb{C}^{m_\rho}$ is the exterior-square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $\mathrm{GL}(m_\rho, \mathbb{C})$ if $G_n$ is a symplectic or even-orthogonal group and $r = \mathrm{Sym}^2\, \mathbb{C}^{m_\rho}$ is the symmetric-square representation of the standard representation on $\mathbb{C}^{m_\rho}$ of $\mathrm{GL}(m_\rho, \mathbb{C})$ if $G_n$ is an odd-orthogonal group. Those two conditions can be analyzed using the theory of $R$-groups (extended to even-orthogonal groups in [Mœ3, A3]) and using explicit formulas for Planceherel measures of [Sh3] (using [Mœ3, A3] for even-orthogonal groups). We immediately obtain

$$\mathrm{Jord}(\mathbf{1}) = \begin{cases} \varnothing & G_0(F) = \mathrm{SO}(1, F), \mathrm{O}(0, F), \\ (1, \mathbf{1}_{F^\times}) & G_0(F) = \mathrm{Sp}(0, F). \end{cases} \tag{3.1}$$

(See also the displayed formula after (12-3) in [MT] for $G_0(F) = \mathrm{SO}(1, F), \mathrm{Sp}(0, F)$.) The third component $\epsilon = \varnothing$ is an empty function.

In general, the discrete series $\sigma \in \operatorname{Irr} G_n(F)$ that we consider here are attached to admissible triples $(\text{Jord}, \mathbf{1}, \epsilon)$ such that Jord consists of pairs $(a, \mu)$, where $\mu$ is a quadratic character of $F^\times$. The above parity condition for $a$ means that $a$ is even if and only if $G_n(F)$ is an odd-orthogonal group. Then Jord is determined completely by requiring that

$$\delta\big(\,[\,|\cdot|^{-(a-1)/2}\mu, |\cdot|^{(a-1)/2}\mu]\,\big) \rtimes \sigma$$

is irreducible. This then gives finiteness of Jord and

$$\sum_{(a,\mu)\in\text{Jord}} a = \begin{cases} 2n & G_n(F) = \mathrm{SO}(2n+1, F), \mathrm{O}(2n, F), \\ 2n+1 & G_n(F) = \mathrm{Sp}(n, F). \end{cases}$$

We write $\text{Jord}_\mu = \{a\, ;\, (a, \mu) \in \text{Jord}\}$, and for $a \in \text{Jord}_\mu$, we write $a_-$ for the largest element of $\text{Jord}_\rho$ that is strictly less than $a$ (if one exists). The inductive definition of the function $\epsilon$ from a subset of $\text{Jord} \cup \text{Jord} \times \text{Jord}$ into $\{\pm 1\}$ follows next (§3.2 and §3.3).

### 3.2 Strongly Positive Discrete Series

These are defined in [Mœ3, §1]. Here we use the explicit description from [MT, §7]. In all cases $\mathbf{1} \in \operatorname{Irr} G_0(F)$ is a strongly positive discrete series. For even-orthogonal groups, this is the only strongly positive discrete series. For symplectic groups, for each $n \in \mathbb{Z}_{>0}$, there is a strongly positive discrete series $\sigma \in \operatorname{Irr} G_n(F)$, the so-called Steinberg representation. It is the unique irreducible subrepresentation of

$$|\cdot|^n \times \cdots \times |\cdot| \rtimes \mathbf{1}.$$

We have $\text{Jord} = \{(2n+1, \mathbf{1}_{F^\times})\}$ and $\epsilon = \varnothing$. For odd-orthogonal groups, we take different quadratic characters $\mu_1, \dots, \mu_k$ of $F^\times$ and integers $l_1, \dots, l_k \in \mathbb{Z}_{>0}$ such that the induced representation

$$|\cdot|^{l_k-1/2}\mu_k \times \cdots \times |\cdot|^{1/2}\mu_k \times \cdots \times |\cdot|^{l_1-1/2}\mu_1 \times \cdots \times |\cdot|^{1/2}\mu_1 \rtimes \mathbf{1} \tag{3.2}$$

has a unique irreducible subrepresentation, say $\sigma$. This representation is strongly positive, $\text{Jord} = \{(2l_1+1, \mu_1), \dots, (2l_k+1, \mu_k)\}$ and $\epsilon(2l_i, \mu_i) = 1$, $i = 1, \dots, k$.

Finally, we compute $D(\sigma)$ when $\sigma \in \operatorname{Irr} G_n(F)$, $n > 0$, is strongly positive. If $\sigma$ is the Steinberg representation of $\mathrm{Sp}(n, F)$, then $D(\sigma)$ is trivial representation, and thus the unique irreducible quotient of $|\cdot|^n \times |\cdot|^{n-1} \times \cdots \times |\cdot| \rtimes \mathbf{1}$.

Now, we compute duals of strongly positive discrete series given by (3.2).

***Lemma 3.1*** *Assume that $\sigma$ is a subrepresentation of* (3.2)*. Then $D(\sigma)$ is the unique irreducible (Langlands) quotient of the induced representation in* (3.2)*:*

$$|\cdot|^{l_k-1/2}\mu_k \times \cdots \times |\cdot|^{1/2}\mu_k \times \cdots \times |\cdot|^{l_1-1/2}\mu_1 \times \cdots \times |\cdot|^{1/2}\mu_1 \rtimes \mathbf{1} \twoheadrightarrow D(\sigma).$$

*(Technically, the characters in the induced representation do not appear in the order required for Langlands quotient data (though may easily be "commuted" into the required order using* [Ze]*). In particular,*

$$\zeta(1/2, l_k - 1/2, \mu_k) \times \cdots \times \zeta(1/2, l_1 - 1/2, \mu_1) \rtimes \mathbf{1} \twoheadrightarrow D(\sigma).$$

**Proof** This follows from the fact that [A, Theorem 1.7(2)], (and its appropriate extension to even-orthogonal groups [Jan1]) shows that $D(\sigma)$ contains $|\cdot|^{-l_k+1/2}\mu_k \otimes \cdots \otimes |\cdot|^{-1/2}\mu_k \otimes \cdots \otimes |\cdot|^{-l_1+1/2}\mu_1 \otimes \cdots \otimes |\cdot|^{-1/2}\mu_1$ in its Jacquet module with respect to the Borel subgroup. The remainder of the lemma is clear. ■

Hence, we have the following corollary:

***Corollary 3.2*** *Assume that $\sigma$ is a subrepresentation of* (3.2)*. Then $D(\sigma)$ is the unique irreducible (Langlands) subrepresentation of the induced representation in* (3.2)*:*

$$D(\sigma) \hookrightarrow |\cdot|^{-l_k+1/2}\mu_k \times \cdots \times |\cdot|^{-1/2}\mu_k \times \cdots \times |\cdot|^{-l_1+1/2}\mu_1 \times \cdots \times |\cdot|^{-1/2}\mu_1 \rtimes \mathbf{1}.$$

*(See the comment after the first displayed formula in Lemma* 3.1.*) In particular,*

$$D(\sigma) \hookrightarrow \zeta(-l_k + 1/2, -1/2, \mu_k) \times \cdots \times \zeta(-l_1 + 1/2, -1/2, \mu_1) \rtimes \mathbf{1}.$$

Now, we consider the normalized intertwining operators defined in the last section. We remark that $\zeta(1/2, l_i - 1/2, \mu_i) = |\det|^{l_i/2}\mu_i \mathbf{1}_{\mathrm{GL}(l_i,F)}$. We also use Lemma 3.1 and Corollary 3.2 to define strongly positive discrete series as follows. We let $\sigma_0 = \mathbf{1}$ and $D(\sigma_0) = \mathbf{1}$. Inductively, we define $\sigma_i$, $(1 \le i \le k)$, as follows:

$$\zeta(1/2, l_i - 1/2, \mu_i) \rtimes D(\sigma_{i-1}) \twoheadrightarrow D(\sigma_i) \quad (1 \le i \le k).$$

Clearly, $\sigma_k \simeq \sigma$ and $\sigma_i \in \mathrm{Irr}\, G_{l_1+\cdots+l_i}(F)$, $(1 \le i \le k)$. Put

$$\begin{aligned}\boldsymbol{\mu}_s = |\cdot|^{s-(l_i-1)/2}\mu_i \otimes \cdots \otimes |\cdot|^{s+(l_i-1)/2}\mu_i \otimes |\cdot|^{-1/2}\mu_{i-1} \otimes \cdots \\ \otimes |\cdot|^{-l_{i-1}+1/2}\mu_{i-1} \otimes \cdots \otimes |\cdot|^{-1/2}\mu_1 \otimes \cdots \otimes |\cdot|^{-l_1+1/2}\mu_1.\end{aligned}$$

We consider the following diagram of intertwining operators:

$$\text{(3.3)} \qquad \begin{array}{ccccc} 0 \longrightarrow & |\det|^s \mathbf{1}_{\mathrm{GL}(l_i,F)}\mu_i \rtimes D(\sigma_{i-1}) & \xrightarrow{i_s} & \mathrm{Ind}_{B_{l_1+\cdots+l_i}}^{G_{l_1+\cdots+l_i}}(\boldsymbol{\mu}_s) \\ & & & \Big\downarrow {\scriptstyle R(\boldsymbol{\mu}_s,\widetilde{w})} \\ & & & \mathrm{Ind}_{B_{l_1+\cdots+l_i}}^{G_{l_1+\cdots+l_i}}(w(\boldsymbol{\mu}_s)), \end{array}$$

where $w \in W$ is any element such that $w(\Delta \setminus \{\alpha_{l_i}\}) > 0$ (see Lemma 1.1). The inclusion $i_s$ does not depend on $s$ if we realize all representations in (3.3) in the compact picture, which is necessary to compute the poles. It is holomorphic in $s$.

***Lemma 3.3*** *The composition of intertwining operators in* (3.3) *is holomorphic at $s = l_i/2$.*

**Proof** In view of Lemma 1.1, at $s = l_i/2$ the vertical operator in (3.3) can be decomposed into a product of operators induced from (normalized) long-intertwining operators of rank one in $\mathrm{GL}(2, F)$ and in $G_1(F)$. Those operators are holomorphic by Theorem 2.1(iv). ■

### 3.3 The General Case

In general, we first present the inductive definition of an admissible triple as well as the description of all discrete series (that are subquotients of principal series).

Let $\sigma \in \operatorname{Irr} G_n(F)$ be a discrete series attached to its admissible triple $(\mathrm{Jord}, \mathbf{1}, \epsilon)$. Let $\mu$ be some quadratic character of $F^\times$ such that there is an $a \in \mathrm{Jord}_\mu$ such that $a_-$ is defined and $\epsilon(a,\mu)\cdot\epsilon(a,\mu)^{-1} = 1$. Put $a = 2b+1$ and $a_- = 2b_- + 1$. Put $\mathrm{Jord}'' = \mathrm{Jord}\setminus\{(a,\mu),(a_-,\mu)\}$, and consider the restriction $\epsilon''$ of $\epsilon$ to $\mathrm{Jord}''$. Let $\sigma''$ be the discrete series attached to the admissible triple $(\mathrm{Jord}'', \mathbf{1}, \epsilon'')$. Then we have the following:

$$\sigma \hookrightarrow \delta\big(\big[|\cdot|^{-b_-}\mu, |\cdot|^{b}\mu\big]\big) \rtimes \sigma''. \tag{3.4}$$

In the appropriate normalized Jacquet module of the induced representation in (3.4), the representation $\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b}\mu])\otimes\sigma''$ is contained exactly twice. Moreover, the induced representation $\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b}\mu]) \rtimes \sigma''$ has exactly two (non-equivalent) irreducible subrepresentations, say $\sigma$ and $\sigma_1$. Next, both irreducible subrepresentations $\sigma$ and $\sigma_1$ are in the discrete series and their triples differ in $\epsilon$ only. Moreover, the induced representation

$$\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b_-}\mu]) \rtimes \sigma''$$

is a direct sum of two non-equivalent tempered representations $\tau_\pm$, and there exists a unique $\tau \in \{\tau_-,\ \tau_+\}$ such that

$$\sigma \hookrightarrow \delta\big(\big[|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu\big]\big) \rtimes \tau. \tag{3.5}$$

Finally, $\sigma$ is the unique irreducible subrepresentation in the induced representation in (3.5) since $\delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu]) \otimes \tau$ appears in appropriate normalized Jacquet module of $\delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu]) \rtimes \tau$ exactly once [MT]. (The other representation $\tau_1 \in \{\tau_-, \tau_+\}$ is attached to $\sigma_1$.)

We have the following lemma:

***Lemma 3.4*** *Using the above notation, we have the following:*

$$\begin{aligned} &D(\sigma) \hookrightarrow \mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2} \rtimes D(\tau),\\ &\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{(b+b_-+1)/2} \rtimes D(\tau) \twoheadrightarrow D(\sigma). \end{aligned} \tag{3.6}$$

*Moreover, $D(\sigma)$ is the unique irreducible subrepresentation (resp., irreducible quotient) of the first (resp., second) induced representation in* (3.6)*. Finally, assuming that $D(\sigma'')$ is unitary, the induced representation*

$$\mu\mathbf{1}_{\mathrm{GL}(2b_-+1,F)} \rtimes D(\sigma'') \tag{3.7}$$

*is a direct sum of two non-equivalent irreducible representations $D(\tau_\pm)$. (Actually, this can be proved without the assumption on unitarity of $D(\sigma'')$, but it is not necessary for our purpose.)*

**Proof** First, [A, Theorem 2.3] implies the following:

$$D(\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b_-}\mu])) = \mu \mathbf{1}_{\mathrm{GL}(2b_-+1,F)},$$
$$D(\delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu])) = \mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{(b+b_-+1)/2}. \tag{3.8}$$

Next, [A, Jan1] imply that induction commutes with the involution $D$ on the level of Grothendieck groups. This immediately shows that the representation $D(\sigma)$ is an irreducible subquotient of the appropriate induced representation in (3.6). Since the appropriate Jacquet modules of $\sigma$ and $\delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu]) \rtimes \sigma''$ contain

$$\delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu]) \otimes \sigma''$$

with multiplicity one, [A, Theorem 1.7(2)] (and its appropriate extension to even-orthogonal groups [Jan1]) shows that $\mu|\det|^{-(b-b_-)/2}\mathbf{1}_{\mathrm{GL}(b+b_-+1,F)} \otimes D(\sigma'')$ is contained in the appropriate Jacquet module of $D(\sigma)$ and $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2} \rtimes D(\tau)$ with multiplicity one. It is also contained in an irreducible subrepresentation of $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2} \rtimes D(\tau)$ by Frobenius reciprocity. Now, the inclusion in (3.6) is clear.

It remains to prove that $D(\sigma)$ is a quotient of $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{(b+b_-+1)/2} \rtimes D(\tau)$. We note that if $G_n(F) = \mathrm{SO}(2n+1,F)$ or $G_n(F) = \mathrm{O}(2n,F)$, then all the irreducible representations of $G_n(F)$ are self-contragredient [MVW]. Thus, the epimorphism in (3.6) follows from the inclusion in (3.6). Let $G_n(F) = \mathrm{Sp}(n,F)$. Assume that $\eta \in G\mathrm{Sp}(n,F)$ (the corresponding similitude group) has similitude $-1$. Then, for $\pi \in \mathrm{Irr}\,\mathrm{Sp}(n,F)$, we have that $\pi^{\eta}(g) := \pi(\eta \cdot g \cdot \eta^{-1})$ is the contragredient of $\pi$ [MVW]. Assume that $D(\tau)$ is a representation of $\mathrm{Sp}(n',F)$; then there is a standard Levi subgroup in $G\mathrm{Sp}(n,F)$ isomorphic to $\mathrm{GL}(n-n',F) \times G\mathrm{Sp}(n',F)$. We take $\eta = (1,\eta')$, where $\eta'$ has similitude $-1$. Now, the inclusion in (3.6) implies

$$D(\sigma)^{\eta} \hookrightarrow \mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2} \rtimes D(\tau)^{\eta'}.$$

This means

$$\widetilde{D(\sigma)} \hookrightarrow \mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2} \rtimes \widetilde{D(\tau)}.$$

Taking contragredients, we obtain the epimorphism in (3.6). The last statement in the lemma has a similar proof. ■

***Lemma 3.5*** *Assume that $D(\sigma'')$ is unitary. Then we have the following:*

$$\mu|\det|^{(b-b_-)/2}\mathbf{1}_{\mathrm{GL}(b+b_-+1,F)} \rtimes D(\sigma'') \twoheadrightarrow D(\sigma) \oplus D(\sigma_1),$$
$$D(\sigma) \oplus D(\sigma_1) \hookrightarrow \mu|\det|^{-(b-b_-)/2}\mathbf{1}_{\mathrm{GL}(b+b_-+1,F)} \rtimes D(\sigma'').$$

**Proof** This proof is similar to the one given in the previous lemma. We just need to check

$$D(\sigma) \oplus D(\sigma_1) \hookrightarrow \mu|\det|^{-(b-b_-)/2}\mathbf{1}_{\mathrm{GL}(b+b_-+1,F)} \rtimes D(\sigma''). \tag{3.9}$$

First, the arguments of Lemma 3.4 can equally well be applied to $\sigma_1$ and $\tau_1$, where $\tau_1 \in \{\tau_-, \tau_+\}$ is unique such that

$$\sigma_1 \hookrightarrow \delta\big(\big[|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu\big]\big) \rtimes \tau.$$

Now, we note that Lemma 3.3 and [Ze] imply

$$\begin{aligned}\zeta(-b, b_-, \mu) \rtimes D(\sigma'') \hookrightarrow \zeta(-b, -b_-1, \mu) \times \zeta(-b_-, b_-, \mu) \rtimes D(\sigma'')\\ \simeq \zeta(-b, -b_-1, \mu) \rtimes \tau \oplus \zeta(-b, -b_-1, \mu) \rtimes \tau_1,\end{aligned}$$

where $\zeta(-b, -b_-1, \mu)$ is another notation for $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{-(b+b_-+1)/2}$. Also, Lemma 3.3 implies

$$D(\sigma) \oplus D(\sigma_1) \hookrightarrow \zeta(-b, -b_-1, \mu) \rtimes \tau \oplus \zeta(-b, -b_-1, \mu) \rtimes \tau_1.$$

Now, we obtain (3.9) since again it follows by counting occurrences of

$$\delta\big(\big[|\cdot|^{-b_-}\mu, |\cdot|^{b}\mu\big]\big) \otimes \sigma'',$$

one each in the Jacquet modules of $\sigma$ and $\sigma_1$ and two in the Jacquet module of $\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b}\mu]) \rtimes \sigma''$. It can also be proved that it appears with multiplicity two in the Jacquet module of $\delta([|\cdot|^{-b_-}\mu, |\cdot|^{b_-}\mu]) \times \delta([|\cdot|^{b_-+1}\mu, |\cdot|^{b}\mu]) \rtimes \sigma''$. The last statement can be proved using methods of [M3]. We omit the details since the proof of a similar statement can be found in the proof of [M3, Theorem 3.1] (see the paragraph where (3.4) appears). ■

***Remark 3.1*** It would be interesting to determine $D(\sigma)$ in the Langlands classification. We plan to return to that problem on some later occasion. The case of general linear groups is considered in [Jan2]. (See also [MW1].)

In the remainder of this section we compute the poles of some normalized intertwining operators defined in the last section on $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{s} \rtimes D(\tau)$, $s = (b+b_-+1)/2$. We remind the reader that we use the notation of Lemma 3.4. In particular, $D(\sigma)$ is the unique irreducible quotient there. The normalized intertwining operators depend on the choice of inclusion

$$D(\tau) \hookrightarrow \mu_1 \times \cdots \times \mu_{n'} \rtimes \mathbf{1}, \tag{3.10}$$

where characters $\mu_i$ are not related to the quadratic characters considered in Lemma 3.1 and will be specified in the proof of Lemma 3.6 below. Next, (3.10) implies

$$\begin{aligned}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^{s} \rtimes D(\tau) \hookrightarrow |\cdot|^{s-(b-b_-1)/2}\mu\times\\ \cdots \times |\cdot|^{s+(b-b_-1)/2}\mu \times \mu_1 \times \cdots \times \mu_{n'} \rtimes \mathbf{1}.\end{aligned} \tag{3.11}$$

We remark that $\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu|\det|^s \rtimes D(\tau)$ is induced from the maximal parabolic subgroup determined by $\Delta \setminus \{\alpha_{b-b_-}\}$ (see Section 1 for notation).

We put

$$\boldsymbol{\mu}_s := |\cdot|^{s-(b-b_--1)/2}\mu \otimes \cdots \otimes |\cdot|^{s+(b-b_--1)/2}\mu \otimes \mu_1 \otimes \cdots \otimes \mu_{n'}.$$

Now, we state and prove the main result of this subsection.

***Lemma 3.6*** *The embeddings* (3.10) *and* (3.11) *are specified by* (3.13) *and* (3.14) *below. (They are actually specified for odd-orthogonal groups only as the notationally most complicated case. The reader can easily make adjustments to other two cases.) Assume that $D(\sigma'')$ is unitarizable. (This is only to be able to use Lemma* 3.4*. Eventually, we will prove the unitarity $D(\sigma)$ by induction, and that is why we allow that kind of assumption here.) Then we have the following:*

(i) *We denote the inclusion from* (3.11) *by $i_s$. (It does not depend on $s$ if we realize all representations in the compact picture, which is necessary to compute the poles. It is holomorphic in $s$.) Then in the diagram*

$$\begin{tikzcd} 0 \arrow[r] & |\det|^s\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau) \arrow[r, "i_s"] & \mathrm{Ind}_{B_n}^{G_n}(\boldsymbol{\mu}_s) \arrow[d, "R(\boldsymbol{\mu}_s,\widetilde{w})"] \\ & & \mathrm{Ind}_{B_n}^{G_n}(w(\boldsymbol{\mu}_s)), \end{tikzcd} \tag{3.12}$$

*the composition $R(\boldsymbol{\mu}_s,\widetilde{w})\circ i_s$ is holomorphic at $s=(b+b_-+1)/2$, for any $w\in W$ such that $w(\Delta\setminus\{\alpha_{b-b_-}\})>0$.*

(ii) *If we denote by $w_0$ the unique longest element such that $w_0(\Delta\setminus\{\alpha_{b-b_-}\})>0$, then*

$$\mathrm{Im}\, R(\boldsymbol{\mu}_\mathbf{s},\widetilde{w}_0)\circ i_s \simeq D(\sigma)$$

*at $s=(b+b_-+1)/2$.*

(iii) *$w_0(\boldsymbol{\mu}_\mathbf{s}) \neq w(\boldsymbol{\mu}_\mathbf{s})$, $w(\Delta\setminus\{\alpha_{b-b_-}\})>0$, $w\neq w_0$, at $s=(b+b_-+1)/2$.*

**Proof** We give the proof in the case of odd-orthogonal groups. The other two cases are similar. First, by the inductive definition of discrete series, Lemma 3.1 and Lemma 3.5, there are quadratic characters $\lambda_1,\dots,\lambda_l$, $\lambda_1',\dots,\lambda_k'$ of $F^\times$, $a_1,\dots,a_l$, $b_1,\dots,b_l,c_1,\dots,c_k \in (1/2)\mathbb{Z}_{\geq 0}$, $b_i<a_i$, $2a_i+1, 2b_i+1 \in \mathrm{Jord}(\sigma'')_{\lambda_i}$, $\epsilon(2a_i+1,\lambda_i)\epsilon(2b_i+1,\lambda_i)^{-1}=1$, $2c_i+1\in\mathrm{Jord}(\sigma'')_{\lambda_i'}$, $\epsilon(2c_i+1,\lambda_i')=1$, such that

$$\begin{aligned} D(\sigma'') \hookrightarrow{} & \zeta(-a_1,b_1,\lambda_1)\times\cdots\times\zeta(-a_l,b_l,\lambda_k) \\ & \times\zeta(-c_1,-1/2,\lambda_1')\times\cdots\times\zeta(-c_k,-1/2,\lambda_k')\rtimes\mathbf{1}, \end{aligned} \tag{3.13}$$

and, if $\lambda_i=\lambda_j$, we also have $[2b_i+1,2a_i+1]\cap[2b_j+1,2a_j+1]=\varnothing$ or $[2b_i+1,2a_i+1]\subset[2b_j+1,2a_j+1]$ $(i<j)$. Next, if $\lambda_i=\lambda_j'$, $[2b_i+1,2a_i+1]\subset$

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$[2, 2c_j+1]$, $[2, 2c_j+1] \subset [2b_i+1, 2a_i+1]$, or $[2b_i+1, 2a_i+1] \cap [2, 2c_j+1] = \varnothing$. Finally, all $a_i$, $b_j$ and $c_s$ are mutually different.

Next, we note that

$$\zeta(-a_i, b_i, \lambda_i) \hookrightarrow |\cdot|^{-a_i}\lambda_i \times \cdots \times |\cdot|^{b_i}\lambda_i;$$
$$\zeta(-c_i, -1/2, \lambda_i') \hookrightarrow |\cdot|^{-c_i}\lambda_i' \times \cdots \times |\cdot|^{-1/2}\lambda_i'$$

by [Ze]. We use this together with (3.13) and Lemma 3.4, which implies $D(\tau) \hookrightarrow |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu \rtimes D(\sigma'')$, to fix the embedding (3.10). Also, the right-hand side of (3.11) is

$$\begin{aligned} &|\cdot|^{s-(b-b_--1)/2}\mu \times \cdots \times |\cdot|^{s+(b-b_--1)/2}\mu \times |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu \\ &\times |\cdot|^{-a_1}\lambda_1 \times \cdots \times |\cdot|^{b_1}\lambda_1 \times \cdots \times |\cdot|^{-a_l}\lambda_l \times \cdots \times |\cdot|^{b_l}\lambda_l \\ &\times |\cdot|^{-c_1}\lambda_1' \times \cdots \times |\cdot|^{-1/2}\lambda_1' \times \cdots \times |\cdot|^{-c_k}\lambda_k' \times \cdots \times |\cdot|^{-1/2}\lambda_k' \rtimes \mathbf{1}. \end{aligned} \tag{3.14}$$

It is induced from the following character:

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{s}} := &|\cdot|^{s-(b-b_--1)/2}\mu \otimes \cdots \otimes |\cdot|^{s+(b-b_--1)/2}\mu \otimes |\cdot|^{-b_-}\mu \otimes \cdots \otimes |\cdot|^{b_-}\mu \\ &\otimes |\cdot|^{-a_1}\lambda_1 \otimes \cdots \otimes |\cdot|^{b_1}\lambda_1 \otimes \cdots \otimes |\cdot|^{-a_l}\lambda_l \otimes \cdots \otimes |\cdot|^{b_l}\lambda_l \\ &\otimes |\cdot|^{-c_1}\lambda_1' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_1' \otimes \cdots \otimes |\cdot|^{-c_k}\lambda_k' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_k'. \end{aligned} \tag{3.15}$$

Now, we explain how elements $w \in W$, $w\left(\Delta \setminus \{\alpha_{b-b_-}\}\right) > 0$, act on $\boldsymbol{\mu}_{\mathbf{s}}$ defined by (3.15). We use Lemma 1.1. First, by Lemma 1.1, that set is divided into the disjoint union of $W_j$, $0 \le j \le b-b_-$. In order to describe the action of elements of $W_j$, $0 \le j \le b-b_-$, on $\boldsymbol{\mu}_{\mathbf{s}}$, we consider the action of a particular element $w_j \in W_j$ first:

$$\begin{aligned} w_j(\boldsymbol{\mu}_{\mathbf{s}}) = &|\cdot|^{s-(b-b_--1)/2}\mu \otimes \cdots \otimes |\cdot|^{s-(b-b_--1)/2+j-1}\mu \otimes |\cdot|^{-s-(b-b_--1)/2}\mu \otimes \cdots \otimes \\ &|\cdot|^{-s+(b-b_--1)/2-j}\mu \otimes |\cdot|^{-b_-}\mu \otimes \cdots \otimes |\cdot|^{b_-}\mu \otimes |\cdot|^{-a_1}\lambda_1 \otimes \cdots \otimes \\ &|\cdot|^{b_1}\lambda_1 \otimes \cdots \otimes |\cdot|^{-a_l}\lambda_l \otimes \cdots \otimes |\cdot|^{b_l}\lambda_l \otimes |\cdot|^{-c_1}\lambda_1' \otimes \cdots \otimes \\ &|\cdot|^{-1/2}\lambda_1' \otimes \cdots \otimes |\cdot|^{-c_k}\lambda_k' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_k'. \end{aligned} \tag{3.16}$$

(The definition $w_0$ given here agrees with one given in the statement of Lemma 3.6(ii).) Now, the orbit of the action of $W_j$ on $\boldsymbol{\mu}_{\mathbf{s}}$ consists of all characters obtained from all shuffles (see [KR, p. 235] for the definition) of the following:

(i) $|\cdot|^{s-(b-b_--1)/2}\mu, \dots, |\cdot|^{s-(b-b_--1)/2+j-1}\mu$,
(ii) $|\cdot|^{-s-(b-b_--1)/2}\mu, \dots, |\cdot|^{-s+(b-b_--1)/2-j}\mu$,
(iii) $|\cdot|^{-b_-}\mu, \dots, |\cdot|^{b_-}\mu, |\cdot|^{-a_1}\lambda_1, \dots, |\cdot|^{b_1}\lambda_1, \dots, |\cdot|^{-a_l}\lambda_l, \dots, |\cdot|^{b_l}\lambda_l, |\cdot|^{-c_1}\lambda_1'$, $\dots, |\cdot|^{-1/2}\lambda_1', \dots, |\cdot|^{-c_k}\lambda_k', \dots, |\cdot|^{-1/2}\lambda_k'$.

Now, one can easily get the proof of (iii). We leave the simple verification to the reader.

In the remainder of the proof we will use the following simple facts about normalized intertwining operators for general linear groups. The proofs can be found in [MW2], for example.

**GL-1** We consider the normalized operator:

$$R(\cdots\otimes|\cdot|^{s}\chi\otimes|\cdot|^{s'}\chi\otimes\cdots,\widetilde{r})\colon \cdots\times|\cdot|^{s}\chi\times|\cdot|^{s'}\chi\times\cdots\rtimes\mathbf{1}\to \cdots\times|\cdot|^{s'}\chi\times|\cdot|^{s}\chi\times\cdots\rtimes\mathbf{1}$$

($\chi$ is a unitary character of $F^{\times}$, $s,s'\in\mathbb{R}$) determined by the simple reflection $r$ that interchanges the positions of $|\cdot|^{s}\chi$ and $|\cdot|^{s'}\chi$. This operator is induced from the normalized intertwining operator $|\cdot|^{s}\chi\times|\cdot|^{s'}\chi\to|\cdot|^{s'}\chi\times|\cdot|^{s}\chi$ of $\mathrm{GL}(2,F)$. The last operator is holomorphic when $s\geq s'$ (standard representation) or when $|\cdot|^{s}\chi\times|\cdot|^{s'}\chi$ is irreducible (that is, when $s-s'\neq\pm1$). If $s'=s+1$, it has a simple pole.

**GL-2** More generally, let $R(\cdots\otimes|\cdot|^{s}\chi\otimes|\cdot|^{s'}\chi'\otimes|\cdot|^{s'+1}\chi'\otimes\cdots\otimes|\cdot|^{s'+m}\chi'\otimes\cdots,\widetilde{r})$ be the normalized intertwining operator:

$$\cdots\times|\cdot|^{s}\chi\times|\cdot|^{s'}\chi'\times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi'\times\cdots\rtimes\mathbf{1} \to\cdots|\cdot|^{s'}\chi'\times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi'\times|\cdot|^{s}\chi\times\cdots\rtimes\mathbf{1}$$

($\chi,\chi'$ are unitary characters of $F^{\times}$, $s,s'\in\mathbb{R}$, $m\in\mathbb{Z}_{\geq0}$) determined by $r\in W$ which is the composition of successive transpositions of $|\cdot|^{s}\chi$ with characters $|\cdot|^{s'}\chi'$, $|\cdot|^{s'+1}\chi',\dots,|\cdot|^{s'+m}\chi'$. It is induced from the normalized intertwining operator

$$R(s)\colon |\cdot|^{s}\chi\times|\cdot|^{s'}\chi'\times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi'\to|\cdot|^{s'}\chi' \times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi'\times|\cdot|^{s}\chi.$$

In fact, combining with Lemma 1.3, when holomorphic, $R(s)$ induces the following commutative diagram:

$$\begin{array}{ccccc} 0 & \to & |\cdot|^{s}\chi\times\zeta(s',s'+m,\chi') & \longrightarrow & |\cdot|^{s'}\chi'\times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi' \\ & & \downarrow & & \downarrow{\scriptstyle R(s)} \\ 0 & \to & \zeta(s',s'+m,\chi')\times|\cdot|^{s}\chi & \to & |\cdot|^{s'}\chi'\times|\cdot|^{s'+1}\chi'\times\cdots\times|\cdot|^{s'+m}\chi'\times|\cdot|^{s}\chi \end{array}$$

Furthermore, if $|\cdot|^{s}\chi\times\zeta(s',s'+m,\chi')$ is irreducible, then $R(s)$ is holomorphic on $|\cdot|^{s}\chi\times\zeta(s',s'+m,\chi')$.

**GL-3** The construction of discrete series shows

$$|\cdot|^{s}\mu\times\zeta(-a_i,b_i,\lambda_i)\simeq\zeta(-a_i,b_i,\lambda_i)\times|\cdot|^{s}\mu$$
$$|\cdot|^{s}\mu\times\zeta(-c_i,-1/2,\lambda_i')\simeq\zeta(-c_i,-1/2,\lambda_i')\times|\cdot|^{s}\mu,$$

for $\pm s\in[b_{-}+1,b]$.

Besides those three facts, we will use the following simple observations obtained from our choice of embeddings (3.10):

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**D-1** Using (3.14), we obtain the following (at $s = (b_- + b + 1)/2$):

$$\begin{aligned}(3.17)\quad |\det|^{(b_-+b+1)/2}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau) &\hookrightarrow |\cdot|^{b_-+1}\mu \times \cdots \times |\cdot|^{b}\mu\\ &\times |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu \times \zeta(-a_1, b_1, \lambda_1) \times \cdots \times \zeta(-a_l, b_l, \lambda_l)\\ &\times \zeta(-c_1, -1/2, \lambda'_1) \times \cdots \times \zeta(-c_k, -1/2, \lambda'_k) \rtimes \mathbf{1}\\ &\hookrightarrow |\cdot|^{b_-+1}\mu \times \cdots \times |\cdot|^{b}\mu \times |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu\\ &\times |\cdot|^{-a_1}\lambda_1 \times \cdots \times |\cdot|^{b_1}\lambda_1 \times \cdots \times |\cdot|^{-a_l}\lambda_l \times \cdots \times |\cdot|^{b_l}\lambda_l\\ &\times |\cdot|^{-c_1}\lambda'_1 \times \cdots \times |\cdot|^{-1/2}\lambda'_1 \times \cdots \times |\cdot|^{-c_k}\lambda'_k \times \cdots \times |\cdot|^{-1/2}\lambda'_k \rtimes \mathbf{1}.\end{aligned}$$

The composition of intertwining operators in the statement of Lemma 3.6(i) factors through that inclusion.

**D-2** The normalized intertwining operator $|\cdot|^{s}\chi \rtimes \mathbf{1} \to |\cdot|^{-s}\chi \rtimes \mathbf{1}$ ($\chi$ is a quadratic character of $F^\times$, $s \in \mathbb{R}$) is holomorphic for $s \geq 0$. (This is well known if $G_1(F) = \mathrm{Sp}(1, F) = \mathrm{SL}(2, F)$ or $G_1(F) = \mathrm{SO}(3, F) = P\,\mathrm{GL}(2, F)$. If $G_1(F) = \mathrm{O}(2, F)$, the claim is trivial.) In particular, the normalized intertwining operator $|\cdot|^{s}\mu \rtimes \mathbf{1} \to |\cdot|^{-s}\mu \rtimes \mathbf{1}$, $s \in [b_- + 1, b]$, is holomorphic.

Fix $j$, $0 \leq j \leq b - b_-$, and $w \in W_j$. The above description of the action of $w$ shows that the action of $R(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w})$ can be described as follows. First, this action can be factorized into actions attached to simple reflections (see Theorem 2.1) that we call *simple moves* here. Several simple moves make *a move*. We have the following:

- We first move $|\cdot|^{b_-+1+j}\mu, \dots, |\cdot|^{b}\mu$ to get them just before $\rtimes\mathbf{1}$. Thus, we get that the image of that intertwining operator, say $R$, satisfies

$$\begin{aligned}\operatorname{Im} R \hookrightarrow |\cdot|^{b_-+1}\mu \times \cdots \times |\cdot|^{b_-+j}\mu \times |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu \times \zeta(-a_1, b_1, \lambda_1)\\ \times \cdots \times \zeta(-a_l, b_l, \lambda_l) \times \zeta(-c_1, -1/2, \lambda'_1) \times \cdots \times \zeta(-c_k, -1/2, \lambda'_k)\\ \times |\cdot|^{b_-+1+j}\mu \times \cdots \times |\cdot|^{b}\mu \rtimes \mathbf{1}.\end{aligned}$$

  We accomplish that using (GL-1) and (GL-3), producing no poles.
- Now, we move around $\rtimes\mathbf{1}$ using (GL-1) and (D-2) (producing no poles on $\operatorname{Im} R$ from the last step), calling the intertwining operator $R_1$. We obtain the following:

$$\begin{aligned}\operatorname{Im} R_1 \hookrightarrow |\cdot|^{b_-+1}\mu \times \cdots \times |\cdot|^{b_-+j}\mu \times |\cdot|^{-b_-}\mu \times \cdots \times |\cdot|^{b_-}\mu \times \zeta(-a_1, b_1, \lambda_1)\\ \times \cdots \times \zeta(-a_l, b_l, \lambda_l) \times \zeta(-c_1, -1/2, \lambda'_1) \times \cdots \times \zeta(-c_k, -1/2, \lambda'_k)\\ \times |\cdot|^{-b}\mu \times \cdots \times |\cdot|^{-b_- -1-j}\mu \rtimes \mathbf{1}.\end{aligned}$$

- Now, we move $|\cdot|^{-b}\mu, \dots, |\cdot|^{-b_--1-j}\mu$ to their final positions. We move $|\cdot|^{-b}\mu$ first. We use (GL-1), (GL-2) and (GL-3) (without producing any poles) if the position of $|\cdot|^{-b}\mu$ is not in some "unfolded" $\zeta(-a_i, b_i, \lambda_i)$ or $\zeta(-c_i, -1/2, \lambda'_i)$. If this is so, we assume $|\cdot|^{-b}\mu$ is in "unfolded" $\zeta(-a_1, b_1, \lambda_1)$. (The other case is treated analogously.) Now, $\lambda_1 = \mu$, and say $|\cdot|^{-b}\mu$ must be immediately after $|\cdot|^{c}\mu$, $-a_1 \le c \le b_1 - 1$. Then we use the fact that $\zeta(-a_1, b_1, \lambda_1) \hookrightarrow \zeta(-a_1, c, \lambda_1) \times \zeta(c+1, b_1, \lambda_1)$ (Lemma 1.3), and

$$\zeta(-a_1, c, \lambda_1) \times \zeta(c+1, b_1, \lambda_1) \times |\cdot|^{-b}\mu \simeq \zeta(-a_1, c, \lambda_1) \times |\cdot|^{-b}\mu \times \zeta(c+1, b_1, \lambda_1),$$

unless $-b = c$. In that case, we use two steps. First,

$$\zeta(-a_1, b_1, \lambda_1) \hookrightarrow \zeta(-a_1, c-1, \lambda_1) \times \zeta(c, b_1, \lambda_1)$$

[Ze] and then

$$\zeta(-a_1, c-1, \lambda_1) \times \zeta(c, b_1, \lambda_1) \times |\cdot|^{-b}\mu \simeq \zeta(-a_1, c-1, \lambda_1) \times |\cdot|^{-b}\mu \times \zeta(c, b_1, \lambda_1).$$

Then we unfold $\zeta(c, b_1, \lambda_1) \hookrightarrow |\cdot|^{c}\mu \times \cdots \times |\cdot|^{b_1}\mu$ and move $|\cdot|^{-b}\mu$ one step back. This way we do not produce a pole. We can repeat the same procedure with $|\cdot|^{b_- + j}\mu, \cdots, |\cdot|^{b-1}\mu$ without producing any poles. The only difference is that in each step the number of $\zeta$'s might increase by one for the reason that we explained for $|\cdot|^{-b}\mu$. Finally, we move $|\cdot|^{b_- + 1}\mu, \ldots, |\cdot|^{b_- + j - 1}\mu$, which is similar to the moving of $|\cdot|^{-b}\mu$. This proves (i).

Now, we prove (ii). Let $A(s, \widetilde{w}_0)$ be the integral intertwining operator

$$|\det|^{s}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau) \to |\det|^{-s}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau).$$

Put

$$R(s, \widetilde{w}_0) = r(\boldsymbol{\mu}_{\mathbf{s}}, w_0)A(s, \widetilde{w}_0),$$

where

$$r(\boldsymbol{\mu}_{\mathbf{s}}, w_0) = \prod_{\alpha \in \Sigma_+, w_0(\alpha) < 0} \frac{L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ \alpha^\vee)\epsilon(1, \boldsymbol{\mu}_{\mathbf{s}} \circ \alpha^\vee, \psi)}{L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ \alpha^\vee)} \tag{3.18}$$

is the normalization factor for the unnormalized intertwining operator $A(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w}_0)$ defined in Section 2. (Here $\psi$ is a non-trivial additive character of $F$.) We have

$$R(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w}_0) = r(\boldsymbol{\mu}_{\mathbf{s}}, w_0)A(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w}),$$

and, as usual, the defining integrals of $A(s, \widetilde{w}_0)$ and $A(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w}_0)$ imply the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & {|\det|^{s}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau)} \arrow[r, "i_s"] \arrow[d, "R(s,\widetilde{w}_0)"] & \mathrm{Ind}_{B_n}^{G_n}(\boldsymbol{\mu}_s) \arrow[d, "R(\boldsymbol{\mu}_{\mathbf{s}},\widetilde{w}_0)"] \\
0 \arrow[r] & {|\det|^{-s}\mathbf{1}_{\mathrm{GL}(b-b_-,F)}\mu \rtimes D(\tau)} \arrow[r, "i_{-s}"] & \mathrm{Ind}_{B_n}^{G_n}(w_0(\boldsymbol{\mu}_s)).
\end{tikzcd}$$

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Since $R(\boldsymbol{\mu}_{\mathbf{s}}, \widetilde{w}_0) \circ i_s$ is holomorphic at $s = (b+b_- +1)/2$, $R(s, \widetilde{w}_0)$ is also holomorphic there. Now, Lemma 3.4 implies (ii), as soon as we show that $R((b+b_- +1)/2, \widetilde{w}_0) \neq 0$. The claim is well known for the unnormalized operator $A(s, \widetilde{w}_0)$ (see, for example, the proof of (10) [W, p. 283]). Now, if $R((b+b_- +1)/2, \widetilde{w}_0) = 0$, then the normalization factor $r(\boldsymbol{\mu}_{\mathbf{s}}, w_0)$ must be zero at $s = (b + b_- + 1)/2$. Since local $L$-functions never vanish, to show that $r(\boldsymbol{\mu}_{\mathbf{s}}, w_0)$ does not vanish at $s = (b + b_- + 1)/2$, we use (3.18) to construct an injection $\alpha \rightsquigarrow \alpha_1$ from the set of all $\alpha \in \Sigma_+$ such that $w_0(\alpha) < 0$ and $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ \alpha^\vee)$ has a pole at $s = (b + b_- + 1)/2$, into the set of all $\beta \in \Sigma_+$, $w_0(\beta) < 0$, such that $L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ \alpha_1^\vee)$ has a pole at $s = (b + b_- + 1)/2$. We list all coroots of the roots $\beta \in \Sigma_+$, $w_0(\beta) < 0$, using Corollary 1.2 and the discussion after the statement of Corollary 1.2:

(1) $\{2\varphi_e \,;\, 1 \leq e \leq b - b_-\}$,
(2) $\{\varphi_e + \varphi_f \,;\, 1 \leq e < f \leq b - b_-\}$,
(3) $\{\varphi_e - \varphi_f \,;\, 1 \leq e \leq b - b_-,\ b - b_- + 1 \leq f \leq n\}$,
(4) $\{\varphi_e + \varphi_f \,;\, 1 \leq e \leq b - b_-,\ b - b_- + 1 \leq f \leq n\}$.

We note that $\boldsymbol{\mu}_{\mathbf{s}} \circ \varphi_e = |\cdot|^{b_- + e}\mu$, $1 \leq e \leq b - b_-$, at $s = (b + b_- + 1)/2$. Now, given $1 \leq e, f \leq b - b_-$, we have

$$L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (2\varphi_e)) = L(2(b_- + e), \mathbf{1}) \neq \infty,$$
$$L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_e + \varphi_f)) = L(2b_- + e + f, \mathbf{1}) \neq \infty$$

at $s = (b + b_- + 1)/2$. Thus, the first two cases do not contribute at all.

Now, we consider case (3). If $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_e - \varphi_f)) = \infty$ ($s = (b+b_- +1)/2$), then $\boldsymbol{\mu}_{\mathbf{s}} \circ \varphi_e = |\cdot|^{b_- + e}\mu$ must belong to $\{|\cdot|^{-a_i}\lambda_i, |\cdot|^{-a_i+1}\lambda_i, \ldots, |\cdot|^{b_i}\lambda_i\}$, for some $i$. Clearly, $|\cdot|^{b_- + e+1}\mu$ belongs to the same set. (Otherwise, $b_- + e = b_i$ would imply that $2b_i + 1 \in [2b_- + 1, 2b + 1]$ and, since $\lambda_i = \mu$, this would violate the assumptions stated after (3.13).) Thus, we see

$$L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_e - \varphi_{f+1})) = \infty (s = (b + b_- + 1)/2).$$

Now, we consider case (4). If $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_e + \varphi_f)) = \infty$ ($s = (b+b_- +1)/2$), then $|\cdot|^{-b_- - e}\mu$ must belong to

$$\{|\cdot|^{-a_i}\lambda_i, |\cdot|^{-a_i+1}\lambda_i, \ldots, |\cdot|^{b_i}\lambda_i\} \quad \text{or} \quad \{|\cdot|^{-c_i}\lambda_i', |\cdot|^{-c_i+1}\lambda_i, \ldots, |\cdot|^{-1/2}\lambda_i\},$$

for some $i$. As in the previous case, we see that $\varphi_e - \varphi_{f-1}$ is the required coroot. ■

### 3.4 Main Theorem

The next theorem is the main result of this paper. The proof will be given in the next section.

***Theorem 3.7*** *Assume that $\sigma \in \operatorname{Irr} G_n(F)$, $n > 0$, is a discrete series representation which is a subquotient of some principal series representation. Then $D(\sigma)$ is a unitarizable representation.*

***Corollary 3.8*** *Assume that $\tau \in \operatorname{Irr} G_n(F)$, $n > 0$, is a tempered representation which is a subquotient of some principal series representation. Then $D(\tau)$ is a unitarizable representation.*

**Proof** There exist unitary characters $\chi_1, \chi_2, \dots, \chi_k$ of $F^\times$, $n_1, n_2, \dots, n_k \in \mathbb{Z}_{>0}$, and a discrete series $\sigma \in \operatorname{Irr} G_{n'}(F)$, such that $n = n_1 + n_2 + \cdots + n_k + n'$ and

$$\tau \hookrightarrow \chi_1 \operatorname{Steinberg}_{\mathrm{GL}(n_1,F)} \times \cdots \times \chi_k \operatorname{Steinberg}_{\mathrm{GL}(n_k,F)} \rtimes \sigma.$$

If $n' = 0$, then we let $D(\mathbf{1}) = \mathbf{1}$. Then [A, Jan1], as well as unitarity of $D(\sigma)$, imply

$$D(\tau) \hookrightarrow \chi_1 \mathbf{1}_{\mathrm{GL}(n_1,F)} \times \cdots \times \chi_k \mathbf{1}_{\mathrm{GL}(n_k,F)} \rtimes D(\sigma).$$

Hence, $D(\tau)$ is unitarizable. ■

## 4 Construction of Square-Integrable Automorphic Forms and Representations

In this section we start the proof of Theorem 3.7, keeping the notation from the last section. So, let $F$ be a local non-archimedean field of characteristic zero. We may take a global field $K$ such that at a finite place $v_0$ we have $K_{v_0} \simeq F$ and all archimedean places are complex. We fix such a $K$ and its place $v_0$.

Any local quadratic character $\mu_{v_0}$ of $F \simeq K_{v_0}$ is a local component of some quadratic grössencharacter $\mu\colon K^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$. Since we are assuming that all archimedean places are complex, we have $\mu_v = 1$ for such places $v$.

Let $\sigma \in \operatorname{Irr} G_n(F)$ be a fixed discrete series representation. Let $(\mathrm{Jord}, \mathbf{1}, \epsilon)$ be its admissible triple. Then, as recalled in the last section, there are only finitely many quadratic characters $\mu_{v_0}$ of $F \simeq K_{v_0}$ that appear in Jord (that is, for some $a \in (1/2)\mathbb{Z}_{\geq 0}$, $(a, \mu_{v_0}) \in \mathrm{Jord}$). For such a quadratic character $\mu_{v_0}$, we fix some extension to a quadratic grössencharacter $\mu\colon K^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$. Let $I$ be the set of all quadratic grössencharacters of $K$ obtained this way.

Now, we fix the set $S = S(I)$ of all finite places of $K$, containing $v_0$, such that outside $S$ we have $\lambda_v = 1$ if $v$ is archimedean and $\lambda_v$ is unramified if $v$ is non-archimedean, for any $\lambda \in I$, and also $\psi_v$ is unramified for our fixed non-trivial additive character $\psi\colon K \setminus \mathbb{A} \to \mathbb{C}^\times$ (see Section 2).

The next proposition shows that the duals of all strongly positive representations from the last section are unitary. In view of Section 3.2, we see that it is enough to do it for $G_n = \mathrm{SO}(2n+1)$. (It is well known that the trivial representation appears in the residual spectrum of any reductive group.)

***Proposition 4.1*** *Let $G_n = \mathrm{SO}(2n+1)$. Assume that $\mu_1, \dots, \mu_k$ are fixed quadratic characters of $F^\times$ and $l_1, \dots, l_k \in \mathbb{Z}_{>0}$. (We assume that $\mu_i = \lambda_{i,v_0}$, $i = 1, \dots, k$, where $\lambda_i \in I$.) Then the global induced representation*

$$|\cdot|^{l_k - 1/2}\lambda_k \times \cdots \times |\cdot|^{1/2}\lambda_k \times \cdots \times |\cdot|^{l_1 - 1/2}\lambda_1 \times \cdots \times |\cdot|^{1/2}\lambda_1 \rtimes \mathbf{1} \tag{4.1}$$

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*has a unique irreducible quotient. This representation appears in the space of square-integrable automorphic forms on $G_n(\mathbb{A})$. In particular, a unique irreducible quotient of*

$$|\cdot|^{l_k-1/2}\mu_k \times \cdots \times |\cdot|^{1/2}\mu_k \times \cdots \times |\cdot|^{l_1-1/2}\mu_1 \times \cdots \times |\cdot|^{1/2}\mu_1 \rtimes \mathbf{1}$$

*is unitarizable. (Applying Lemma* 3.1*, this proves Theorem* 3.7 *in the case of strongly positive discrete series.)*

**Proof** We prove the proposition by induction on $k$. We actually prove only the inductive step, since the base of induction is proved similarly. We use notation introduced immediately after Theorem 2.1; the reader should review the notation there.

We let $\mu = \lambda_k$ (see (2.7)). By the inductive assumption, we have an automorphic realization $(V, \Pi)$ of the quotient of $|\cdot|^{l_{k-1}-1/2}\lambda_{k-1} \times \cdots \times |\cdot|^{1/2}\lambda_{k-1} \times \cdots \times |\cdot|^{l_1-1/2}\lambda_1 \times \cdots \times |\cdot|^{1/2}\lambda_1 \rtimes \mathbf{1}$ so that constant term of $V$ gives the embedding (compare to (2.14)),

$$\Pi \hookrightarrow |\cdot|^{-1/2}\lambda_{k-1} \times \cdots \times |\cdot|^{-l_{k-1}+1/2}\lambda_{k-1} \times \cdots \times |\cdot|^{-1/2}\lambda_1 \times \cdots \times |\cdot|^{-l_1+1/2}\lambda_1 \rtimes \mathbf{1}.$$

Now, as explained at the end of Section 2 (see Lemma 2.3), we may compute the constant term (2.9) using normalized intertwining operators. Thus, we take $f_{\boldsymbol{\mu}_s} = \bigotimes_v f_{\boldsymbol{\mu}_{v,s}}$ in

$$\begin{aligned}(4.2)\qquad |\det|^s \mathbf{1}_{\mathrm{GL}(l_k,\mathbb{A})}\lambda_k \rtimes \Pi \hookrightarrow\ & |\cdot|^{s-(l_k-1)/2}\lambda_k \times \cdots \times |\cdot|^{s+(l_k-1)/2}\lambda_k \\ & \times |\cdot|^{-1/2}\lambda_{k-1} \times \cdots \times |\cdot|^{-l_{k-1}+1/2}\lambda_{k-1} \\ & \times \cdots \times |\cdot|^{-1/2}\lambda_1 \times \cdots \times |\cdot|^{-l_1+1/2}\lambda_1 \rtimes \mathbf{1},\end{aligned}$$

such that $f_{\boldsymbol{\mu}_{v,s}}$, $v \notin S$, is unramified and normalized by $f_{\boldsymbol{\mu}_{v,s}}(1) = 1$, where we write

$$\begin{aligned}\boldsymbol{\mu}_{\mathbf{s}} = |\cdot|^{s-(l_k-1)/2}\lambda_k \otimes \cdots \otimes |\cdot|^{s+(l_k-1)/2}\lambda_k \otimes |\cdot|^{-1/2}\lambda_{k-1} \\ \otimes \cdots \otimes |\cdot|^{-l_{k-1}+1/2}\lambda_{k-1} \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_1 \otimes \cdots \otimes |\cdot|^{-l_1+1/2}\lambda_1\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{\mu}_{v,\mathbf{s}} = |\cdot|^{s-(l_k-1)/2}\lambda_{k,v} \otimes \cdots \otimes |\cdot|^{s+(l_k-1)/2}\lambda_{k,v} \otimes |\cdot|^{-1/2}\lambda_{k-1,v} \\ \otimes \cdots \otimes |\cdot|^{-l_{k-1}+1/2}\lambda_{k-1,v} \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_{1,v} \otimes \cdots \otimes |\cdot|^{-l_1+1/2}\lambda_{1,v}.\end{aligned}$$

So, $\boldsymbol{\mu}_{\mathbf{s}} = \bigotimes_v \boldsymbol{\mu}_{v,\mathbf{s}}$.

Now, Theorem 2.1(iii) implies $R(\boldsymbol{\mu}_{v,\mathbf{s}}, w) f_{\boldsymbol{\mu}_{v,s}} = f_{w(\boldsymbol{\mu}_{v,s})}$. Thus, the constant term (2.9) can be rewritten as follows:

$$\sum_{w,w(\Delta\setminus\{\alpha_{l_k}\})>0} \bigotimes_{v\notin S} f_{w(\boldsymbol{\mu}_{v,s})} \bigotimes_{v\in S} R(\boldsymbol{\mu}_{v,\mathbf{s}}, w) f_{\boldsymbol{\mu}_{v,s}} \cdot \prod_{\alpha\in\Sigma_+, w(\alpha)<0} \frac{L(0, \boldsymbol{\mu}_s \circ \alpha^\vee)}{L(1, \boldsymbol{\mu}_s \circ \alpha^\vee)\epsilon(1, \boldsymbol{\mu}_s \circ \alpha^\vee)}.$$

By the results of the last section (see Lemma 3.3), a pole at $s = l_k/2$ comes from

$$\prod_{\alpha\in\Sigma_+, w(\alpha)<0} \frac{L(0, \boldsymbol{\mu}_s \circ \alpha^\vee)}{L(1, \boldsymbol{\mu}_s \circ \alpha^\vee)\epsilon(1, \boldsymbol{\mu}_s \circ \alpha^\vee)}.$$

Now, as in Lemma 1.1, we decompose the set $\{w\,;\, w(\Delta\setminus\{\alpha_{l_k}\}) > 0\}$ into the disjoint union $\bigcup_{0\le j\le l_k} W_j$. Let $w = p\epsilon \in W_j$ (see Section 1). (This means $w(\phi_t) = \phi_{p(t)}$, $1 \le t \le j$ or $l_k + 1 \le t$, and $w(\phi_t) = \phi^{-1}_{p(t)}$, $j \le t \le l_k$.) Since $\epsilon(1, \boldsymbol{\mu}_s \circ \alpha^\vee)$ is an exponential function of $s$, we compute the order of pole at $s = l_k/2$ of

$$\prod_{\alpha\in\Sigma_+, w(\alpha)<0} \frac{L(0, \boldsymbol{\mu}_s \circ \alpha^\vee)}{L(1, \boldsymbol{\mu}_s \circ \alpha^\vee)}. \tag{4.3}$$

First, the set of all coroots of all $\alpha \in \Sigma_+$ such that $w^{-1}(\alpha) < 0$ is given by the union of the following cases, which are given together with their contribution to (4.3) (see Corollary 1.2):

- $\{2\varphi_t\,;\, j + 1 \le t \le l_k\}$; note that $\boldsymbol{\mu}_s \circ \varphi_t = |\cdot|^{s-(l_k-1)/2+t-1}$ for $1 \le t \le l_k$

$$\frac{L(2s - (l_k - 1) + 2j, 1)L(2s - (l_k - 1) + 2j + 2, 1)\cdots L(2s + (l_k - 1), 1)}{L(1 + 2s - (l_k - 1) + 2j, 1)L(1 + 2s - (l_k - 1) + 2j + 2, 1)\cdots L(1 + 2s + (l_k - 1), 1)}.$$

  Now, at $s = l_k/2$ we see that the term is holomorphic if $j > 0$ and it has a simple pole if $j = 0$. (The reader should now refer to the beginning of Section 2 for properties of $L$-functions.)
- $\{\varphi_t + \varphi_r;\; j + 1 \le t < r \le l_k\}$;

$$\prod_{j+1\le t<r\le l_k} \frac{L(2s - l_k + t + r - 1, 1)}{L(2s - l_k + t + r, 1)}.$$

  Now, at $s = l_k/2$, everything is holomorphic.
- $\{\varphi_t + \varphi_r;\; 1 \le t \le j,\; j + 1 \le r \le l_k,\; p(r) < p(t)\}$; again

$$\prod_{1\le t\le j,\; j+1\le r\le l_k,\; p(r)<p(t)} \frac{L(2s - l_k + t + r - 1, 1)}{L(2s - l_k + t + r, 1)}.$$

  Now, at $s = l_k/2$, everything is holomorphic.

The other three cases

- $\{\varphi_t - \varphi_r\,;\, j + 1 \le t \le l_k,\; l_k + 1 \le r \le n\}$,
- $\{\varphi_t - \varphi_r\,;\, 1 \le t \le j,\; l_k + 1 \le r \le n,\; p(r) < p(t)\}$,
- $\{\varphi_t + \varphi_r\,;\, j + 1 \le t \le i,\; l_k + 1 \le r \le n,\; p(r) > p(t)\}$,

do not contribute, since the $L$-functions $L(s, \mu)$ of grössencharacters $\mu \ne 1$ are holomorphic on $\mathbb{C}$ and never vanish at integral points.

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Now, we multiply Eisenstein series (2.8) by $s - l_k/2$. Letting $s \to l_k/2$, we see that such a normalized Eisenstein series is holomorphic and non-zero. The representation generated by its image in the space of automorphic forms is isomorphic to the representation generated by its constant term:

$$\lim_{s \to l_k/2} (s - l_k/2) \sum_{w, w(\Delta \setminus \{\alpha_{l_k}\}) > 0} \bigotimes_{v \notin S} f_{w(\boldsymbol{\mu}_{v,s})} \bigotimes_{v \in S} R(\boldsymbol{\mu}_{v,\mathbf{s}}, w) f_{\boldsymbol{\mu}_{v,s}} \cdot \prod_{\alpha \in \Sigma_+, w(\alpha) < 0} \frac{L(0, \boldsymbol{\mu}_s \circ \alpha^\vee)}{L(1, \boldsymbol{\mu}_s \circ \alpha^\vee) \epsilon(1, \boldsymbol{\mu}_s \circ \alpha^\vee, \psi_v)}.$$

By the Langlands classification applied to each local place $v$, it is isomorphic to the unique irreducible quotient of (4.1). Thus the normalized Eisenstein series has image isomorphic to the unique irreducible quotient of (4.1). The image is square-integrable, since we easily check that the exponents of constant term satisfy square-integrability criterion of Langlands [MW3]. ■

***Remark 4.1*** If $G_n = \mathrm{Sp}(n)$, then the trivial representation of $G_n(\mathbb{A})$ is a quotient of $|\cdot|^n \times |\cdot|^{n-1} \times \cdots \times |\cdot| \rtimes \mathbf{1}$.

In the remainder of this section, we prove Theorem 3.7 by induction on the number of Jordan blocks in Jord starting from the case, already considered, of strongly positive discrete series. In order to explain our inductive proof, and for definiteness, we consider the most (notationally) difficult case: $G_n = \mathrm{SO}(2n+1)$. We leave it to the reader to adjust the proof in the other two cases. Now, our fixed $\sigma \in \mathrm{Irr}\, G_n(F)$ can be obtained by the following procedure:

(i) we start from some strongly positive discrete series $\sigma'$ (perhaps $\mathbf{1} \in \mathrm{Irr}\, G_0(F)$);

(ii) by the inductive definition of discrete series, Lemma 3.4 and Lemma 3.5, there are quadratic characters $\mu_1, \dots, \mu_k$ of $F^\times$, $0 < a_1, \dots, a_k, b_1, \dots, b_k \in 1/2 + \mathbb{Z}$, $b_i < a_i$, such that we can build $D(\sigma)$ using the following inductive procedure:

   (a) $\sigma_0 = \sigma'$.

   (b) There exists an irreducible subrepresentation $D(\tau_i) \hookrightarrow \zeta(-b_i, b_i, \mu_i) \rtimes D(\sigma_i)$ such that $\zeta(b_i+1, a_i, \mu_i) \rtimes D(\tau_i) \twoheadrightarrow D(\sigma_{i+1})$, where $[2a_i+1,\ 2b_i+1] \cap \mathrm{Jord}(\sigma_i)_{\mu_i} = \varnothing$, $\epsilon(2a_i+1, \mu_i)\epsilon(2b_i+1, \mu_i)^{-1} = 1$. Going in the opposite direction, at each step we take the largest possible $2a_i + 1$ subject to all other conditions (assuming that $a_k, b_k, \dots, a_{i+1}, b_{i+1}$ are already chosen). This is possible by the inductive construction of discrete series (see [MT] or [M3, §2]). We will refer to this as taking the "largest possible" in the proof of Proposition 4.2.

(iii) $\sigma \simeq \sigma_{k+1}$.

Now, we globalize this construction. By Proposition 4.1, there is a global representation $\bigotimes_v D(\sigma'_v)$, that is a quotient of (4.1) and its automorphic realization in the space of square-integrable automorphic forms. Moreover, the computation of the corresponding constant term implies

$$\bigotimes_v D(\sigma'_v) \hookrightarrow |\cdot|^{-c_1}\lambda'_1 \times \cdots \times |\cdot|^{-1/2}\lambda'_1 \times \cdots \times |\cdot|^{-c_l}\lambda'_l \times \cdots \times |\cdot|^{-1/2}\lambda'_l \rtimes \mathbf{1}, \tag{4.4}$$

where $c_1, \ldots, c_l \in 1/2+\mathbb{Z}$. This will be used to compute the poles of certain Eisenstein series in the proof of Proposition 4.1 below. (See Lemma 2.3 and the comment after the statement of that lemma.)

We remark that here and later in this section the notation like $D(\sigma'_v)$ for $v|\infty$ is only formal. Also, $\sigma'_{v_0} \simeq \sigma'$ and $\sigma'_v$ is a strongly positive discrete series for $v < \infty$, and $D(\sigma'_v)$ is spherical for $v \notin S$.

Next, we choose quadratic characters $\lambda_i \in I$, $1 \leq i \leq k$, such that $\lambda_{i,v_0} = \mu_i$. Thus, Theorem 3.7 will be proved if we prove the following proposition:

***Proposition 4.2*** *For any $i$, $0 \leq i \leq k+1$, there exists a finite set of finite places $S_i$, containing $S$, a global representation $\bigotimes_v D(\sigma_{i,v})$ and its automorphic realization $V_i$ in the space of square-integrable automorphic forms such that:*

(i) *$D(\sigma_{i,v})$, $v \notin S_i$, is irreducible and spherical.*

(ii) *$\sigma_{i,v_0} \simeq \sigma_i$, and for any other place $v < \infty$, $\sigma_{i,v}$ is a discrete series built using a procedure similar to the one above for $\sigma$:*

  (a) *$\sigma_{0,v} = \sigma'_v$;*

  (b) *for $0 \leq j \leq i-1$, there exists an irreducible subrepresentation $D(\tau_{j,v}) \hookrightarrow \zeta(-b_j, b_j, \lambda_{j,v}) \rtimes D(\sigma_{j,v})$ such that $\zeta(b_j+1, a_j, \lambda_{j,v}) \rtimes D(\tau_{j,v}) \twoheadrightarrow D(\sigma_{j+1,v})$, where*

$$[2a_j+1, 2b_j+1] \cap \mathrm{Jord}(\sigma_{j,v})_{\lambda_{j,v}} = \varnothing, \quad \epsilon(2a_j+1, \lambda_{j,v})\epsilon(2b_j+1, \lambda_{j,v})^{-1} = 1.$$

(iii) *There is an inclusion*

$$\begin{aligned}(4.5)\quad \bigotimes_v D(\sigma_{i,v}) \hookrightarrow \prod_{1\leq j\leq i-1} (|\,|^{-a_j}\lambda_j \times \cdots \times |\,|^{b_j}\lambda_j) \times |\cdot|^{-c_1}\lambda'_1 \\ \times \cdots \times |\cdot|^{-1/2}\lambda'_1 \times \cdots \times |\cdot|^{-c_l}\lambda'_l \times \cdots \times |\cdot|^{-1/2}\lambda'_l \rtimes \mathbf{1}\end{aligned}$$

*obtained from the computation of the constant term in $V_i$ (compare to* (2.14)*).*

**Proof** This proof is similar to the proof of the previous proposition. Also, the proof of the inductive step is analogous to the proof of the base of induction. Hence we discuss only the inductive step. Thus we assume that $\bigotimes_v D(\sigma_{i,v})$ and $S_i$ have been constructed, and we construct $\bigotimes_v D(\sigma_{i+1,v})$ with its realization $V_{i+1}$ in the space of square-integrable automorphic forms and $S_{i+1}$. First, we construct some automorphic representations.

***Lemma 4.3*** *For any finite place $v$, we can choose irreducible representations $\tau_{i,v}$ (see Proposition* 4.2(ii)*) and spherical representations $D(\tau_{i,v})$, $v|\infty$, such that the subrepresentation*

$$\bigotimes_v D(\tau_{i,v}) \hookrightarrow \mathbf{1}_{\mathrm{GL}(2b_i+1,\mathbb{A})}\lambda_i \rtimes (\bigotimes_v D(\sigma_{i,v}))$$

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*has automorphic realization $V_i'$ such that the inclusion (compare* (4.5)*)*

$$(4.6)\quad \bigotimes_v D(\tau_{i,v}) \hookrightarrow |\ |^{-b_i}\lambda_i \times \cdots \times |\ |^{b_i}\lambda_i \times \prod_{1\le j\le i-1} (|\ |^{-a_j}\lambda_j \times \cdots \times |\ |^{b_j}\lambda_j) \\ \times |\cdot|^{-c_1}\lambda_1' \times \cdots \times |\cdot|^{-1/2}\lambda_1' \times \cdots \times |\cdot|^{-c_l}\lambda_l' \times \cdots \times |\cdot|^{-1/2}\lambda_l' \rtimes \mathbf{1}$$

*is obtained by computing the constant term of $V_i'$. We define $S_{i+1}$ to be a finite set of finite places such that $D(\tau_{i,v})$ is spherical for $v \notin S_{i+1}$. Moreover, the representation $\bigotimes_v D(\tau_{i,v})$ can be chosen so that $S_i \subseteq S_{i+1}$.*

We defer the proof of this lemma to the end of the proof of the proposition. Now, we continue with the proof of the proposition. First, we form global induced representations

$$(4.7)\quad |\det|^s 1_{\mathrm{GL}(a_i-b_i,\mathbb{A})}\lambda_i \rtimes (\bigotimes_v D(\tau_{i,v})) \hookrightarrow \\ |\cdot|^{s-(a_i-b_i-1)/2}\lambda_i \times \cdots \times |\cdot|^{s+(a_i-b_i-1)/2}\lambda_i \times |\cdot|^{-b_i}\lambda_i \times \cdots \times |\cdot|^{b_i}\lambda_i \times \\ \prod_{1\le j\le i-1} \big(|\ |^{-a_j}\lambda_j \times \cdots \times |\cdot|^{b_j}\lambda_j\big) \times |\cdot|^{-c_1}\lambda_1' \times \cdots \times |\cdot|^{-1/2}\lambda_1' \times \cdots \times \\ |\cdot|^{-c_l}\lambda_l' \times \cdots \times |\cdot|^{-1/2}\lambda_l' \rtimes \mathbf{1}.$$

Now, we are ready to proceed as indicated at the end of Section 2 (and as was done in the proof of previous proposition). We consider the Eisenstein series (2.8) at the point $s = (a_i + b_i + 1)/2$. We use the embedding in (4.7) to compute its poles. We consider only normalized intertwining operators attached to $w \in W$, $w(\Delta \setminus \{\alpha_{a_i-b_i}\}) > 0$. We put

$$(4.8)\quad \boldsymbol{\mu}_{\mathbf{s}} = |\cdot|^{s-(a_i-b_i-1)/2}\lambda_i \otimes \cdots \otimes |\cdot|^{s+(a_i-b_i-1)/2}\lambda_i \otimes |\cdot|^{-b_i}\lambda_i \otimes \cdots \otimes |\cdot|^{b_i}\lambda_i \\ \otimes \prod_{1\le j\le i-1} \big(|\cdot|^{-a_j}\lambda_j \otimes \cdots \otimes |\cdot|^{b_j}\lambda_j\big) \\ \otimes |\cdot|^{-c_1}\lambda_1' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_1' \otimes \cdots \otimes |\cdot|^{-c_l}\lambda_l' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_l'.$$

The constant term of the Eisenstein series (2.8) is given by sum over $w \in W$, $w(\Delta \setminus \{\alpha_{a_i-b_i}\}) > 0$, of normalized intertwining operators times the inverse of their normalization factors. Thus, for each $w \in W$, $w(\Delta \setminus \{\alpha_{a_i-b_i}\}) > 0$, we need to calculate the order of pole at $s = (a_i + b_i + 1)/2$ of the quotient of $L$-functions given by (4.3):

$$(4.9)\qquad \prod_{\alpha\in\Sigma_+, w(\alpha)<0} \frac{L(0, \boldsymbol{\mu}_s \circ \alpha^\vee)}{L(1, \boldsymbol{\mu}_s \circ \alpha^\vee)}.$$

Let $w = p\epsilon \in W_j$ (see Section 1), $0 \le j \le a_i - b_i$. (This means $w(\phi_t) = \phi_{p(t)}$, $1 \le t \le j$ or $a_i - b_i + 1 \le t$, and $w(\phi_t) = \phi_{p(t)}^{-1}$, $j \le t \le a_i - b_i$.) Note that

$\boldsymbol{\mu}_s \circ \varphi_t = |\cdot|^{s-(a_i-b_i-1)/2+t-1}\lambda_i$ for $1 \leq t \leq a_i - b_i$. Now, the set of all coroots of roots $\alpha \in \Sigma_+$ such that $w^{-1}(\alpha) < 0$ is given by the union of the following cases, which are given together with their contribution to (4.9):

(1) $\{2\varphi_t \,;\, j+1 \leq t \leq a_i - b_i\}$;

$$\prod_{j+1\leq t\leq a_i-b_i} \frac{L(2s-(a_i-b_i)+2t-1,1)}{L(2s-(a_i-b_i)+2t,1)}.$$

(2) $\{\varphi_t + \varphi_r \,;\, j+1 \leq t < r \leq a_i - b_i\}$;

$$\prod_{j+1\leq t<r\leq a_i-b_i} \frac{L(2s-(a_i-b_i)+t+r-1,1)}{L(2s-(a_i-b_i)+t+r,1)}.$$

(3) $\{\varphi_t + \varphi_r \,;\, 1 \leq t \leq j,\ j+1 \leq r \leq a_i - b_i,\ p(r) < p(t)\}$;

$$\prod_{\substack{1\leq t\leq j\\ j+1\leq r\leq a_i-b_i\\ p(r)<p(t)}} \frac{L(2s-(a_i-b_i)+t+r-1,1)}{L(2s-(a_i-b_i)+t+r,1)}.$$

In all three cases, the $L$-functions are holomorphic and non-zero at $s = (a_i+b_i+1)/2$. We leave the simple verification to the reader. Next, we consider two cases:

(4) $\{\varphi_t - \varphi_r \,;\, j+1 \leq t \leq a_i - b_i, a_i - b_i + 1 \leq r\}$;

$$\prod_{\substack{j+1\leq t\leq a_i-b_i\\ a_i-b_i+1\leq r}} L(0,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r))/L(1,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)). \tag{4.10}$$

(5) $\{\varphi_t - \varphi_r \,;\, 1 \leq t \leq j, a_i - b_i + 1 \leq r,\ p(r) < p(t)\}$;

$$\prod_{\substack{1\leq t\leq j\\ a_i-b_i+1\leq r\\ p(r)<p(t)}} L(0,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r))/L(1,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)). \tag{4.11}$$

If $L(0,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)) = \infty$ in (4.10) or (4.11), then $\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r) = |\cdot|^{\epsilon}, \epsilon \in \{0,1\}$. Now, if $\epsilon = 0$, then $L(1,\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)) = \infty$. Thus, they will cancel each other in (4.10) or (4.11). Hence $\epsilon = 1$. Since $\boldsymbol{\mu}_{\mathbf{s}} \circ \varphi_t = |\cdot|^{s-(a_i-b_i-1)/2+t-1}\lambda_i$, we see that $\boldsymbol{\mu}_s \circ \varphi_r = |\cdot|^{s-(a_i-b_i-1)/2+t-1-\epsilon}\lambda_i$ at $s = (a_i+b_i+1)/2$. Hence $\boldsymbol{\mu}_s \circ \varphi_r = |\cdot|^{b_i+t-1}\lambda_i$ and $\boldsymbol{\mu}_s \circ \varphi_t = |\cdot|^{b_i+t}\lambda_i$ at $s = (a_i+b_i+1)/2$. Since the indices $r$ and $t$ refer to positions of the corresponding characters from left to right in (4.8), we see that $\boldsymbol{\mu}_s \circ \varphi_r = |\cdot|^{b_i+t-1}\lambda_i$ (at $s = (a_i+b_i+1)/2$) must be one of the characters in

- $|\cdot|^{-b_i}\lambda_i, \ldots, |\cdot|^{b_i}\lambda_i$ ,
- $|\cdot|^{-a_{j'}}\lambda_{j'}, \ldots, |\cdot|^{b_{j'}}\lambda_{j'}, \quad 1 \leq j' \leq i-1.$

Hence, we see that there exists $j'$, $1 \le j' \le i$, such that $b_i + t - 1 \le b_{j'}$, $\lambda_{j'} = \lambda_i$. We claim $j' = i$. If not, $j' < i$, so $b_i < b_{j'}$. Since by the construction of the elements $a_i$ and $b_i$, $]2b_i + 1, 2a_i + 1[ \cap \mathrm{Jord}(\sigma_{v_0})_{\lambda_i} = \varnothing$, we must have $b_i < a_i < b_{j'} < a_{j'}$. (If $a, b \in \mathbb{R}$, then we write $]a,\ b[$ for the set of all $x \in \mathbb{R}$ such that $a < x < b$.) Now, by the construction of discrete series, this means that there exists $u$, $1 \le u < i$, such that $\lambda_u = \lambda_i$, $]2b_u + 1,\ 2a_u + 1[ \cap \mathrm{Jord}(\sigma_{v_0})_{\lambda_i} = \varnothing$ and $[b_u, a_u] \subset [a_j, b_j]$. If we take $2a_u + 1$ largest possible, this contradicts our assumption that $2a_i + 1$ is largest possible available after we have chosen $a_k, b_k, \dots, a_{i+1}, b_{i+1}$. Thus, $j' = i$. Now, $b_i + t - 1 \le b_{j'}$ implies $t = 1$. Hence $\boldsymbol{\mu}_s \circ \varphi_r = |\cdot|^{b_i}\lambda_i$ and $\boldsymbol{\mu}_s \circ \varphi_t = |\cdot|^{b_i+1}\lambda_i$ (at $s = (a_i + b_i + 1)/2$). It is clear that $r = a_i + b_i + 1$. Modifying the above discussion, it is also easy to see that $L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)) = \infty$ implies $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t - \varphi_r)) = \infty$. Thus, the terms (4.10) and (4.11) contribute to (4.9) as follows:

- (4.10) contributes with a simple pole if and only if $j = 0$;
- (4.11) contributes with a simple pole if and only if $p(a_i + b_i + 1) < p(1)$ (note that $j > 0$).

***Claim 1*** *The product of* (4.10) *and* (4.11) *is non-zero and it has at most a simple pole at* $s = (a_i + b_i + 1)/2$.

Finally, we consider the case:

(6) $\{\varphi_t + \varphi_r\ ;\ j + 1 \le t \le a_i - b_i, a_i - b_i + 1 \le r, p(r) > p(t)\}$;

$$\prod_{\substack{j+1 \le t \le a_i - b_i \\ a_i - b_i + 1 \le r \\ p(r) > p(t)}} L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r))/L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r)). \tag{4.12}$$

Put

$$A(s, w) := \sideset{}{'}\prod_{\substack{j+1 \le t \le a_i - b_i \\ a_i - b_i + 1 \le r \\ p(r) > p(t)}} L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r)),$$

$$B(s, w) := \sideset{}{'}\prod_{\substack{j+1 \le t \le a_i - b_i \\ a_i - b_i + 1 \le r \\ p(r) > p(t)}} L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r)),$$

where $\prod'$ means that we throw out $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r))$ and $L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r))$ such that $\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r) = |\cdot|^0$ in $A(s, w)$ and $B(s, w)$, respectively. Since $L$-functions attached to grössencharacters are non-zero at integral points, no term appearing in $A(s, w)$ and $B(s, w)$ can be zero at $s = (a_i + b_i + 1)/2$.

If $L(0, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r)) = \infty$ at $s = (a_i + b_i + 1)/2$, appearing in $A(s, w)$, then $\boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r) = |\cdot|^{\epsilon}$, $\epsilon \in \{0, 1\}$. If $\epsilon = 0$, then $L(1, \boldsymbol{\mu}_{\mathbf{s}} \circ (\varphi_t + \varphi_r)) = \infty$. Thus, they will cancel each other in the product (4.12). Hence $\epsilon = 1$ and $\boldsymbol{\mu}_s \circ \varphi_r = |\cdot|^{-b_i - t + 1}\lambda_i$ and $\boldsymbol{\mu}_s \circ \varphi_t = |\cdot|^{b_i + t}\lambda_i$ at $s = (a_i + b_i + 1)/2$. There are three possibilities (see (4.8)) for where $\boldsymbol{\mu}_s \circ \varphi_r$ can appear:

(1) $|\cdot|^{-b_i}\lambda_i,\ldots,|\cdot|^{b_i}\lambda_i$. Now, $-b_i-t+1\geq -b_i$. Hence $j+1\leq t\leq 1$. This is equivalent to $j=0$ and $t=1$. Also, $r=a_i-b_i+1$. Since $p(r)>p(t)$ and $j=0$, Lemma 1.1 implies that $p(a_i-b_i)=1, p(a_i-b_i-1)=2,\ldots,p(1)=a_i-b_i, p(a_i-b_i+1)=a_i-b_i+1, p(a_i-b_i+2)=a_i-b_i+2,\ldots$. Hence $w=w_0$; ($w_0$ is defined by Lemma 3.6(ii).)

(2) $(1\leq j'\leq i-1)$ $|\cdot|^{-a_{j'}}\lambda_{j'},\ldots,|\cdot|^{b_{j'}}\lambda_{j'}$. First, as before $\lambda_i=\lambda_{j'}$ and $-a_{j'}\leq -b_i-t+1$. This means that $b_i+t-1\leq a_{j'}$. Hence $b_i\leq a_{j'}$. By construction of discrete series, $b_i<a_i<a_{j'}$. The case $b_{j'}>a_i$ is not possible because of "choosing the largest". Thus, we must have $b_{j'}<b_i<a_i<a_{j'}$. We denote by $\{a_{j_1},b_{j_1}\},\ldots,\{a_{j_u},b_{j_u}\}$ all such pairs.

(3) $j'$, $1\leq j'\leq l$, such that $-c_{j'}\leq -b_i-t+1$ and $\lambda'_{j'}=\lambda_i$. This means that $b_i+t-1\leq c_{j'}$. Hence $b_i\leq c_{j'}$. Hence $b_i<a_i<c_{j'}$. We denote by $u'$ the unique such $j'$ if it exists.

The above discussion shows the following:

***Claim 2*** *The order of the pole at $s=(a_i+b_i+1)/2$ of $A(s,w)$, $w\neq w_0$, is determined by the number of coroots in $\{\varphi_t+\varphi_r\ ;\ j+1\leq t\leq a_i-b_i, a_i-b_i+1\leq r, p(r)>p(t)\}$, where $r$ ranges over positions of characters in* (4.8) *corresponding to the exponents in $[-a_{j_1},b_{j_1}],\ldots,[-a_{j_u},b_{j_u}],[-c_{u'},-1/2]$ such that $\boldsymbol{\mu}_s\circ\varphi_r=|\cdot|^{-b_i-t+1}\lambda_i$, $\boldsymbol{\mu}_s\circ\varphi_t=|\cdot|^{b_i+t}\lambda_i$ and $p(r)>p(t)$. If $w=w_0$, this number must be increased by* 1.

Similarly, we obtain the following:

***Claim 3*** *The order of the pole at $s=(a_i+b_i+1)/2$ of $B(s,w)$, is determined by the number of coroots in $\{\varphi_t+\varphi_r\ ;\ j+1\leq t\leq a_i-b_i, a_i-b_i+1\leq r, p(r)>p(t)\}$, where $r$ ranges over positions of characters in* (4.8) *corresponding to the exponents in $[-a_{j_1},b_{j_1}],\ldots,[-a_{j_u},b_{j_u}],[-c_{u'},-1/2]$ such that $\boldsymbol{\mu}_s\circ\varphi_r=|\cdot|^{-b_i-t-1}\lambda_i$, $\boldsymbol{\mu}_s\circ\varphi_t=|\cdot|^{b_i+t}\lambda_i$ and $p(r)>p(t)$.*

Next, we have the following two claims which are easy to verify using Lemma 1.1.

***Claim 4*** *Let us fix one of the segments $[-a_{j_1},b_{j_1}],\ldots,[-a_{j_u},b_{j_u}],[-c_{u'},-1/2]$. Let us consider only indices $r$ corresponding to the position of characters corresponding to exponents in that segment relative to* (4.8)*. Assume that there exists $t$, $j+1\leq t\leq a_i-b_i$, and $r$, such that $p(r)>p(t)$, $\boldsymbol{\mu}_s\circ\varphi_r=|\cdot|^{-b_i-t+1}\lambda_i$ and $\boldsymbol{\mu}_s\circ\varphi_t=|\cdot|^{b_i+t}\lambda_i$. Then this segment contributes to $A(s,w)/B(s,w)$ with a double pole at $s=(a_i+b_i+1)/2$.*

***Claim 5*** *Let us fix one of the segments $[-a_{j_1},b_{j_1}],\ldots,[-a_{j_u},b_{j_u}],[-c_{u'},-1/2]$, and let us consider only indices $r$ corresponding to the position of characters corresponding to exponents in that segment relative to* (4.7)*. Assume that there exist $t$, $j+1\leq t\leq a_i-b_i$, and $r$, such that $p(r)>p(t)$, $\boldsymbol{\mu}_s\circ\varphi_r=|\cdot|^{-b_i-t-1}\lambda_i$ and $\boldsymbol{\mu}_s\circ\varphi_t=|\cdot|^{b_i+t}\lambda_i$. Then this segment contributes to $A(s,w)/B(s,w)$ with a double pole at $s=(a_i+b_i+1)/2$.*

The above discussion shows that for the inverse normalization factor (see (4.9)), $w\in W$, $w(\Delta\setminus\{\alpha_{a_i-b_i}\})>0$, the order of pole at $s=(a_i+b_i+1)/2$ is largest (and strictly greater than any other) for $w=w_0$. Let $m$ be the order of pole of the inverse normalization factor (4.9) at $s=(a_i+b_i+1)/2$ for $w=w_0$.

Now, we multiply the Eisenstein series in (2.8) by $(s-(a_i+b_i+1)/2)^m$. The constant term of such normalized Eisenstein series is given by

$$(s-(a_i+b_i+1)/2)^m \sum_{w, w(\Delta\setminus\{\alpha_{a_i-b_i}\})>0} \bigotimes_{v\notin S_{i+1}} f_{w(\boldsymbol{\mu}_{v,s})} \bigotimes_{v\in S_{i+1}} R(\boldsymbol{\mu}_{v,\mathbf{s}}, w) f_{\boldsymbol{\mu}_{v,s}} \cdot \prod_{\alpha\in\Sigma_+, w(\alpha)<0} \frac{L(0,\boldsymbol{\mu}_s\circ\alpha^\vee)}{L(1,\boldsymbol{\mu}_s\circ\alpha^\vee)\epsilon(1,\boldsymbol{\mu}_s\circ\alpha^\vee)}.$$

Thus, when $s=(a_i+b_i+1)/2$, the normalized Eisenstein series generated has constant term:

$$\lim_{s\to(a_i+b_i+1)/2}(s-(a_i+b_i+1)/2)^m f_{w_0(\boldsymbol{\mu}_{v,s})} \bigotimes_{v\in S_{i+1}} R(\boldsymbol{\mu}_{v,\mathbf{s}}, w_0) f_{\boldsymbol{\mu}_{v,s}} \cdot \prod_{\alpha\in\Sigma_+, w_0(\alpha)<0} \frac{L(0,\boldsymbol{\mu}_s\circ\alpha^\vee)}{L(1,\boldsymbol{\mu}_s\circ\alpha^\vee)\epsilon(1,\boldsymbol{\mu}_s\circ\alpha^\vee)}. \tag{4.13}$$

Now, Lemma 3.6 shows the representation generated by the constant term (4.13) is not zero. Hence, the normalized Eisenstein series is non-trivial at $s=(a_i+b_i+1)/2$. Next, the representation generated by the constant term (4.13), by Lemma 3.6(ii) applied to $v\in S_{i+1}$, must be of the form:

$$\bigotimes_{v\in S_{i+1}} D(\sigma_v) \bigotimes_{v\notin S_{i+1}} X_v, \tag{4.14}$$

where $X_v$ is the subrepresentation of the induced representation (compare to (4.5))

$$\prod_{1\le j\le i} (|\cdot|^{-a_j}\lambda_{j,v}\times\cdots\times|\cdot||^{b_j}\lambda_{j,v})\times|\cdot|^{-c_1}\lambda'_{1,v}\times\cdots \times|\cdot|^{-1/2}\lambda'_{1,v}\times\cdots\times|\cdot|^{-c_l}\lambda'_{l,v}\times\cdots\times|\cdot|^{-1/2}\lambda'_{l,v}\rtimes\mathbf{1} \tag{4.15}$$

generated by $f_{w_0(\boldsymbol{\mu}_{v,s})}$, $s=(a_i+b_i+1)/2$. Clearly, $D(\sigma_v)$, as the unique irreducible spherical subquotient of (4.15), is a subquotient of $X_v$.

Clearly, any irreducible subquotient of the representation in (4.14) is automorphic. The corresponding automorphic forms must be in the space generated by our normalized Eisenstein series intersected with the space of square-integrable automorphic forms. Let us call $Y$ that intersection. This is so since, for $v\in S_{i+1}$, $D(\sigma_v)$ is a local component of any irreducible subquotient of (4.14). Hence, we see the exponents of corresponding automorphic forms, with respect to the Borel subgroup are in $-\mathbb{R}_{>0}\alpha_1\oplus\cdots\oplus-\mathbb{R}_{>0}\alpha_n$. (We are thankful to Colette Mœglin for explaining this to us.) This completes the proof of the existence of the global representation $\bigotimes_v D(\sigma_v)$ satisfying (i) and (ii) of Proposition 4.2. We show that it also satisfies (iii). First, clearly (4.14) is a quotient of $Y$. Since $Y$ is semisimple (being an admissible subrepresentation in the space of square-integrable automorphic forms), we see that the representation in (4.14) must be semisimple. In particular, $X_v$ is semisimple. On the other hand, as we remarked above, it is generated by a spherical element $f_{w_0(\boldsymbol{\mu}_{v,s})}$, $s=(a_i+b_i+1)/2$. Hence $D(\sigma_v)\simeq X_v$. ■

Now, we prove Lemma 4.3. We use Lemma 2.2 and Lemma 2.3 again. As part of the inductive assumption of Proposition 4.2, $\bigotimes_v D(\sigma_{i,v})$ comes with an automorphic realization $V_i$ such that the computation of constant term gives the embedding (4.5). We put

$$\begin{aligned}(4.16)\qquad \boldsymbol{\mu}_{\mathbf{s}} = |\cdot|^{-b_i+s}\lambda_i \otimes \cdots \otimes |\cdot|^{b_i+s}\lambda_i \bigotimes_{1\le j\le i-1} (|\cdot|^{-a_j}\lambda_j \otimes \cdots \otimes |\cdot|^{b_j}\lambda_j)\\ \otimes |\cdot|^{-c_1}\lambda_1' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_1' \otimes \cdots \otimes |\cdot|^{-c_l}\lambda_l' \otimes \cdots \otimes |\cdot|^{-1/2}\lambda_l'\end{aligned}$$

We first prove the following claim:

***Claim 6*** *Let $w$ be such that $w(\Delta \setminus \{\alpha_{2b_i+1}\}) > 0$. Also, let $w_0$ denote the longest element such that $w_0(\Delta \setminus \{\alpha_{2b_i+1}\}) > 0$. Then $w(\boldsymbol{\mu}_{\mathbf{s}}) = \boldsymbol{\mu}_{\mathbf{s}}$ at $s = 0$ if and only if $w = id$ or $w = w_0$.*

**Proof** We use Lemma 1.1. Let $w = p\epsilon \in W_j$ (see Section 1), $0 \le j \le 2b_i + 1$. Now, we look at how we get the $(2b_i + 1)$-th element (from the left), $|\cdot|^{b_i}\lambda_i$, in (4.16) ($s$=0). First, it cannot be produced by moving some character in $|\cdot|^{-a_j}\lambda_j, \dots, |\cdot|^{b_j}\lambda_j$ $(1 \le j \le i-1)$ left to the position $2b_i + 1$, since then we would have $b_i \le b_j$, and using Lemma 1.1, we would also have that $|\cdot|^{-a_j}\lambda_j$ is one of the characters $|\cdot|^{-b_i}\lambda_i, \dots, |\cdot|^{b_i}\lambda_i$. Hence we obtain $-a_j > -b_i$. Thus, $a_j < b_i$. Now, $a_j < b_i \le b_j < a_j$. This is a contradiction. It is easy to complete the proof of the claim using the explicit description of $w = p\epsilon$. ■

Now, we form the global induced representation (see (2.7))

$$(4.17)\qquad |\det|^s \mathbf{1}_{\mathrm{GL}(2b_i+1,\mathbb{A})}\lambda_i \rtimes (\bigotimes_v D(\sigma_{i,v})).$$

We form Eisenstein series as in (2.8). By a deep result of Langlands, it is holomorphic at $s = 0$. We use that fact to complete the proof of the lemma. First, we use $f_s$ so that its constant term $(f_s)_0$ is in (4.17) (compare to (2.13)). Now, Lemma 2.3 and Claim 6 imply that

$$(4.18)\qquad (f_s)_0 + M(\boldsymbol{\mu}_{\mathbf{s}}, w_0)(f_s)_0$$

is holomorphic at $s = 0$. We can write $(f_s)_0 = \bigotimes_v f_{\boldsymbol{\mu}_{v,\mathbf{s}}}$. We also put (see Section 2 for notation)

$$r(s, w_0) = \prod_v r(\boldsymbol{\mu}_{v,\mathbf{s}}, w_0).$$

Now, one can rewrite (4.18) as follows:

$$(4.19)\qquad \bigotimes_v f_{\boldsymbol{\mu}_{v,\mathbf{s}}} + r(s, w_0)^{-1} \cdot \bigotimes_v R_v(s, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,\mathbf{s}}}.$$

We remark that in (4.19), $R_v(s, \widetilde{w}_0)$ is defined by Lemma 3.6(ii). Also, for all but finitely many places $v$, $f_{\boldsymbol{\mu}_{v,s}}$ is $K_v$-invariant and, using Theorem 2.1(iii), we obtain

$$R_v(s, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,s}} = f_{\boldsymbol{\mu}_{v,s}} \quad (s = 0).$$

To analyze (4.19), we observe that $D(\sigma_{i,v})$ is unitary and that

$$R_v(0, \widetilde{w}_0)\colon \mathbf{1}_{\mathrm{GL}(2b_i+1,K_v)}\lambda_{i,v} \rtimes D(\sigma_{i,v}) \to \mathbf{1}_{\mathrm{GL}(2b_i+1,K_v)}\lambda_{i,v} \rtimes D(\sigma_{i,v})$$

is a Hermitian operator such that $R_v(0, \widetilde{w}_0)^2 = id$. Therefore it is holomorphic. (I would like to thank Erez Lapid for explaining this.) It follows that $\bigotimes_v R_v(s, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,s}}$ is holomorphic at $s = 0$ and clearly non-zero for $f_0 \neq 0$. Now, $r(s, w_0)^{-1}$ is holomorphic at $s = 0$. We write $r$ for its value at $s = 0$. Now, (4.19) at $s = 0$ is written as follows:

$$\bigotimes_v f_{\boldsymbol{\mu}_{v,s}} + r \cdot \bigotimes_v R_v(0, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,s}}. \tag{4.20}$$

Hence, for $v \in S_i$, $v \neq v_0$, we choose an arbitrary irreducible subrepresentation $D(\tau_{i,v}) \hookrightarrow \mathbf{1}_{\mathrm{GL}(2b_i+1,K_v)}\lambda_{i,v} \rtimes D(\sigma_{i,v})$. If $v|\infty$, then $D(\tau_{i,v}) \hookrightarrow \mathbf{1}_{\mathrm{GL}(2b_i+1,K_v)}\lambda_{i,v} \rtimes D(\sigma_{i,v})$ is the unique irreducible spherical subrepresentation. Now, we choose $f_{\boldsymbol{\mu}_{v,s}}$, $v \in S_i$, such that it belongs to $D(\tau_{i,v})$ when $s = 0$. Now, we have the following ($s = 0$)

$$\bigotimes_{v \in S_i} R_v(0, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,s}} = \pm \bigotimes_{v \in S_i} f_{\boldsymbol{\mu}_{v,s}}.$$

Hence ($s = 0$)

$$\bigotimes_{v \in S_i} f_{\boldsymbol{\mu}_{v,s}} + r \cdot \bigotimes_{v \in S_i} R_v(0, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v,s}} = \bigotimes_{v \in S_i} f_{\boldsymbol{\mu}_{v,s}} (1 \pm r). \tag{4.21}$$

If the expression in (4.21) is not zero, then we choose $f_{\boldsymbol{\mu}_{v,s}}$, $v \notin S_i$, $v < \infty$, to be $K_v$-spherical and normalized as in Theorem 2.1(iii). Now, the Eisenstein series (2.8) is nonzero at $s = 0$, giving the required embedding in Lemma 4.3. We take $S_{i+1} = S_i$.

If the expression in (4.21) is zero, then we choose $v_1 \notin S_i$, $v_1 < \infty$. We note that $R_{v_1}(0, \widetilde{w}_0)$ is not the identity. (See for example [Mœ2, p. 692]). Thus we can take $f_{\boldsymbol{\mu}_{v_1,s}}$ such that $R_{v_1}(0, \widetilde{w}_0) f_{\boldsymbol{\mu}_{v_1,s}} = -f_{\boldsymbol{\mu}_{v_1,s}}$ at $s = 0$. We put $S_{i+1} = S_i \cup \{v_1\}$ and take $f_{\boldsymbol{\mu}_{v,s}}$, $v \notin S_{i+1}$, $v < \infty$, to be $K_v$-spherical and normalized as in Theorem 2.1(iii). Now, the Eisenstein series (2.8) is non-zero at $s = 0$, giving the required embedding in Lemma 4.1. This completes the proof of Lemma 4.3. ■

## 5 Unitarity of Spherical Unipotent Representations over $\mathbb{R}$ and $\mathbb{C}$

Let $F$ be $\mathbb{R}$ or $\mathbb{C}$. The dual groups of the split (over $F$) groups $G_n = \mathrm{Sp}(n), \mathrm{SO}(2n+1)$, and $\mathrm{O}(2n)$ are $\hat{G}_n(\mathbb{C}) = \mathrm{SO}(2n+1, \mathbb{C}), \mathrm{Sp}(n, \mathbb{C})$, and $\mathrm{O}(2n, \mathbb{C})$, respectively. We write $|\cdot|$ for normalized absolute value of $F$. (It is the ordinary absolute value for $F = \mathbb{R}$ and the square of the usual module of a complex number: $|z| = z \cdot \bar{z}$, $z \in \mathbb{C}$.)

We look at algebraic (or holomorphic) homomorphisms of $\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \hat{G}_n(\mathbb{C})$. We say that $\varphi$ is spherical unipotent if its image is not contained in any proper Levi factor of $\hat{G}_n(\mathbb{C})$. Using standard geometric arguments [M2, Mœ1], they might be classified combinatorially as follows. First, there is a one-to-one correspondence between the set of conjugacy classes of homomorphisms $\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \hat{G}_n(\mathbb{C})$ such that its image is not contained in any proper Levi factor $\hat{G}_n(\mathbb{C})$ and

(i) (assume $G_n = \mathrm{SO}(2n+1)$) the sequences $(m_1,\dots,m_k)$, $m_i \in 2\mathbb{Z}$, $0 < m_1 < \cdots < m_k$, $\sum_i m_i = 2n$. The corresponding spherical Harish-Chandra module $\sigma$ is a subquotient of

$$|\cdot|^{-(m_k-1)/2} \times \cdots \times |\cdot|^{(m_{k-1}-1)/2} \times \cdots \times |\cdot|^{-(m_2-1)/2} \times \cdots \times |\cdot|^{(m_1-1)/2} \rtimes \mathbf{1},$$

if $k$ is even and of

$$|\cdot|^{-(m_k-1)/2} \times \cdots \times |\cdot|^{(m_{k-1}-1)/2} \times \cdots \times |\cdot|^{-(m_3-1)/2} \times \cdots \times |\cdot|^{(m_2-1)/2} \times |\cdot|^{-(m_1-1)/2} \times \cdots \times |\cdot|^{-1/2} \rtimes \mathbf{1}, \tag{5.1}$$

if $k$ is odd;

(ii) (assume $G_n = \mathrm{O}(2n)$) the sequences $(m_1,\dots,m_k)$, $m_i \in 2\mathbb{Z}_{>0}+1$, $0 < m_1 < \cdots < m_k$, $\sum_i m_i = 2n$. $k$ is even here;

(iii) (assume $G_n = \mathrm{Sp}(2n)$) the sequences $(m_1,\dots,m_k)$, $m_i \in 2\mathbb{Z}_{>0}+1$, $0 < m_1 < \cdots < m_k$, $\sum_i m_i = 2n+1$.

For (ii) and (iii), the corresponding spherical Harish-Chandra module $\sigma$ is a subquotient of

$$|\cdot|^{-(m_k-1)/2} \times \cdots \times |\cdot|^{(m_{k-1}-1)/2} \times \cdots \times |\cdot|^{-(m_2-1)/2} \times \cdots \times |\cdot|^{(m_1-1)/2} \rtimes \mathbf{1},$$

if $k$ is even and of

$$|\cdot|^{-(m_k-1)/2} \times \cdots \times |\cdot|^{(m_{k-1}-1)/2} \times \cdots \times |\cdot|^{-(m_3-1)/2} \times \cdots \times |\cdot|^{(m_2-1)/2} \times |\cdot|^{-(m_1-1)/2} \times \cdots \times |\cdot|^{-1} \rtimes \mathbf{1}, \tag{5.2}$$

if $k$ is odd.

Now, we have the following theorem:

***Theorem 5.1*** *The representation $\sigma$ is a subrepresentation of the corresponding induced representation above. Moreover, $\sigma$ is unitarizable.*

**Proof** This theorem follows at once from Propositions 4.1 and 4.2 if $F = \mathbb{C}$. A minor problem is that in Section 4 we assumed that all archimedean places are complex. That was only to ensure that we can work with spherical representations on archimedean places. If all grössencharacters from the set $I$ defined in the last section are trivial characters, then we can remove the assumption from the beginning of that section that all archimedean places are complex. The proofs in that section do not require any further modification. For example, in the case (i) above, when $k$ is odd we build discrete series $D(\sigma_v)$, $v \in S$, as follows ($k = 2l+1$):

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- $D(\sigma_{1,\nu}) \hookrightarrow \zeta(-(m_1-1)/2, -1/2, 1) \rtimes \mathbf{1}$;
- there exists an irreducible subrepresentation

$$D(\tau_{j,\nu}) \hookrightarrow \zeta\big(-(m_{2j}-1)/2, (m_{2j}-1)/2, 1\big) \rtimes D(\sigma_{j,\nu})$$

  such that $\zeta(m_{2j}-1)/2+1, (m_{2j+1}-1)/2, 1) \rtimes D(\tau_{j,\nu}) \twoheadrightarrow D(\sigma_{j+1,\nu})$, $2 \leq j \leq l-1$;
- $D(\sigma_{l,\nu}) \simeq D(\sigma_\nu)$.

■

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