Can. J. Math., Vol. XXII, No. 6, 1970, pp. 1266–1283

IDEALS IN RINGS OF ANALYTIC FUNCTIONS WITH SMOOTH BOUNDARY VALUES

B. A. TAYLOR AND D. L. WILLIAMS

1. Introduction. Let A denote the Banach algebra of functions analytic in the open unit disc D and continuous in $\bar{D}$. If f and its first m derivatives belong to A, then the boundary function $f(e^{i\theta})$ belongs to $C^m(\partial D)$. The space A^m of all such functions is a Banach algebra with the topology induced by $C^m(\partial D)$. If all the derivatives of f belong to A, then the boundary function belongs to $C^\infty(\partial D)$, and the space A^∞ of all such functions is a topological algebra with the topology induced by $C^\infty(\partial D)$. In this paper we determine the structure of the closed ideals of A^∞ (Theorem 5.3).

Beurling and Rudin (see e.g. [**7**, pp. 82–89; **10**]) have characterized the closed ideals of A, and their solution suggests a possible structure for the closed ideals of A^∞. To a closed ideal I in A^∞, associate S, the greatest common divisor of the singular inner factors of the non-zero functions in I, and $Z(I) = \{Z^n(I)\}$, where

$$Z^n(I) = \bigcap_{f \in I} \{z \in \bar{D}: f^{(k)}(z) = 0, k = 0, \ldots, n\},$$

$n = 0, 1, \ldots$. Let $I(Z(I))$ denote the closed ideal of all functions $f \in A^\infty$ with $f^{(n)}(z) = 0$ for $z \in Z^n(I)$, $n = 0, 1, \ldots$. We show that

$$I = \{f \in I(Z(I)): S|f\} = S \cdot I(Z(I)).$$

The proofs given here parallel the proof of the Beurling-Rudin Theorem for A, as presented in [**7**, pp. 82–89], except that the role of the F. and M. Riesz Theorem is replaced by certain estimates for subharmonic functions.

In studying the ideal problem, the question of factorization of A^∞ functions arises. We show, in particular, that if an inner function S divides the inner part of an A^∞ function f, then f/S belongs to A^∞ (Theorem 4.1).

Since zero sets play a prominent role in the ideal structure of A^∞, it is of interest to characterize the zero sets of A^∞ functions. Carleson [**2**] has shown that the boundary zero sets of analytic functions in A^m, or even satisfying a Lipschitz condition, are the closed sets $E \subset \partial D$ such that the function $\log \rho(e^{i\theta}, E)$ is integrable. Here, $\rho(e^{i\theta}, E)$ is the distance from $e^{i\theta}$ to E. Such sets are called Carleson sets. Novinger [**9**] and ourselves have independently

Received April 2, 1970. The research of the first author was supported by the Sloan Foundation while he was a visiting member at the Courant Institute of Mathematical Sciences, New York University, and by the National Science Foundation at the University of Michigan. The research of the second author was supported by the Air Force, Contract No. AF 49 (638)-1401.

shown that the Carleson sets are also the boundary zero sets of A^∞ functions. Also, L. Carleson and S. Jacobs have recently proved that if $F \in A$ is an outer function with $|F| \in C^\infty(\partial D)$, then $F \in A^\infty$ (unpublished). This result can be used to easily construct A^∞ functions vanishing on a given Carleson set. In our proof of the theorem characterizing the closed ideals of A^∞, we have found it necessary to construct outer functions in A^∞ whose zero sets are a given Carleson set and which have some additional properties (see Theorem 3.3). We also note that the sets in $\bar{D}$ which are zero sets of A^∞ functions have been characterized as follows [**13**]. A closed set $Z \subset \bar{D}$ is the zero set of an A^∞ function, or a function satisfying a Lipschitz condition, if and only if

$$\sum_{z_n \in D \cap Z} (1 - |z_n|) < +\infty \quad \text{and} \quad \int_0^{2\pi} \log \rho(e^{i\theta}, Z)\, d\theta > -\infty.$$

The techniques we use to obtain the ideal structure of A^∞ may be applied to obtain information about the ideal structure of other algebras of analytic functions satisfying some regularity condition on ∂D. In § 6, we comment on the ideal structure of A^m. In particular, we determine the structure of those closed ideals of A^m whose functions have at most a finite number of common zeros of order m on ∂D. This result is closely related to some recent work of Kahane [**8**] and Gurariĭ [**6**].

Acknowledgement. We wish to thank Professors Allen L. Shields and Harold S. Shapiro for helpful conversations concerning this work.

2. Definitions and duality. Let D denote the open unit disc in the complex plane $\mathbf{C}$ and let A be the Banach algebra of functions f analytic in D and continuous in $\bar{D}$ with $||f||_\infty = \sup\{|f(z)|: z \in \partial D\}$.

2.1. *Definition. The space A^m, $m = 1, 2, \ldots$, is the algebra of functions $f \in A$ such that $f^{(n)} \in A$, $n = 0, 1, \ldots, m$. The space A^∞ is the algebra of functions $f \in A$ such that $f^{(n)} \in A$ for $n = 1, 2, \ldots$, i.e.,*

$$A^\infty = \bigcap \{A^m: m = 1, 2, \ldots\}.$$

We now give a brief account of the topology and the dual space of A^∞. Let $C^\infty(\partial D)$ be the space of infinitely differentiable complex-valued functions on the unit circle. We provide $C^\infty(\partial D)$ with the usual locally convex topology defined by the seminorms

$$||f||_m = \sum_{k=0}^{m} ||f^{(k)}||_\infty, \qquad m = 0, 1, 2, \ldots.$$

Each $f \in C^\infty(\partial D)$ has a Fourier series expansion $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$, convergent in the topology of $C^\infty(\partial D)$, where $|c_n| = O(|n|^{-m})$ for all positive integers m. The seminorms

$$||f||_m' = \sup_n |c_n| (|n| + 1)^m, \qquad m = 0, 1, 2, \ldots,$$

also describe the topology of $C^\infty(\partial D)$.

https://doi.org/10.4153/CJM-1970-143-x Published online by Cambridge University Press

By restricting each $f \in A^\infty$ to ∂D, we may identify A^∞ with the closed subalgebra of $C^\infty(\partial D)$ consisting of those functions with vanishing negative Fourier coefficients.

Since the topologies of $C^\infty(\partial D)$ and A^∞ are given by a countable collection of seminorms, they are Fréchet and hence barrelled spaces. Also, the closed bounded sets in $C^\infty(\partial D)$, and hence in A^∞, are compact. Thus $C^\infty(\partial D)$ and A^∞ are Montel spaces.

The dual of $C^\infty(\partial D)$ is $\mathscr{D}'(\partial D)$, the space of Schwartz distributions on the circle ∂D. The value of a distribution T at $f \in C^\infty(\partial D)$ is denoted by (f, T). Every distribution $T \in \mathscr{D}'(\partial D)$ has a Fourier series representation

$$T(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta},$$

converging to T in the strong topology of $\mathscr{D}'(\partial D)$, where $d_n = (e^{-in\theta}, T)$ and $|d_n| = O(|n|^m)$ for some sufficiently large integer m [**11**, p. 224]. Conversely, every trigonometric series with coefficients satisfying this growth condition is the Fourier series of a unique distribution. If

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta} \in C^\infty(\partial D) \qquad \text{and} \qquad T(e^{i\theta}) = \sum_{n=-\infty}^{\infty} d_n e^{in\theta} \in \mathscr{D}'(\partial D),$$

then

$$(f, T) = \sum_{n=-\infty}^{\infty} c_n d_{-n}.$$

Let B' be the strongly closed subspace of $\mathscr{D}'(\partial D)$ consisting of the distributions with vanishing positive Fourier coefficients. The space $C^\infty(\partial D)$ is the topological direct sum of A^∞ and the subspace of $C^\infty(\partial D)$ functions with vanishing non-negative Fourier coefficients. Likewise, $\mathscr{D}'(\partial D)$ in the strong topology is the topological direct sum of B' and the subspace of distributions with vanishing non-positive Fourier coefficients. Thus B' is the dual of A^∞; if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in A^\infty \qquad \text{and} \qquad T(e^{i\theta}) = \sum_{n=0}^{\infty} b_n e^{-in\theta} \in B',$$

then the value of T at f is $(f, T) = \sum_{n=0}^{\infty} a_n b_n$.

For each $T \in B'$, the (Borel) transform $T(\zeta) = (f_\zeta, T)$, where $f_\zeta(z) = \zeta(\zeta - z)^{-1}$, is a function analytic for $|\zeta| > 1$. If T has the Fourier series $\sum_{n=0}^{\infty} b_n e^{-in\theta}$, then $\sum_{n=0}^{\infty} b_n \zeta^{-n}$ is the Laurent expansion of $T(\zeta)$. A short calculation shows that if $|b_n| \leqq (n+1)^m$, then

$$|T(\zeta)| \leqq \text{const} \cdot [(|\zeta| - 1)^{-m-1} + 1], \qquad |\zeta| > 1.$$

On the other hand, if $U(\zeta)$ is any function analytic for $|\zeta| > 1$ such that

$$|U(\zeta)| \leqq (|\zeta| - 1)^{-m} + 1,$$

then its Laurent coefficients $\{d_n\}$ satisfy

$$|d_n| \leqq \text{const} \cdot (n+1)^m, \qquad n = 0, 1, 2, \ldots.$$

Consequently, $U(\zeta)$ determines uniquely the element $U(e^{i\theta}) = \sum_{n=0}^{\infty} d_n e^{-in\theta}$ of B'. Therefore, we may identify B' with the space of all functions $T(\zeta)$ analytic for $|\zeta| > 1$, including ∞, such that

$$|T(\zeta)| = O((|\zeta| - 1)^{-m}), \qquad |\zeta| \to 1^+,$$

for some $m > 0$. We will regard $T \in B'$ as the boundary value of the analytic function $T(\zeta)$ as $|\zeta| \to 1^+$. It is easy to verify that for $T \in B'$ and $f \in A^\infty$,

$$(f, T) = \lim_{r\to 1^+} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) T(re^{i\theta})\, d\theta.$$

2.2. *Remark.* The space B' can also be identified, by Fourier transform, with the space of all entire functions $F(z)$ such that

$$|F(z)| \leqq C(1 + |z|)^m e^{|z|}.$$

See [**12**].

3. Construction of A^∞ outer functions. To establish the characterization of the closed ideals of A^∞ (Theorem 5.3) we have found it necessary to use certain well-behaved outer functions in A^∞. In this section the existence of such outer functions is proved by a modification of a construction of Carleson [**2**, Theorem 1].

Let E be a closed subset of ∂D and let $\rho(z) = \rho(z, E)$ denote the distance from z to E.

3.1. *Definition. The closed set $E \subset \partial D$ is a Carleson set if*

$$\int_{-\pi}^{\pi} \log \rho(e^{i\theta}, E)\, d\theta > -\infty. \tag{3.1}$$

Now, if $F \in A^1$ (or even if F satisfies a Lipschitz condition of order $\alpha > 0$), $F \not\equiv 0$, and F vanishes on E, then

$$\log|F(z)| \leqq \alpha \log \rho(z, E) + K$$

for some α, $K > 0$. Thus, (3.1) holds since $\log F(e^{i\theta})$ is integrable for $-\pi \leqq \theta \leqq \pi$. As mentioned earlier, the converse is true. For our purposes, we need a slight extension of the converse.

3.2. *Definition. For $f \in A^\infty$ let*

$$Z^n(f) = \{z \in \bar{D} : f^{(k)}(z) = 0,\ k = 0, 1, \ldots, n\}, \qquad n = 0, 1, 2, \ldots,$$

and let

$$Z^\infty(f) = \bigcap_{n=0}^{\infty} Z^n(f).$$

3.3. THEOREM. *Let E be a Carleson set. Then there exist outer functions $F_k \in A^\infty$, $k = 1, 2, \ldots$, such that*

$$Z^0(F_k) = Z^\infty(F_k) = E$$

and for every $h \in A^\infty$ with $Z^\infty(h) \supset E$, the sequence $\{F_k h\}$ converges to h in A^∞.

To construct such outer functions F_k we consider real-valued functions φ on ∂D which satisfy the following conditions:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi(e^{i\theta})|\, d\theta \leqq M < +\infty; \tag{3.2}$$

(3.3) φ is infinitely differentiable on $\partial D \sim E$ and

$$\left|\frac{d^n}{d\theta^n}\varphi(e^{i\theta})\right| \leqq C_n \rho(e^{i\theta})^{-p_n}, \qquad n = 0, 1, 2, \ldots,$$

for some constants C_n, $p_n \geqq 0$;

(3.4) $\varphi \geqq 0$ and for every $C > 0$,

$$\varphi(e^{i\theta}) + C \log \rho(e^{i\theta}) \to +\infty \text{ as } \rho(e^{i\theta}) \to 0.$$

Now define

$$G(z) = G(z, \varphi) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\varphi(e^{i\theta})\, d\theta, \qquad z \in D. \tag{3.5}$$

Provided that φ satisfies (3.2)–(3.4), it will be shown that $F = \exp(-G) \in A^\infty$. To do this and to prove Theorem 3.3 we use the following two lemmas.

3.4. Lemma. *Assume that φ satisfies* (3.2) *and* (3.3) *and that G is defined by* (3.5). *Then*

$$|G^{(n)}(z)| \leqq D_n \rho(z)^{-q_n}, \qquad n = 0, 1, 2, \ldots,$$

for some constants D_n, $q_n \geqq 0$ which depend only on M, C_n, and p_n.

The proof of this lemma follows closely Carleson's proof of [**2**, Theorem 1], and is exactly the same as that of [**13**, Lemma 2.3]. We shall not reproduce it here.

3.5. Lemma. *Assume that $\varphi \geqq 0$ and satisfies* (3.2). *Then for $z = re^{i\theta}$ with* $\frac{1}{4} < r < 1$,

$$\operatorname{Re} G(z) \geqq \eta(r, \theta),$$

where

$$\eta(r, \theta) = \tfrac{1}{4}r^{-1/2} \inf\{\varphi(e^{it}) : |t - \theta| \leqq r^{-1/2}(1 - r)\}.$$

If in addition (3.4) *holds, then*

$$|F(z)| \leqq \rho(z)^{\tau(z)}, \qquad z \in D,$$

where $\tau(z) \to +\infty$ as $\rho(z) \to 0$.

Proof. Setting $\delta = \delta(r) = r^{-1/2}(1 - r)$ and using the Poisson kernel

$$P(r, t) = (1 - r^2)(1 - 2r\cos t + r^2)^{-1},$$

we have

$$\operatorname{Re} G(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, t)\varphi(e^{i(\theta - t)})\, dt$$
$$\geqq \frac{1}{2\pi}\int_{|t| \leqq \delta} P(r, t)\varphi(e^{i(\theta - t)})\, dt.$$

Since $P(r, t) \geqq (1 - r^2)[(1 - r)^2 + rt^2]^{-1}$ and

$$\frac{1}{2\pi}\int_{-\delta}^{\delta} (1 - r^2)[(1 - r)^2 + rt^2]^{-1}\,dt = \tfrac{1}{4}r^{-1/2}(1 + r),$$

the first inequality of the lemma follows easily. The second inequality is a consequence of the first.

Proof of Theorem 3.3. It follows easily from Lemma 3.4 and the second inequality of Lemma 3.5 that every outer function $F = \exp(-G)$, where φ satisfies (3.2)–(3.4), belongs to A^∞. Moreover, F vanishes exactly on E and all its derivatives vanish there too. We will construct our sequence F_k in the form $F_k(z) = \exp(-G(z, \varphi_k))$ by selecting suitable functions φ_k.

To this end, introduce the following slight modification of $\rho(e^{i\theta})$. Let $\{(e^{ia_n}, e^{ib_n}): a_n < b_n\}$ be the complementary arcs of the Carleson set E and define

$$\tilde{\rho}(\theta) = \begin{cases} \left(\dfrac{1}{\theta - a_n} + \dfrac{1}{b_n - \theta}\right)^{-1}, & \theta \in (a_n, b_n), \\ 0, & e^{i\theta} \in E. \end{cases}$$

Note that

$$\text{const} \cdot \rho(e^{i\theta}) \leqq \tilde{\rho}(\theta) \leqq \text{const} \cdot \rho(e^{i\theta}).$$

Therefore $\log \tilde{\rho}(\theta)$ is integrable on $[-\pi, \pi]$.

Choose a positive, increasing, infinitely differentiable function $\omega(x)$, $-\infty < x < +\infty$, such that

$$x^{-1}\omega(x) \to +\infty \quad \text{as } x \to +\infty; \tag{3.6}$$

$$\omega^{(n)}(x) \leqq C_n'(1 + |x|^2), \qquad n = 0, 1, 2, \ldots; \tag{3.7}$$

$$\int_{-\pi}^{\pi} \omega(-\log \tilde{\rho}(\theta))\, d\theta < +\infty. \tag{3.8}$$

It is easy to obtain an increasing function ω_1 satisfying (3.6) and (3.8). Then ω may be obtained as the convolution

$$\omega(x) = (\omega_1 * \chi)(x) = \int_{-\infty}^{+\infty} \omega(t)\chi(x - t)\, dt$$

of ω_1 with a non-negative infinitely differentiable function χ with compact support in $x \geqq 0$ and $\int_{-\infty}^{+\infty} \chi(t)\, dt = 1$. Next, let ψ be an infinitely differentiable function defined for $-\infty < x < +\infty$, with $0 \leqq \psi(x) \leqq 1$, $\psi(x) = 0$ for $x \leqq 1$, and $\psi(x) = 1$ for $x \geqq 2$. Define $\omega_k(x) = \psi(x/k)\omega(x)$ and $\varphi_k(e^{i\theta}) = \omega_k(-\log \tilde{\rho}(\theta))$. We assert that the associated F_k are suitable functions.

To prove this, first note that the φ_k satisfy (3.2)–(3.4) uniformly in k. That is, the constants M, C_n, p_n may be chosen independent of k, as is routinely verified. Consequently, Lemma 3.4 implies the existence of constants D_n, q_n such that

$$|G^{(n)}(z, \varphi_k)| \leqq D_n\rho(z)^{-q_n}, \qquad n = 0, 1, 2, \ldots, \tag{3.9}$$

for all $z \in D$ and $k = 1, 2, \ldots$.

Let $h \in A^\infty$ with $Z^\infty(h) \supset E$. To prove $F_k h \to h$ in A^∞, it suffices to show that $F_k h \to h$ pointwise and that

$$\sup\left\{\left|\frac{d^m}{dz^m} F_k(z)h(z)\right|: \quad z \in D, \quad k = 1, 2, \ldots\right\}$$

is finite for every $m = 0, 1, 2, \ldots$. For $e^{i\theta} \notin E$, $\lim_{k\to\infty} \varphi_k(e^{i\theta}) = 0$ and $\varphi_k(e^{i\theta}) \leqq \omega(-\log \tilde{\rho}(\theta))$. Because of (3.8) and the dominated convergence theorem, it follows that $\lim_{k\to\infty} G(z, \varphi_k) = 0$ for each $z \in D$; hence

$$F_k(z)h(z) \to h(z)$$

for each $z \in D$. The uniform boundedness of the mth derivatives of the $F_k h$ follows from (3.9) and the fact that for all positive integers n and l,

$$|h^{(n)}(z)| = O(\rho(z)^l) \quad \text{as } \rho(z) \to 0.$$

(This estimate for h follows immediately by writing an appropriate Taylor series expansion with remainder about points of E. (See Proposition 4.4.))

The proof of the theorem is now complete.

4. Factorization. Every function $f \in A$ has a factorization $f = SF$ into an inner function S and an outer function F. Moreover, if $S = S_1 \cdot S_2$, where S_1 and S_2 are inner functions, then $S_2 F \in A$ (see e.g. [7, pp. 69, 70]). Here we establish the same result for A^∞ functions.

When we say that an inner function S divides an H^1 function f, we mean that the quotient of the inner part of f by S is an inner function.

4.1. THEOREM. *If $f = SF \in A^\infty$, where S is an inner function dividing f, then $F \in A^\infty$. In particular, the outer part of an A^∞ function is an A^∞ function. Moreover, the set $\{F \in A^\infty: f = SF, S \text{ inner}\}$ is a bounded subset of A^∞.*

Proof. Represent f and S with their Taylor series expansions

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad S(z) = \sum_{j=0}^{\infty} b_j z^j.$$

Since $|S(e^{i\theta})| = 1$ a.e. on ∂D, $S(e^{i\theta})^{-1} = \overline{S(e^{i\theta})}$ a.e. on ∂D. Because $F \in A$, it is defined on ∂D, and we have $F(e^{i\theta}) = f(e^{i\theta})\overline{S(e^{i\theta})}$ a.e. on ∂D. Let $F(e^{i\theta}) = \sum_{n=0}^\infty c_n e^{in\theta}$ be the Fourier series of F on ∂D. The Fourier series of f and $\bar{S}$ are $\sum_{k=0}^\infty a_k e^{ik\theta}$ and $\sum_{j=0}^\infty \bar{b}_j e^{-ij\theta}$, respectively. Hence

$$c_n = \sum_{k=n}^{\infty} a_k \bar{b}_{k-n}, \qquad n = 0, 1, 2, \ldots .$$

Since S is bounded by 1, the Cauchy inequalities imply that $|b_j| \leqq 1$, $j = 0, 1, 2, \ldots$. Choose any integer $m > 0$ and, using the assumption $f \in A^\infty$, choose $M > 0$ such that $|a_k| \leqq Mk^{-m-2}$, $k = 1, 2, \ldots$. Then

$$|c_n| \leqq \sum_{k=n}^{\infty} |a_k \bar{b}_{k-n}| \leqq \sum_{k=n}^{\infty} Mk^{-m-2} \leqq Mn^{-m} \sum_{k=1}^{\infty} k^{-2}$$

for $n = 1, 2, \ldots$. Hence $F \in A$ has a C^∞ boundary function; thus $F \in A^\infty$. The last assertion of the theorem follows immediately from the above inequality.

In the discussion that follows we will make use of the following two results.

4.2. THEOREM (Rudin). *If S is the singular inner function determined by the positive singular measure μ, then S is analytic everywhere in* **C** *except on the support of μ in ∂D and $|S|$ cannot be extended continuously from D to any point in the support of μ.*

4.3. THEOREM (Caughran). *If $f \in A$ and $f' \in H^p$ for some $p > 1$, then the singular inner part of f divides f'.*

A proof of Theorem 4.2 may be found in [**7**, p. 68] and a proof of Theorem 4.3 may be found in [**3**].

Next we state three simple propositions concerning the zeros of A^∞ functions. If $f \in A^m$ and $a, z \in \bar{D}$, we have the Taylor expansion

$$f(z) = \sum_{n=0}^{m-1} \frac{1}{n!} f^{(n)}(a)(z-a)^n + \frac{1}{(m-1)!} \int_a^z f^{(m)}(\zeta)(z-\zeta)^{m-1} d\zeta,$$

where the path of integration in the remainder term is the straight line from a to z. The first two propositions follow easily from this representation. We omit their proofs.

4.4. PROPOSITION. *If $f \in A^\infty$, then for $n = 0, 1, 2, \ldots$,*

$$|f(z)| = O(\rho(z, Z^n(f))^{n+1}) \quad \text{as } \rho(z, Z^n(f)) \to 0.$$

4.5. PROPOSITION. *If $f \in A^\infty$, the following are equivalent for $n = 0, 1, 2, \ldots$:*
(i) $a \in Z^n(f)$,
(ii) $|f(z)| = O(|z-a|^{n+1})$, *as* $z \to a$,
(iii) $f(z)(z-a)^{-n-1} \in A^\infty$.

4.6. PROPOSITION. *If $f = SF \in A^\infty$, where S is an inner function dividing f, then $Z^n(f) \cap \partial D = Z^n(F) \cap \partial D$ for $n = 0, 1, 2, \ldots$. Also, $Z^\infty(F) = Z^\infty(f)$.*

Proof. By Theorem 4.1, $F \in A^\infty$ and so $Z^n(F)$ is defined. Writing

$$f(z)(z-a)^{-n} = S(z)F(z)(z-a)^{-n},$$

the proposition follows immediately from Proposition 4.5 and the fact that $|S| = 1$ a.e. on ∂D.

The following theorem clarifies the role of singular inner functions in the ideal structure of A^∞.

4.7. THEOREM. *Let S be a singular inner function with μ as its associated positive singular measure on ∂D. The set $\mathscr{D}(S) = \{f \in A^\infty: Sf \in A^\infty\}$ is equal to the closed set $\{f \in A^\infty:$ support $\mu \subset Z^\infty(f)\}$. The operation of multiplication* *by S is a continuous linear one-to-one operator from $\mathscr{D}(S)$ into A^∞ with range equal to the closed set $\{f \in A^\infty: S|f\} \subset \mathscr{D}(S)$. The inverse of multiplication by S is continuous.*

Proof. Suppose that $f \in \mathscr{D}(S)$; that is, both f and $g = Sf$ belong to A^∞. It is clear from Theorem 4.2 that a function of class A vanishes on the support of the measure associated with its singular inner part. By Theorem 4.3, $S|g^{(n)}$ for $n = 0, 1, 2, \ldots$. Since $g^{(n)} \in A$, the support of μ is contained in $Z^n(g)$. It follows from Proposition 4.6 that the support of μ is contained in $Z^n(f)$ for $n = 0, 1, 2, \ldots$. Thus,

$$\mathscr{D}(S) \subset \{f \in A^\infty: \text{support } \mu \subset Z^\infty(f)\}.$$

Now suppose that $f \in A^\infty$ and the support of μ is contained in $Z^\infty(f)$. To show that $Sf \in A^\infty$, it suffices, by Theorem 4.2, to show that $(Sf)^{(n)}(z) \to 0$ as z approaches a point in the support of μ. Now

$$Sf(z) = f(z) \exp\left[- \int \frac{e^{it} + z}{e^{it} - z} \, d\mu(t) \right].$$

On taking the nth derivative of Sf, one obtains a finite sum of terms, each of which is obviously bounded by a function of the form

$$|f^{(k)}(z)| \rho(z, \text{support } \mu)^{-j}, \qquad z \in \bar{D} \sim \text{support } \mu,$$

where $0 \leqq k \leqq n$ and $0 \leqq j \leqq 2n$. It follows from the estimate of Proposition 4.4 that $(Sf)^{(n)} \in A$. Thus

$$\mathscr{D}(S) = \{f \in A^\infty: \text{support } \mu \subset Z^\infty(f)\}.$$

Now it is clear that $\mathscr{D}(S)$ is closed; and, since convergence in A^∞ implies pointwise convergence, the operation of multiplication by S has a closed graph. By the closed graph theorem, this operator is continuous.

That the range of the operator of multiplication by S is $\{f \in A^\infty: S|f\}$ is clear from Theorem 4.1. It is known [**7**, p. 84] that if a sequence of functions $f_n \in A$ with $S|f_n$, S an inner function, converges uniformly to $f \in A$, then $S|f$. Therefore, $\{f \in A^\infty: S|f\}$ is closed in A^∞. By the open mapping theorem, the operation of multiplication by $1/S$ is continuous on $\{f \in A^\infty: S|f\}$. The fact that $\{f \in A^\infty: S|f\} \subset \mathscr{D}(S)$ follows from Proposition 4.6.

4.8. Corollary. *A singular inner function S divides some non-trivial A^∞ function if and only if the support of the singular measure associated with S is a Carleson set.*

4.9. Corollary. *If I is a closed ideal in A^∞ and the singular inner function S divides the g.c.d. of the inner factors of the non-zero elements of I, then $\{f \in A^\infty: Sf \in I\}$ is a closed ideal in A^∞.*

4.10. *Remark.* Several other mathematicians have made contributions to the types of problems considered in this section. In particular, we wish to point out the papers [**3**; **4**] of Caughran and the paper [**14**] of Wells.

https://doi.org/10.4153/CJM-1970-143-x Published online by Cambridge University Press

5. Closed ideals in A^∞. In this section we present the main result of the paper, the characterization of the closed ideals of A^∞.

5.1. *Definition. If $I \subset A^\infty$ is an ideal, let $Z^n(I) = \bigcap\{Z^n(f): f \in I\}$, $n = 0, 1, 2, \ldots$. Let $Z^\infty(I) = \bigcap\{Z^\infty(f): f \in I\}$ and $Z(I) = \{Z^0(I), Z^1(I), \ldots\}$.*

Each $Z^n(I)$ is closed, $Z^{n+1}(I) \subset Z^n(I)$, and $Z^\infty(I) = \bigcap_{n=0}^\infty Z^n(I)$.

5.2. *Definition. For $Z = \{Z^0 \supset Z^1 \supset \ldots\}$, a family of closed subsets of $\bar{D}$, define*

$$I(Z) = \{f \in A^\infty: Z^n(f) \supset Z^n, n = 0, 1, \ldots\}.$$

Clearly $I(Z)$ is a closed ideal.

5.3. Theorem. *Let I be a closed ideal in A^∞. If S is the g.c.d. of the singular inner factors of the non-zero functions in I, then $I = S \cdot I(Z(I))$.*

5.4. *Remark.* It is an immediate consequence of Theorem 4.7 that $S \cdot I(Z(I))$ is a closed ideal in A^∞. Proposition 4.6 implies that

$$S \cdot I(Z(I)) = \{f \in I(Z(I)): S|f\}.$$

Before giving the proof of the ideal theorem we collect some lemmas.

In studying a closed ideal I in A^∞, we will want to consider the subspace $I^\perp \subset B'$ which annihilates I. We will regard the elements of B' as functions analytic in $\mathbf{C} \sim \bar{D}$ with distribution boundary values as discussed in § 2.

5.5. Lemma. *Let I be an ideal in A^∞. If $T \in B'$ and $(f, T) = 0$ for all $f \in I$, then $T(\zeta)$ can be continued analytically to the complement of $Z^0(I)$.*

Proof. Consider $T \in I^\perp \subset B'$ with the Fourier series $\sum_{n=0}^\infty b_n e^{-in\theta}$ on ∂D. We wish to show that $T(\zeta) = \sum_{n=0}^\infty b_n \zeta^{-n}$ can be continued analytically to the complement of $Z^0(I)$. For $f(z) = \sum_{n=0}^\infty a_n z^n \in I$, fT is a well-defined element of $\mathscr{D}'(\partial D)$ with the Fourier series $fT(e^{i\theta}) = \sum_{n=-\infty}^\infty c_n e^{in\theta}$, where $c_n = \sum_{k=0}^\infty a_k b_{k-n}$ for $n < 0$ and $c_n = \sum_{k=n}^\infty a_k b_{k-n}$ for $n \geqq 0$. However, since I is an ideal and $f \in I$, $(g, fT) = (gf, T) = 0$ for all $g \in A^\infty$; in particular, fT has vanishing negative Fourier coefficients. Now choose a positive integer m such that $b_n = O(n^m)$. For any positive integer j, $|a_n| = O(n^{-j})$. Take an integer $p > 0$ and set $j = m + p + 2$. Then

$$|c_n| \leqq \sum_{k=n}^\infty |a_k|\,|b_{k-n}| = O\left(\sum_{k=n}^\infty k^{-j}(k-n)^m\right)$$
$$= O\left(\sum_{k=n}^\infty k^{m-j}\right) = O\left(\sum_{k=n}^\infty k^{-2-p}\right) = O(n^{-p})$$

for $n = 1, 2, \ldots$. Hence $fT \in A^\infty$.

Let $[e^{i\alpha}, e^{i\beta}] \subset \partial D$ lie in a complementary interval of $Z^0(f) \cap \partial D$. Choose $0 < r_0 < 1$ such that $f(re^{i\theta}) \neq 0$ for $r_0 \leqq r < 1$ and $\theta \in [e^{i\alpha}, e^{i\beta}]$, and choose

https://doi.org/10.4153/CJM-1970-143-x Published online by Cambridge University Press

a function $\psi \in C^\infty(\partial D)$ which is equal to one on a neighbourhood of $[e^{i\alpha}, e^{i\beta}]$ and equal to zero on a neighbourhood of $Z^0(f) \cap \partial D$. For $r_0 \leqq r < 1$, let

$$T_r(e^{i\theta}) = \frac{\psi(e^{i\theta})}{f(re^{i\theta})} fT(re^{i\theta}),$$

and for $s > 1$, let $T_s(e^{i\theta}) = T(se^{i\theta})$. The functions T_r and T_s are to be regarded as elements of $\mathscr{D}'(\partial D)$. If it can be shown that for all $\varphi \in C^\infty(\partial D)$ with support in $(e^{i\alpha}, e^{i\beta})$,

$$\lim_{r\to 1^-} (\varphi, T_r) = (\varphi, T) = \lim_{s\to 1^+} (\varphi, T_s), \tag{5.1}$$

then it follows from a well-known theorem (see e.g. [**1**]) that $T(z)$, $|z| > 1$, and $fT(z)/f(z)$, $|z| < 1$, are analytic continuations of each other across $(e^{i\alpha}, e^{i\beta})$.

Suppose that $\varphi(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$. Clearly

$$\lim_{s\to 1^+} (\varphi, T_s) = \lim_{s\to 1^+} \sum_{n=0}^{\infty} \alpha_n b_{-n} s^{-n} = \sum_{n=0}^{\infty} \alpha_n b_{-n} = (\varphi, T).$$

The other half of (5.1) is immediate since $fT \in A^\infty$.

We conclude that T and fT/f are continuations of each other to the complement of $Z^0(f)$. Since f was an arbitrary function of I, it follows that T can be continued analytically to the complement of $Z^0(I)$.

5.6. *Remark*. Let I and T be as in the above lemma. If

$$a \in [Z^n(I) \sim Z^{n+1}(I)] \cap D,$$

it is clear from the representations $T(z) = fT(z)/f(z)$, $f \in I$, that T has at worst a pole of order $n + 1$ at a. Suppose that

$$a \in [Z^n(I) \sim Z^{n+1}(I)] \cap \partial D.$$

Then T has an isolated singularity at a and it can be shown that it has at worst a pole of order $n + 1$. We argue as follows. Proposition 4.5 implies that $I_1 = \{g \in A^\infty : g(z) = f(z)z^{n+1}(z - a)^{-n-1}, f \in I\}$ is a closed ideal in A^∞ with $a \notin Z^0(I_1)$. Now $(1 - ae^{-i\theta})^{n+1} \in C^\infty(\partial D)$, and so

$$U(e^{i\theta}) = (1 - ae^{-i\theta})^{n+1} T(e^{i\theta})$$

is a distribution on ∂D. It is easy to check that $U \in B'$ and its continuation to the exterior of D is $U(z) = T(z)(z - a)^{n+1}z^{-n-1}$. For $g \in I_1$ write $g(z) = f(z)z^{n+1}(z - a)^{-n-1}$, $f \in I$. Then $0 = (f, T) = (g, U)$. Thus $U \in I_1^\perp$. Since $a \notin Z^0(I_1)$, U is analytic at a; and therefore, T has at worst a pole of order $n + 1$ at a.

In the exterior of D, any $T \in B'$ of order N satisfies the growth condition

$$|T(z)| = O((|z| - 1)^{-N-1})$$

https://doi.org/10.4153/CJM-1970-143-x Published online by Cambridge University Press

as $|z| \to 1^+$. Given an ideal I in A^∞ with $Z^0(I) \subset \partial D$, the following four lemmas are used to establish a growth restriction on $T \in I^\perp$ as $\rho(z, Z^0(I)) \to 0$, $z \in \mathbf{C} \sim (D \cup Z^0(I))$.

5.7. LEMMA. *Let I be an ideal in A^∞ with $Z^0(I) \subset \partial D$. Let $T \in I^\perp \subset B'$ and let S be the g.c.d. of the singular inner parts of the non-zero functions in I. Then for every $\epsilon > 0$, there exists a constant C_ϵ such that*

$$\log|S(z)T(z)| \leqq \epsilon(1 - |z|)^{-1} + C_\epsilon, \qquad z \in D. \tag{5.2}$$

Proof. It suffices to prove (5.2) locally; that is, to prove that for each $\epsilon > 0$ and each $a \in \partial D$ there is a constant $C_{\epsilon,a}$ such that

$$\log|S(z)T(z)| \leqq \epsilon(1 - |z|)^{-1} + C_{\epsilon,a} \tag{5.3}$$

for $z \in D$ and in some neighbourhood of a. Then (5.2) follows by a standard compactness argument.

To establish (5.3) we use the representation $T(z) = (fT)(z)/f(z)$, $z \in D$, $f \in I$, that was derived in the proof of Lemma 5.5. It is no loss of generality to assume that $a = 1$. Choose $f \in I$ as follows. Let ν be the positive singular measure on ∂D associated with S. Given $\epsilon > 0$, there exist $f_1 \in I$ and $\eta > 0$ such that if μ is the singular measure associated with the singular inner part of f_1, then

$$\mu\{e^{i\theta} \in \partial D\colon |\theta| < \eta\} < \epsilon + \nu\{e^{i\theta} \in \partial D\colon |\theta| < \eta\}. \tag{5.4}$$

This is because the g.c.d. of the singular inner parts of the non-zero functions in I being S is equivalent to the g.c.d. of the set of all the associated positive singular measures being ν, the latter g.c.d. being taken in the lattice of positive measures [**7**, p. 85].

Write $f_1 = BS_1F$, where B is the Blaschke factor of f_1, S_1 is the singular inner factor of f_1, and F is the outer factor of f. Next, note that $f = S_1F = f_1/B \in I$. To prove this, it is enough to show that $f_1/B_N \in I$, where B_N is the Nth partial product of B. For, by Theorem 4.1 the sequence f_1/B_N is bounded in A^∞. Also, f_1/B_N converges to f uniformly on compact subsets of D. Therefore, f_1/B_N converges to f in A^∞, and I is closed; thus $f \in I$ if $f_1/B_N \in I$ for all N. To see that $f_1/B_N \in I$, observe that each factor of B_N is of the form $z - a$ times a unit in A^∞. Thus, it suffices to show that if $g \in I$, $g(a) = 0$, then $g(z)/(z - a) \in I$. The following argument for this was shown to us by L. A. Rubel (see [**12**, p. 456]). Choose $h \in I$ with $h(a) \neq 0$. Then

$$g(z)/(z-a) = -h(a)^{-1}[g(z)((h(z) - h(a))/(z-a)) - h(z)g(z)/(z-a)] \in I$$

since $h, g \in I$.

Thus, we have $f = S_1F \in I$ with μ satisfying (5.4). With this f, represent $T(z) = (fT)(z)/f(z)$. Then

$$\log|S(z)T(z)| = \log|fT(z)| - \log|S_1(z)/S(z)| - \log|F(z)|.$$

As was shown in the proof of Lemma 5.5, $fT \in A^\infty$; hence $\log|fT|$ is bounded above. Choose $\delta = \eta/2$ and r_0 such that

$$P(r, t) = (1 - r^2)(1 - 2r\cos t + r^2)^{-1} \leqq 2\pi \quad \text{for } |t| \geqq \delta, r_0 < r < 1.$$

Then for $z = re^{i\theta}$, $r_0 < r < 1$ and $|\theta| < \delta$,

$$\begin{aligned} |\log|S_1(z)/S(z)|| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \theta - t)\, d(\mu - \nu)(t)\right| \\ &\leqq \left|\frac{1}{2\pi}\int_{|t|<\eta} P(r, \theta - t)\, d(\mu - \nu)(t)\right| \\ &\qquad + \left|\frac{1}{2\pi}\int_{|t|\geqq\eta} P(r, \theta - t)\, d(\mu - \nu)(t)\right|. \end{aligned}$$

From the trivial estimate $0 \leqq P(r, t) \leqq 2/(1 - r)$ and (5.4), the first term does not exceed $2\epsilon/(1 - r)$. The second term does not exceed $||\mu - \nu||$ since $r_0 < r < 1$ and $|\theta - t| \geqq \delta$. Thus

$$|\log|S_1(z)/S(z)|| \leqq 2\epsilon/(1 - r) + ||\mu - \nu||, \qquad r_0 < r < 1, |\theta| < \delta.$$

Similarly we can derive the estimate

$$|\log|F(re^{i\theta})|| \leqq \frac{2\epsilon}{1 - r} + \frac{1}{2\pi}\int_{-\pi}^{\pi} |\log|F(e^{i\theta})||\, d\theta$$

for $r_0 < r < 1$, $|\theta| < \delta$, possibly with different r_0 and δ. We just have to replace (5.4) by an analogous estimate based on the absolute continuity of the integral of $\log|F(e^{i\theta})|$. This completes the proof of the lemma.

Lemma 5.8 below is a consequence of results of Domar [**5**] and Beurling. However, for the case in question here, it is technically simpler to reprove Domar's results, using the argument of [**5**], than to deduce it from the theorems stated in [**5**]. We wish to thank Professor H. S. Shapiro for calling our attention to these results.

5.8. LEMMA. *Let E be a closed subset of ∂D and let $G = \mathbf{C} \sim E$. If u is subharmonic on G and satisfies $u(z) \leqq ||z| - 1|^{-1}$, then $u(z) \leqq$ const $\cdot\, \rho(z, E)^{-1}$, $z \in G$.*

Proof. For technical reasons which will become apparent in the proof, we prove the lemma with E replaced by $E \cup \{0\}$, which involves no loss of generality. The first observation in the argument is essentially a special case of [**5**, p. 434, Lemma 2]. We assert that

(*) If $z_0 \in G$, $|z_0| \geqq \frac{1}{2}$, and $u(z_0) \geqq e^v$, then each disk of radius $R > (4e^3e^{-v}/(e - 1))$ contains either a point z with $u(z) > e^{v+1}$ or a point of the complement of G.

For, if $\{|z - z_0| \leqq R\} \subset G$ and $u(z) \leqq e^{v+1}$, then

$$e^v \leqq u(z_0) \leqq \frac{1}{\pi R^2}\int_{|\xi|\leqq R} u(z_0 + \xi)\, d\lambda(\xi),$$

https://doi.org/10.4153/CJM-1970-143-x Published online by Cambridge University Press

by the mean-value property for subharmonic functions (λ represents Lebesgue measure). Break this last integral up into the sum of integrals over the sets where $u \leqq e^{v-1}$ and $u > e^{v-1}$. The first of these is dominated by e^{v-1}, and the second is dominated by

$$\frac{e^{v+1}}{\pi R^2}\lambda\{z: u(z) > e^{v-1}, |z - z_0| \leqq R\}$$

$$\leqq \frac{e^{v+1}}{\pi R^2}\lambda\{z = re^{i\theta}: |z - z_0| \leqq R, |1 - r|^{-1} > e^{v-1}\}$$

$$\leqq \frac{e^{v+1}}{\pi R^2}\lambda\left\{z = re^{i\theta}: |1 - r| \leqq e^{-(v-1)}, |\theta - \arg z_0| \leqq \sin^{-1}\frac{R}{|z_0|}\right\}$$

since $R \leqq |z_0|$ (this is why we replaced E by $E \cup \{0\}$). The measure of this last set is no larger than $2e^{-(v-1)}R\pi/|z_0|$. Consequently, $e^v \leqq e^{v-1} + 2e^2/(R|z_0|)$, or

$$R \leqq \frac{2e^{3-v}}{(e-1)|z_0|}.$$

The assertion (*) follows from this.

It follows from (*) that $u(z_0) \geqq e^v$ implies

$$\rho(z_0, E) \leqq \text{const}\sum_{k \geqq v} e^{-k} = \text{const}\, e^{-v},$$

and the lemma follows from this.

5.9. Lemma. *Let E be a closed subset of ∂D and let T be analytic on $\mathbf{C} \sim E$. If there exist constants $N > 0$, $C > 0$, and $K > 0$ such that*

$$(5.5) \qquad |T(z)| = O((|z| - 1)^{-N}), \qquad |z| \to 1^+,$$

and

$$(5.6) \qquad \log|T(z)| \leqq C\rho(z, E)^{-1} + K, \qquad z \in \mathbf{C} \sim E,$$

then

$$|T(z)| = O(\rho(z, E)^{-2N}), \qquad |z| > 1, \rho(z, E) \to 0.$$

Proof. Consider one of the complementary intervals of $\partial D \sim E$. Without loss of generality we may assume that the interval is of the form $(e^{-i\delta}, e^{i\delta})$ where $0 < \delta < \pi/12$. The case in which δ is larger requires only trivial modifications. Let Ω be the domain in $\mathbf{C}$ bounded by segments of the straight lines

$$\Gamma_1: \arg(z - e^{-i\delta}) = \pi/3, \qquad \Gamma_2: \arg(z - e^{i\delta}) = \pi/6,$$

$$\Gamma_3: \mathrm{Re}[(e^{-i\delta} + z)(e^{-i\delta} - z)^{-1}] = -1, \quad \Gamma_4: \mathrm{Re}[(e^{i\delta} + z)(e^{i\delta} - z)^{-1}] = -1.$$

Note that Γ_3 and Γ_4 are tangent to ∂D at $e^{-i\delta}$ and $e^{i\delta}$, respectively.

Observe that the lemma follows if we prove the desired estimate for $z \in \Omega$, $|z| > 1$. For, if V is the union of all such Ω over all the complementary intervals of $\partial D \sim E$, then clearly $(|z| - 1)^{-1} = O(\rho(z, E)^{-2})$ as $\rho(z, E) \to 0$ in $\mathbf{C} \sim (\bar{D} \cup V)$.

To prove the estimate in Ω for $|z| > 1$ we use an argument of Phragmén-Lindelöf type. Consider the subharmonic function

$$u(z) = \log|T(z)| - 2N \log|e^{-i\delta} - z|^{-1} - 2N \log|e^{i\delta} - z| - 2C \operatorname{Re}\left[\frac{e^{-i\delta} + z}{e^{-i\delta} - z} + \frac{e^{i\delta} + z}{e^{i\delta} - z}\right] - C',$$

where C' is a constant to be chosen. From (5.6),

$$\log|T(z)| \leqq C\rho(z, E)^{-1} + K \leqq 2C \operatorname{Re}\frac{e^{-i\delta} + z}{e^{-i\delta} - z} + K$$

for $z \in \Gamma_1 \cap \partial\Omega$. From (5.5) there exist constants C_1 and C_2, independent of δ, such that for $z \in \Gamma_3 \cap \partial\Omega$,

$$\log|T(z)| \leqq C_1 + N \log(|z| - 1)^{-1} \leqq C_1 + C_2 + N \log|e^{-i\delta} - z|^{-2}.$$

Similar inequalities hold on $\Gamma_2 \cap \partial\Omega$ and $\Gamma_4 \cap \partial\Omega$. Thus, C' can be chosen independently of δ so that $u(z) \leqq 0$ for $z \in \partial\Omega$, $z \neq e^{\pm i\delta}$.

We now claim that $u(z) \leqq 0$ for $z \in \Omega$. To see this consider the function

$$h(z) = \operatorname{Re}\left[\frac{-1}{(e^{i\delta} - z)^2} - \frac{1}{(z - e^{-i\delta})^2}\right] = -|e^{i\delta} - z|^{-2} \cos(2 \arg(e^{i\delta} - z)) - |z - e^{-i\delta}|^{-2} \cos(2 \arg(z - e^{-i\delta})).$$

For $z \in \Omega$,

$$h(z) \geqq \tfrac{1}{2}(|e^{i\delta} - z|^{-2} + |z - e^{-i\delta}|^{-2}).$$

Now h is harmonic in Ω, and so for every $\epsilon > 0$, $v_\epsilon = u - \epsilon h$ is subharmonic in Ω. We have $v_\epsilon(z) \leqq 0$ for $z \in \partial\Omega$, $z \neq e^{\pm i\delta}$; and by (5.6), v_ϵ is bounded in Ω. Thus $v_\epsilon \leqq 0$ for $z \in \Omega$ and for all $\epsilon > 0$. Hence $u(z) \leqq 0$ for $z \in \Omega$, and the lemma follows immediately from this.

5.10. Lemma. *Let I and T be as in Lemma 5.7 and let N be the order of T. Then for $|z| > 1$,*

$$|T(z)| = O(\rho(z, Z^0(I))^{-2N-2}), \qquad \rho(z, Z^0(I)) \to 0.$$

Proof. Since T is of order N,

$$|T(z)| = O((|z| - 1)^{-N-1}), \qquad |z| \to 1^+.$$

By Lemma 5.7 and the obvious estimate on $\log|S|$, there is a constant $C_0 > 0$ such that

$$\log|T(z)| \leqq C_0(1 - |z|)^{-1}, \qquad z \in D.$$

Applying Lemma 5.8 to $\log|T(z)|$, there exist constants $C, K > 0$ such that T satisfies (5.6). Application of Lemma 5.9 completes the proof.

The next lemma (plus Theorem 4.1, Proposition 4.4, and the Hahn-Banach Theorem) actually suffices to establish Theorem 5.3 in the case $Z^0(I) = Z^\infty(I)$.

5.11. LEMMA. *Let $T \in B'$ have order N. Assume that E, the set of singularities of T, lies in ∂D. Let I be an ideal in A^∞ which is orthogonal to T, and let S be the g.c.d. of the singular inner factors of the non-zero functions in I. If $f = gF \in A^\infty$, where $g \in H^2$, S divides g, $F \in A^\infty$, and*

$$|F(z)| = O(\rho(z, E)^{2N+2}), \qquad z \in \bar{D},\ \rho(z, E) \to 0, \tag{5.7}$$

then $(f, T) = 0$.

Proof. Let G be any function in A^∞ satisfying (5.7). Then

$$(G, T) = \lim_{r\to 1^+} \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})T(re^{i\theta})\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{i\theta})T(e^{i\theta})\, d\theta$$

by the bounded convergence theorem since, by Lemma 5.10,

$$|G(e^{i\theta})T(re^{i\theta})| = O(\rho(e^{i\theta}, E)^{2N+2}\rho(re^{i\theta}, E)^{-2N-2}) = O(1) \quad \text{as } r \to 1^+.$$

By Beurling's invariant subspace theorem [**7**, p. 99], there exists a sequence of functions $g_n \in I$ such that $g_n \to g$ in H^2. Therefore

$$(f, T) = (gF, T) = \int_{-\pi}^{\pi} gFT = \lim_n \int_{-\pi}^{\pi} g_nFT = \lim_n (g_nF, T) = 0.$$

5.12. *Remark.* It is possible to avoid the appeal to Beurling's theorem by studying the structure of $T(z)$ in more detail.

Proof of Theorem 5.3. Let I be a closed ideal in A^∞ and let $I_0 = S \cdot I(Z(I))$, where S is the g.c.d. of the singular inner factors of the non-zero functions in I. We must show that $I = I_0$.

Let

$$J = [I_0{:}I] = \{f \in A^\infty : fI_0 \subset I\}.$$

Now J is a closed ideal; for, if $\{f_n\} \subset J$, $f_n \to f$ in A^∞, and $g \in I_0$, then $f_ng \in I$ and $f_ng \to fg$ in A^∞. Since I is closed, $fg \in I$; thus $f \in J$.

We claim that $Z^0(J) \subset Z^\infty(I)$. To show this, choose $a \in Z^n(I) \sim Z^{n+1}(I)$ and take $f \in I$ such that $a \in Z^n(f) \sim Z^{n+1}(f)$. Let $g(z) = f(z)(z-a)^{-n-1}$. By Proposition 4.5, $g \in A^\infty$. For $h \in I$ we have, again by Proposition 4.5, $h(z) = (z-a)^{n+1}H(z)$, where $H \in A^\infty$. Hence $gh = fH \in I$, and so $g \in J$. Since $g(a) \neq 0$, $a \notin Z^0(J)$.

We also claim that S_J, the g.c.d. of the inner factors of the non-zero functions in J, is 1. To see this consider a function $f = SF \in I$. By Theorem 4.1, $F \in A^\infty$. Let $h = SH$ be a function in I_0. Now $Fh = FSH = fH \in I$; thus $F \in J$. It follows that $S_J \equiv 1$.

Following Theorem 3.3, let $\{F_n\} \subset A^\infty$ be a sequence of outer functions such that for each n, $Z^0(F_n) = Z^\infty(F_n) = Z^\infty(I)$ and for each $g \in A^\infty$ with $Z^\infty(g) \supset Z^\infty(I)$, $F_n g \to g$ in A^∞. If we show that $F_n \in J$, then the theorem is proved. For, if $f \in I_0$, then $F_n f \in I$ and $F_n f \to f$ in A^∞ which implies that $f \in I$ since I is closed.

To see that $F_n \in J$, we will apply the Hahn-Banach theorem. Let T belong to $J^\perp$. Proposition 4.4 implies that each F_n satisfies the hypothesis of Lemma 5.11 with respect to T. Therefore $(F_n, T) = 0$. By the Hahn-Banach Theorem, $F_n \in J$. This completes the proof.

6. Remarks on the A^m case. In this section we point out what our methods yield concerning the ideal structure of A^m. The analogues of all the lemmas of § 5 may be established and, consequently, the structure of the closed ideals of A^m could be given if an approximation theorem analogous to Theorem 3.3 could be proved. We do not know how to do this.

Note that for the algebras A^m the zero sets $Z^n(f)$, $Z^n(I)$ must be defined in a slightly different way than for A^∞. The zero sets $Z^n(f)$ may be defined as before when $n \leqq m$ but for $n > m$ we may only talk about $f^{(n)}(z)$ when $|z| < 1$. Thus, $Z^n(f) = \{z \in D: f^{(k)}(z) = 0, 0 \leqq k \leqq n\}$ is a subset of D, with a similar modification for $Z^n(I)$, when $n > m$.

Our methods then enable us to prove the following.

6.1. THEOREM. *Let I be a closed ideal in A^m. If S is the g.c.d. of the singular inner factors of the non-zero functions in I, then I contains*

$$\{f \in A^m: S|f, f \in I(Z(I)), \text{ and } |f^{(m)}(z)| = O(\rho(z, Z^m(I))^{m+1})\}.$$

As a corollary of Theorem 6.1, we can obtain the following.

6.2. THEOREM. *If I is a closed ideal in A^m with $Z^m(I) \cap \partial D$ a finite set, then $I = \{f \in A^m: S|f \text{ and } f \in I(Z(I))\}$.*

To prove these theorems, we basically repeat the steps of § 5. There are, however, several technical problems which arise and require fairly straightforward but lengthy modifications.

REFERENCES

1. Felix E. Browder, *On the "edge of the wedge" theorem*, Can. J. Math. *15* (1963), 125–131.
2. Lennart Carleson, *Sets of uniqueness for functions regular in the unit circle*, Acta Math. *87* (1952), 325–345.
3. James G. Caughran, *Factorization of analytic functions with H^p derivative*, Duke Math. J. *36* (1969), 153–158.
4. ——— *Zeros of analytic functions with infinitely differentiable boundary values*, Proc. Amer. Math. Soc. *24* (1970), 700–704.
5. Yngve Domar, *On the existence of a largest subharmonic minorant of a given function*, Ark. Mat. *3* (1957), 429–440.
6. V. P. Gurariĭ, *The structure of primary ideals in the rings of functions integrable with increasing weight on the half axis*, Soviet Math. Dokl. *9* (1968), 209–213.

7. Kenneth Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, N.J., 1962).
8. J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 50 (Springer-Verlag, Berlin, 1970).
9. W. P. Novinger, *Holomorphic functions with infinitely differentiable boundary values* (to appear in Illinois J. Math.).
10. W. Rudin, *The closed ideals in an algebra of analytic functions*, Can. J. Math. *9* (1957), 426–434.
11. Laurent Schwartz, *Théorie des distributions* (Hermann, Paris, 1966).
12. B. A. Taylor, *Some locally convex spaces of entire functions*, Proc. Sympos. Pure Math., Vol. 11, pp. 431–467 (Amer. Math. Soc., Providence, R.I., 1968).
13. B. A. Taylor and D. L. Williams, *Zeros of Lipschitz functions analytic in the unit disc* (to appear).
14. J. H. Wells, *On the zeros of functions with derivatives in H_1 and H_∞*, Can. J. Math. *22* (1970), 342–347.

Courant Institute of Mathematical Sciences,
New York University,
New York, New York;
University of Michigan,
Ann Arbor, Michigan;

Syracuse University,
Syracuse, New York