Canad. J. Math. Vol. **52** (4), 2000 pp. 804–814

The Distributions in the Invariant Trace Formula Are Supported on Characters

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Abstract. J. Arthur put the trace formula in invariant form for all connected reductive groups and certain disconnected ones. However his work was written so as to apply to the general disconnected case, modulo two missing ingredients. This paper supplies one of those missing ingredients, namely an argument in Galois cohomology of a kind first used by D. Kazhdan in the connected case.

Let F be a number field and let G be a connected component of a reductive group over F. Assume that $G(F) \neq \varnothing$. In [Art88a], [Art88b] Arthur puts the trace formula for G in invariant form, provided that G is the connected component of the identity or arises from cyclic base change for an inner form of $\mathrm{GL}(n)$. However the two papers [Art88a], [Art88b] are written so as to apply to the general case, aside from two missing ingredients. One is the trace Paley-Wiener theorem for $G(F_w)$ at the infinite places w of F. We have nothing new to say about this and will simply assume the validity of Proposition 1.1 of [Art88a] in the general case. The second missing ingredient is an argument in Galois cohomology, along the lines of Kazhdan's proof of Theorem 1 in the appendix to [Kaz86]. Our purpose here is to supply the required argument, thus proving Theorem 5.1 of [Art88b] in the general case, assuming the validity of the trace Paley-Wiener theorem.

In fact our proof of Theorem 5.1 of [Art88b] is unconditionally valid for cyclic base change over p-adic fields, because then in the global arguments we are free to use totally complex base fields. Since twisted harmonic analysis reduces to ordinary harmonic analysis at places where the cyclic extension splits completely, no difficulty arises at the infinite places.

We use the following notation. For a field F we write $\overline{F}$ for an algebraic closure of F. For a number field F and a finite place v of F we write F_v^{un} for a maximal unramified extension of F_v and $\mathfrak{o}_v$ (respectively, $\mathfrak{o}_v^{un}$) for the valuation ring of F_v (respectively, F_v^{un}). For a number field F, a variety X over F and a place v of F we sometimes write X_v for the variety over F_v obtained from X by extension of scalars. For a connected reductive group G we write G_{der} for the derived group of G, G_{sc} for the simply connected cover of G_{der}, and G_{ad} for the adjoint group of G. We write $\widehat{G}$ for the connected Langlands dual group of G and $Z(\widehat{G})$ for its center. For a group G over a number field F we write $\ker^1(F, G)$ for the kernel of

$$H^1(F, G) \to \prod_v H^1(F_v, G),$$

the product being taken over all places v of F.

Received by the editors August 7, 1999.

The research of the first author was partially supported by NSF Grant DMS 8601121. The research of the second author was partially supported by NSF Grant DMS 8703288 and a Sloan Foundation Fellowship.

AMS subject classification: Primary: 22E50; secondary: 11S37, 10D40.

1 Weak Approximation

Let F be a number field and let S be a finite set of places of F. We write F_S for the F-algebra $\prod_{v\in S} F_v$. Let G be a connected reductive group over F. Consider the closure $G(F)^-$ of $G(F)$ in $G(F_S)$. By strong approximation for semisimple simply connected groups, $G(F)^-$ contains the image of $G_{\mathrm{sc}}(F_S)$, a normal subgroup of $G(F_S)$ with abelian quotient. Therefore $G(F)^-$ is a normal subgroup of $G(F_S)$ with abelian quotient.

Elements of $H^1\big(F, Z(\widehat{G})\big)$ define continuous characters on $G(\mathbb{A})$ [Lan89], trivial on $G(F)$. Let $H^1\big(F, Z(\widehat{G})\big)_S$ denote the subgroup of $H^1\big(F, Z(\widehat{G})\big)$ consisting of elements that are trivial in $H^1\big(F_v, Z(\widehat{G})\big)$ for every place v outside of S (and similarly for $H^1\big(W_F, Z(\widehat{G})\big)_S$, where W_F denotes the Weil group of $\overline{F}/F$, and $H^1\big(L/F, Z(\widehat{G})\big)_S$, where L denotes a finite Galois extension of F in $\overline{F}$ such that $\mathrm{Gal}(\overline{F}/L)$ acts trivially on $Z(\widehat{G})$). There is an obvious pairing between

$$G(F)^- \setminus G(F_S)$$

and

$$H^1\big(F, Z(\widehat{G})\big)_S / \ker^1\big(F, Z(\widehat{G})\big).$$

The following lemma is a variant of a result of Sansuc (see [San81], Thm. 8.12) and is an easy consequence of strong approximation and the Hasse principle.

Lemma 1

(a) The abelian groups $G(F)^- \setminus G(F_S)$ and

$$H^1\big(F, Z(\widehat{G})\big)_S / \ker^1\big(F, Z(\widehat{G})\big)$$

are finite and the natural pairing between them is perfect.

(b) Suppose that there exists a place $w \notin S$ and a finite Galois extension K of F in $\overline{F}$ such that K_w is a field and $\mathrm{Gal}\,(\overline{F}/K)$ acts trivially on $Z(\widehat{G})$. Then $G(F)$ is dense in $G(F_S)$.

Proof Let L be any finite Galois extension of F in $\overline{F}$ such that $\mathrm{Gal}\,(\overline{F}/L)$ acts trivially on $Z(\widehat{G})$. Using class-field theory for L, it is easy to see that

$$H^1\big(W_F, Z(\widehat{G})\big)_S = H^1\big(L/F, Z(\widehat{G})\big)_S,$$

and hence that

$$H^1\big(F, Z(\widehat{G})\big)_S = H^1\big(L/F, Z(\widehat{G})\big)_S$$

as well.

We draw three conclusions from this discussion. The first is that (b) follows from (a). Indeed, by hypothesis

$$H^1\big(K/F, Z(\widehat{G})\big) \to H^1\big(K_w/F_w, Z(\widehat{G})\big)$$

is an isomorphism, and therefore $H^1\big(K/F, Z(\widehat{G})\big)_S$ is trivial, since $w \notin S$.

https://doi.org/10.4153/CJM-2000-034-6 Published online by Cambridge University Press

The second conclusion is that $H^1\big(F, Z(\widehat{G})\big)_S$ is finite. In fact it is easy to see that $H^1(H, A)$ is finite for any finite group H and any diagonalizable $\mathbb{C}$-group A on which H acts (algebraically).

The third conclusion is that $H^1\big(F, Z(\widehat{G})\big)_S$ does not change if we enlarge S by adding all of the infinite places of F. We simply use that the decomposition group of any infinite place of L is cyclic and hence, by the Tchebotarev Density Theorem, is also the decomposition group of an infinite number of places of L. An easy reduction step now shows that it is enough to prove (a) when S contains every infinite place of F; we assume this for the rest of the proof.

First we consider the case in which G is a torus T. The group of continuous quasi-characters on $T(F)^- \setminus T(F_S)$ is the same as the group of continuous quasi-characters on $T(F) \setminus T(\mathbb{A})$ that are trivial on $T(F_v)$ for all $v \notin S$. By [Lan97] (see also [Lab84]) this group of quasi-characters is equal to

$$H^1(W_F, \widehat{T})_S / \ker^1(W_F, \widehat{T}),$$

which is in turn equal to

$$H^1(F, \widehat{T})_S / \ker^1(F, \widehat{T}).$$

This proves (a) for T.

Next we consider the case in which G_{der} is simply connected. Let $D = G/G_{\text{der}}$. Since $Z(\widehat{G}) = \widehat{D}$ and D is a torus, it is enough to prove that

$$G(F)^- \setminus G(F_S) \to D(F)^- \setminus D(F_S)$$

is an isomorphism. By what has already been proved for tori we may omit the infinite places from S while proving the surjectivity of this arrow, but then even

$$G(F_S) \to D(F_S)$$

is surjective by Kneser's vanishing theorem for H^1 for semisimple simply connected p-adic groups. To prove the injectivity consider an element $g \in G(F_S)$ whose image d in $D(F_S)$ belongs to $D(F)^-$. We want to show that any open neighborhood U of g in $G(F_S)$ contains an element of $G(F)$. Let V be the image of U in $D(F_S)$. There exists an element $d_0 \in D(F)$ in V since $d \in D(F)^-$. The coboundary of d_0 for the exact sequence

$$1 \to G_{\text{sc}} \to G \to D \to 1$$

is locally trivial at every infinite place of F (since these places belong to S) and is therefore trivial by the Hasse principle for H^1 of G_{sc}. We conclude that d_0 is the image of some $g_0 \in G(F)$. Then $g_0^{-1}U$ meets $G_{\text{sc}}(F_S)$. Applying the strong approximation property for G_{sc}, we see that $g_0^{-1}U$ contains an element of $G_{\text{sc}}(F)^-$. Therefore U contains an element of $G(F)^-$.

Finally we consider the general case. Choose a z-extension $H \to G$ (see [Kot82, Section 1]). Then

$$H(F)^- \setminus H(F_S) \to G(F)^- \setminus G(F_S)$$

is surjective. On the other hand the maps

$$H^1\big(F, Z(\widehat{G})\big)_S \to H^1\big(F, Z(\widehat{H})\big)_S,$$
$$\ker^1\big(F, Z(\widehat{G})\big) \to \ker^1\big(F, Z(\widehat{H})\big)$$

are isomorphisms (see [Kot84, Lemma 4.3.2 and its proof]). Thus it is enough to prove (a) for H, and this has already been done. ■

2 Approximation of a Local Automorphism by a Global One

Let F_1 be a local field of characteristic 0, let G_1 be a connected reductive group over F_1, and let $\theta_1 \in \mathrm{Aut}_{F_1}(G_1)$.

Lemma 2 *There exist a number field F, a connected reductive group G over F, an automorphism $\theta \in \mathrm{Aut}_F(G)$, a place v of F and an isomorphism $(F_v, G_v) \xrightarrow{\sim} (F_1, G_1)$ such that θ_v and θ_1 differ by an inner automorphism* $\mathrm{Int}\, g$ *for some $g \in G_1(F_1)$ (of course we transport θ_v to G_1 via $G_v \xrightarrow{\sim} G_1$).*

Proof Choose a finite Galois extension K_1/F_1 that splits G_1. Then there exist a finite Galois extension K/F of number fields and a place v of F such that K_v is a field and K_v/F_v is isomorphic to K_1/F_1. We may as well replace F_1 by F_v. There exists a quasi-split group G^* over F such that G^*_v is an inner form of G_1. Since

$$H^1(F, G^*_{\mathrm{ad}}) \to H^1(F_v, G^*_{\mathrm{ad}})$$

is surjective (see [BH78, Thm. 1.7]), there exists an inner form G of G^* and an isomorphism $G_v \xrightarrow{\sim} G_1$. Thus we may as well assume that $G_1 = G_v$.

Choose a finite Galois extension L of F that splits G and choose a place w of L lying over v. Replacing F by the decomposition field of w, we may assume that L_v is a field, so that $\mathrm{Gal}(L/F) = \mathrm{Gal}(L_v/F_v)$. Choose $\alpha \in \mathrm{Aut}_{\overline{F}}(G)$ such that α, θ_1 differ by an inner automorphism. Then since $\mathrm{Gal}(L/F) = \mathrm{Gal}(L_v/F_v)$ and L splits G, the automorphisms $x_\tau := \alpha^{-1}\tau(\alpha)$ for $\tau \in \mathrm{Gal}(\overline{F}/F)$ are all inner, and therefore (x_τ) defines an element of $H^1(F, G_{\mathrm{ad}})$ that is trivial in $H^1(F_v, G_{\mathrm{ad}})$. Replacing F by a suitably large finite extension in which v splits completely, we may assume that (x_τ) is trivial in $H^1(F_w, G_{\mathrm{ad}})$ for every place w of F. By the Hasse principle for H^1 of adjoint groups (see [San81, Cor. 5.4]) (x_τ) is trivial in $H^1(F, G_{\mathrm{ad}})$. Modifying our choice of α, we may assume that $x_\tau = 1$ for all τ, which just says that $\alpha \in \mathrm{Aut}_F(G)$. Then α and θ_1 differ by an element of $G_{\mathrm{ad}}(F_v)$. But $G_{\mathrm{ad}}(F)$ is dense in $G_{\mathrm{ad}}(F_v)$ (see [San81, Rem. 5.5]); thus we can modify α by an element of $G_{\mathrm{ad}}(F)$ so as to obtain $\theta \in \mathrm{Aut}_F(G)$ such that θ, θ_1 differ by an element of the open subgroup $\mathrm{im}\big(G(F_v)\big)$ of $G_{\mathrm{ad}}(F_v)$. ■

3 Existence of Suitable Global Situations

Let F be a field, arbitrary for the moment. We consider extensions $\widetilde{G}$ of $\mathbb{Z}/r\mathbb{Z}$ by a connected reductive group G:

$$1 \to G \to \widetilde{G} \to \mathbb{Z}/r\mathbb{Z} \to 1.$$

https://doi.org/10.4153/CJM-2000-034-6 Published online by Cambridge University Press

We consider only those extensions $\widetilde{G}$ for which $\widetilde{G}(F)$ maps onto $\mathbb{Z}/r\mathbb{Z}$. For any positive integer s we can pull back the extension via the canonical surjection

$$\mathbb{Z}/rs\mathbb{Z} \to \mathbb{Z}/r\mathbb{Z}$$

to obtain an extension of $\mathbb{Z}/rs\mathbb{Z}$ by G; we refer to this process as *inflating* $\widetilde{G}$ by s.

Now suppose that we have such an extension

$$1 \to G_1 \to \widetilde{G}_1 \to \mathbb{Z}/r\mathbb{Z} \to 1$$

over a local field F_1 of characteristic 0. We say that $\widetilde{G}_1$ can be *globalized* if there exist a number field F, an extension $\widetilde{G}$ of $\mathbb{Z}/r\mathbb{Z}$ by a connected reductive group over F such that $\widetilde{G}(F)$ maps onto $\mathbb{Z}/r\mathbb{Z}$, a place v of F and an isomorphism

$$(F_v, \widetilde{G}_v) \xrightarrow{\sim} (F_1, \widetilde{G}_1).$$

Lemma 3 *There exists a positive integer s such that the inflation of $\widetilde{G}_1$ by s can be globalized.*

Proof By Lemma 2 we may assume that (F_1, G_1) is of the form (F_v, G_v) for some number field F, some connected reductive group G over F and some place v of F. Moreover we may assume (still by Lemma 2) that there exists an automorphism θ of G over F and an element $x_v \in \widetilde{G}_1(F_v)$ such that x_v maps to the standard generator of $\mathbb{Z}/r\mathbb{Z}$ and the restriction of $\operatorname{Int}(x_v)$ to $G_1 = G_v$ coincides with θ. Let $y_v = x_v^r \in G(F_v)$. Note that $\theta(y_v) = y_v$.

Write Z for the center of G. In order to construct a suitable extension of $\mathbb{Z}/r\mathbb{Z}$ by G it would be enough to find $z_v \in Z(F_v)$ such that $(x_v z_v)^r \in G(F)$. Since we allow inflation as well it is enough to find $z_v \in Z(F_v)$ and a positive integer s such that $(x_v z_v)^{rs} \in G(F)$.

Consider the embedding

$$G^\theta / Z^\theta \to G_{\mathrm{ad}},$$

where G^θ (respectively, Z^θ) denotes the invariants of θ in G (respectively, Z). Then θ^r belongs to $G_{\mathrm{ad}}(F)$ and lies in the image of G^θ/Z^θ (check this over F_v); let t denote this element of $(G^\theta/Z^\theta)(F)$. Since $H^1(F, Z^\theta)$ is a torsion group, some power of t lies in the image of $G^\theta(F)$. Replacing r by a suitable positive multiple we may as well assume that there exists $y \in G^\theta(F)$ such that $\operatorname{Int}(y) = \theta^r$. Then y and y_v differ by an element of $Z^\theta(F_v)$. Replacing r by a suitable positive multiple we may assume that y and y_v differ by an element of $(Z^\theta)^\circ(F_v)$.

We are free to replace (F, v) by (F', v'), where F' is a finite extension of F and v' is a place of F' lying over v such that $F_v \to (F')_{v'}$ is an isomorphism. In fact we do this twice. First we choose a finite Galois extension K of F that splits $(Z^\theta)^\circ$ and replace F by the decomposition field of some place of K lying over v. Thus we may assume that K_v is a field. Then we replace F by any quadratic extension F' in which v splits and replace K by $K \otimes_F F'$, which is necessarily a field (look at what happens at v). At this point we may assume that $(Z^\theta)^\circ$ is split by a finite Galois extension K of F such that K_w is a field for some place $w \neq v$. Then by Lemma 1 $(Z^\theta)^\circ(F)$ is dense in $(Z^\theta)^\circ(F_v)$, and by modifying our choice of y we may assume that y and y_v differ by an element of the open subgroup

$$\{z^r \mid z \in (Z^\theta)^\circ(F_v)\}$$

of $(Z^\theta)^\circ(F_v)$. Therefore there exists $z_v \in (Z^\theta)^\circ(F_v)$ such that $(x_v z_v)^r = y_v z_v^r = y \in G(F)$. This completes the proof. ■

4 A Lemma on the Galois Cohomology of Diagonalizable Groups

Let F be a number field, let v be a place of F, and let D be a diagonalizable group over F (diagonalizable over $\overline{F}$, that is). Recall that the cup product

$$H^1(F_v, D) \times H^1\big(F_v, X^*(D)\big) \to H^2(F_v, \mathbb{G}_m) \hookrightarrow \mathbb{Q}/\mathbb{Z}$$

is a perfect pairing between the finite abelian groups $H^1(F_v, D)$ and $H^1\big(F_v, X^*(D)\big)$ (see [Mil86, Cor. 2.4]). Let S be a finite set of places of F including v, all the infinite places of F, and all the places where the $\mathrm{Gal}(\overline{F}/F)$-module $X^*(D)$ is ramified. For $w \notin S$ the group D_w extends naturally to a group scheme over $\mathfrak{o}_w$ with diagonalizable geometric fibers, and we denote by $H^1(\mathfrak{o}_w, D)$ the group $H^1\big(\mathrm{Gal}(F_w^{un}/F_w), D(\mathfrak{o}_w^{un})\big)$, a subgroup of $H^1(F_w, D)$ (see [Mil86, Thm. 2.6]]). We denote by $H^1(\mathbb{A}, D)$ the restricted direct product over all places of F of the groups $H^1(F_w, D)$, the restriction being with respect to the subgroups $H^1(\mathfrak{o}_w, D)$ for $w \notin S$.

The local pairings induce a continuous pairing between the locally compact group $H^1(\mathbb{A}, D)$ and the discrete group $H^1\big(F, X^*(D)\big)$. By the local theory the annihilator of $H^1(\mathbb{A}, D)$ in $H^1\big(F, X^*(D)\big)$ is $\ker^1\big(F, X^*(D)\big)$. Milne shows (see [Mil86, Thm. 4.20]) that the annihilator of $H^1\big(F, X^*(D)\big)$ in $H^1(\mathbb{A}, D)$ is equal to the image of $H^1(F, D)$ in $H^1(\mathbb{A}, D)$; in particular this image is a closed subgroup of $H^1(\mathbb{A}, D)$.

Lemma 4 *Suppose that the restriction map*

$$H^1\big(F, X^*(D)\big) \to H^1\big(F_v, X^*(D)\big)$$

is surjective. Then there exists a finite set S' of places of F, containing S, such that for any element $d \in H^1(F, D)$ whose image in $H^1(F_w, D)$ is trivial for all $w \in S' - \{v\}$ and belongs to $H^1(\mathfrak{o}_w, D)$ for all $w \notin S'$, the image of d in $H^1(F_v, D)$ is trivial.

Proof Consider the commutative diagram

$$\begin{array}{ccc} H^1(\mathbb{A}, D)/\operatorname{im} H^1(F, D) & \longrightarrow & H^1\big(F, X^*(D)\big)^D \\ {\scriptstyle\alpha}\big\uparrow & & {\scriptstyle\gamma}\big\uparrow \\ H^1(F_v, D) & \xrightarrow{\ \beta\ } & H^1\big(F_v, X^*(D)\big)^D \end{array}$$

where $(\quad)^D$ denotes Pontryagin dual. The map β is an isomorphism, and the map γ is injective by our hypothesis on the dual map. Therefore α is injective, which means that

$$H^1(F_v, D) \cap \operatorname{im} H^1(F, D) = \{1\}.$$

Since $H^1(F_v, D)$ is finite and $\operatorname{im} H^1(F, D)$ is closed in $H^1(\mathbb{A}, D)$ we can find an open subgroup U of $H^1(\mathbb{A}, D)$ such that if $x \in H^1(F_v, D)$, $x \neq 1$, then

$$xU \cap \operatorname{im} H^1(F, D) = \varnothing.$$

By shrinking U we may assume that it is of the form

$$\prod_{w \notin S'} H^1(\mathfrak{o}_w, D)$$

for some finite set S' of places of F, containing S. This proves the lemma. ■

https://doi.org/10.4153/CJM-2000-034-6 Published online by Cambridge University Press

5 Stable Conjugacy in an Unramified Situation (Crude Version)

Let G be a (possibly disconnected) reductive group over a number field F, and let γ be a semisimple element of $G(F)$. Let X denote the conjugacy class of γ under G°, let i denote the inclusion of X in G, and let f denote the morphism $g \mapsto g\gamma g^{-1}$ from G° to X. Then f is smooth and surjective, and i is a closed immersion, since γ is semisimple. There exists a positive integer N such that G, G°, X, γ, i, f come from objects over $\mathfrak{o}_F[\frac{1}{N}]$. Replacing N by a suitable positive multiple we may assume that f is smooth and surjective and i is a closed immersion, over $\mathfrak{o}_F[\frac{1}{N}]$ (see EGA IV (8.10.5) and (17.7.8)).

Lemma 5 *For every finite place v of F not dividing N and for every $\delta \in G(\mathfrak{o}_v)$ such that δ is conjugate under $G^\circ(\overline{F}_v)$ to γ there exists $y \in G^\circ(\mathfrak{o}_v^{un})$ such that $y\gamma y^{-1} = \delta$.*

Proof By hypothesis δ belongs to $G(\mathfrak{o}_v) \cap X(F_v)$, which is equal to $X(\mathfrak{o}_v)$, since i is a closed immersion. The fiber of $f\colon G^\circ \to X$ over $\delta \in X(\mathfrak{o}_v)$ is a smooth scheme Y of finite type over $\mathfrak{o}_v$, and the structural morphism $Y \to \mathrm{Spec}(\mathfrak{o}_v)$ is surjective. Therefore Y has a point in some finite extension of the residue field of $\mathfrak{o}_v$, and hence by smoothness has a point in the valuation ring of some finite unramified extension of F_v; this shows that $Y(\mathfrak{o}_v^{un})$ is non-empty, which just means that there exists $y \in G^\circ(\mathfrak{o}_v^{un})$ such that $y\gamma y^{-1} = \delta$. ■

6 Arthur's Distributions Are Supported on Characters

In this section we adhere strictly to the notation and conventions in Arthur's papers [Art88a], [Art88b]. Let F_1 be a local field of characteristic 0, and let G_1 be a connected component of a reductive group over F_1. We assume that $G_1(F_1)$ is non-empty, and we write G_1^+ for the group generated by G_1, and G_1° for the identity component of G_1^+. Our goal is to prove

Theorem 1 *For any Levi subset M_1 of G_1 over F_1 and any $\gamma_1 \in M_1(F_1)$ the distribution $I_{M_1}(\gamma_1)$ is supported on characters.*

Proof If $G_1 = G_1^\circ$ or G_1 is an inner twist of a component

$$G^* = \big(\mathrm{GL}(n) \times \cdots \times \mathrm{GL}(n)\big) \rtimes \theta^*,$$

this theorem is Theorem 5.1 of [Art88b], and our goal here is simply to check that Arthur's proof of his Theorem 5.1 can be made to work in the general case. In fact most of Arthur's proof is perfectly general; it is just the parts involving Galois cohomology that need to be extended.

Our proof cannot be read independently of Arthur's papers since it is part of the long inductive argument that occurs throughout [Art88a], [Art88b]. For this reason we have kept our proof as close as possible to that of Arthur's Theorem 5.1. In fact we have simply copied or summarized Arthur's arguments whenever possible. We should also recall that we are operating under the assumption that the trace Paley-Wiener theorem is valid; this has yet to be verified in the archimedean case (for disconnected groups).

Now we begin the proof of Theorem 1. Fix a positive integer N_1, and assume that the theorem is valid for any F_1, G_1 with $\dim_{F_1} G_1 < N_1$. Having made this induction assumption, we fix G_1 and F_1 such that $\dim_{F_1} G_1 = N_1$. If $L_1 \in \mathcal{L}_0(M_1)$, the distributions $I^{L_1}_{M_1}(\gamma_1)$ are by hypothesis supported on characters. This matches the induction assumption of Section 2 of [Art88a] which must be satisfied in order to define $I_{M_1}(\gamma_1)$.

Let f_1 be a fixed function in $\mathcal{H}\big(G_1(F_1)\big)$ such that $f_{1,G_1} = 0$. We must show that the distributions all vanish on f_1. It is convenient to fix M_1 and make a second induction assumption that

$$I_{L_1}(\delta_1, f_1) = 0 \quad \delta_1 \in L_1(F_1)$$

for any $L_1 \in \mathcal{L}(M_1)$ with $L_1 \neq M_1$. We must show that

$$I_{M_1}(\gamma_1, f_1) = 0 \tag{6.1}$$

for all $\gamma_1 \in M_1(F_1)$. At this point Arthur uses the two induction hypotheses and some results from [Art88a, (2.2), (2.3) and Corollary 8.3] to show that it is enough to verify (6.1) for all $\gamma_1 \in U$, where U is the set of G_1-regular semisimple elements of $M_1(F_1)$ that are elliptic in $M_1(F_1)$. In fact by continuity it is enough to verify (6.1) for any dense subset of U, and this is what we will do, using a global argument introduced by Kazhdan [Kaz86].

There is no harm in inflating G_1; thus by Lemma 3 we can find a number field F, a reductive group G^+ over F, a component G of G^+ such that $G(F) \neq \varnothing$, a place v of F and an isomorphism

$$(F_v, G^+_v, G_v) \xrightarrow{\sim} (F_1, G^+_1, G_1).$$

Choose a maximal F_1-split torus A_1 of M°_1 and choose a maximal F_1-torus T_1 of M°_1 containing A_1. Changing the isomorphism $G^+_v \xrightarrow{\sim} G^+_1$ by an inner automorphism obtained from an element of $G^\circ_1(F_1)$, we may assume that there exists a maximal F-torus T of G° such that $G^\circ_v \xrightarrow{\sim} G^\circ_1$ carries T_v into T_1. Let K be a finite Galois extension of F that splits T. Replacing F by the decomposition field of some place of K lying over v, we may assume that K_v is a field. Replacing F by a finite extension F' in which v splits completely (and replacing K by $K \otimes_F F'$, which is necessarily a field), we may assume that there is another place v' of F besides v such that $K_{v'}$ is a field. Let A be the maximal F-split torus in T. We have arranged that A is a maximal F-split torus of G° and that A_v is a maximal F_v-split torus of G°_v; therefore the standard (for A) Levi subsets of G correspond bijectively with the standard Levi subsets of G_v. We now replace (F_1, G^+_1, G_1) by (F_v, G^+_v, G_v). Note that M_1 is of the form M_v for a standard Levi subset M of G. Moreover $M(F)$ is dense in $M(F_v)$ by Lemma 1, since K splits M° and $K_{v'}$ is a field.

Therefore it is enough to prove that $I_{M_v}(\gamma, f_1) = 0$ for all G-regular semisimple $\gamma \in M(F)$ such that γ is elliptic in $M(F_v)$. Fix such an element γ and let D denote its centralizer in G°. It is enough to consider γ which are G-regular in the strongest sense; for these γ the group D is a diagonalizable group over F (but not necessarily a torus). All we have left is to show that $I_{M_v}(\gamma, f_1) = 0$ for our fixed γ, and in doing so we are free to replace (F, v) by (F', v'), where F' is a finite extension of F and v' is a place of F' lying over v such that $F_v \xrightarrow{\sim} F'_{v'}$.

By making such a replacement we may assume that

$$H^1\big(F, X^*(D)\big) \to H^1\big(F_v, X^*(D)\big)$$

is surjective. Indeed, since $H^1\big(F_v, X^*(D)\big)$ is finite, there exists a finite Galois extension L of F and a place v' of L lying over v such that

$$H^1\big(L_{v'}/F_v, X^*(D)\big) \to H^1\big(F_v, X^*(D)\big)$$

is surjective. Replacing F by the decomposition field of v' does the job.

Let S be a finite set of places of F including all the infinite places and all the places where the $\mathrm{Gal}(\overline{F}/F)$-module $X^*(D)$ is ramified. It is convenient to assume that S contains v and that $S - \{v\}$ contains at least one finite place. By Lemma 4 we can enlarge S so that the following statement becomes true: if $d \in H^1(F, D)$ is an element whose image in $H^1(F_w, D)$ is trivial for all $w \in S - \{v\}$ and belongs to $H^1(\mathfrak{o}_w, D)$ for all $w \notin S$, then the image of d in $H^1(F_v, D)$ is trivial.

Next we apply the considerations of Section 5 to γ. Lemma 5 tells us that there exists a positive integer N and $\mathfrak{o}_F[\frac{1}{N}]$ structures on G and G° such that for every place v of F not dividing N and for every $\delta \in G(\mathfrak{o}_v)$ such that δ is conjugate under $G^\circ(\overline{F}_v)$ to γ, there exists $y \in G^\circ(\mathfrak{o}_v^{un})$ such that $y\gamma y^{-1} = \delta$. Then by enlarging S so as to include all places dividing N we can make the following statement hold: if $\delta \in G(F)$ is conjugate under $G^\circ(F_w)$ to an element of $G(\mathfrak{o}_w)$ for all $w \notin S$ and is conjugate under $G^\circ(F_w)$ to γ for all $w \in S - \{v\}$, then δ is conjugate under $G^\circ(F_v)$ to γ. To prove this one considers the element of $H^1(F, D)$ corresponding to δ.

Now we apply the trace formula to certain functions $f \in \mathcal{H}\big(G(\mathbb{A})\big)$, all of the form $\prod_w f_w$ with $f_w \in \mathcal{H}\big(G(F_w)\big)$. We take f_v to be our fixed function f_1. For $w \notin S$ we take f_w to be the characteristic function of $G(\mathfrak{o}_w)$. At each finite place $w \in S - \{v\}$ we fix a compact neighborhood C_w of γ in the subset of regular semisimple elements in $G(F_w)$, and we take f_w to be the characteristic function of some compact open neighborhood U_w of γ with $U_w \subset C_w$. At each infinite place $w \in S - \{v\}$ we fix a compact neighborhood C_w of γ in $G(F_w)$ such that $K_w C_w K_w = C_w$, and we take f_w to be any left and right K_w-finite C^∞-function on $G(F_w)$ that is supported in C_w.

At this point Arthur uses the hypothesis $f_{1,G_v} = 0$ to show that the trace formula for any f as above reduces to the equality

$$\sum_{L \in \mathcal{L}} |W_0^L|\,|W_0^G|^{-1} \sum_{\delta \in (L(F))_{L,S'}} a^L(S', \delta) I_L(\delta, f) = 0 \tag{6.2}$$

for some finite set of places S' containing S. It is part of Thm. 3.3 of [Art88b] that the set S' may be chosen to be independent of f and that the sums over δ can be taken over finite sets, independent of f. Suppose that $L \in \mathcal{L}$ and $\delta \in L(F)$. Then Arthur uses the splitting formula (see [Art88a, Corollary 9.2]) and the hypothesis that $f_{1,G_v} = 0$ to show that

$$I_L(\delta, f) = I_L(\delta, f_v) \prod_{w \neq v} I_G(\delta, f_w).$$

Looking at any finite place $w \in S - \{v\}$, we see that $I_L(\delta, f)$ is 0 unless δ is G-regular semisimple, in which case

$$a^L(S', \delta) = |L_\delta(F) \setminus L(F, \delta)|^{-1} \operatorname{vol}\big(L_\delta(F) \setminus L_\delta(\mathbb{A})^1\big),$$

a positive number independent of S'. We conclude that there is a finite set $\mathcal{S}$ of pairs (L,δ), containing the pair (M,γ), as well as positive numbers $c(L,\delta)$, such that

$$\sum_{(L,\delta)\in\mathcal{S}} c(L,\delta) I_L(\delta,f_v) \prod_{w\neq v} I_G(\delta,f_w) = 0 \tag{6.3}$$

for all f as above. We may assume that δ is G-regular semisimple for all $(L,\delta)\in\mathcal{S}$. Let T denote the set of infinite places of F, other than v if v happens to be infinite, and let

$$G(F_T) = \prod_{w\in T} G(F_w), \quad C_T = \prod_{w\in T} C_w, \quad K_T = \prod_{w\in T} K_w, \quad f_T = \prod_{w\in T} f_w.$$

In deriving the equality (6.3) we were obliged to use left and right K_T-finite C^∞-functions on $G(F_T)$, supported on C_T. Such functions are dense in the space of C^∞-functions on $G(F_T)$, supported on C_T, where we use the supremum norm on this space of functions. But the orbital integrals $I_G(\delta,f_T)$ are defined on this bigger function space and are continuous for the supremum norm topology (here we use that semisimple conjugacy classes are closed to see that the orbital integrals can be taken over a compact set depending only on C_T). Therefore (6.3) remains valid if we drop the requirement that f_T be left and right K_T-finite. This allows us to assume that for each $w\in T$ the function f_w has non-negative values, is non-zero at γ, and is supported on an arbitrarily small neighborhood of γ.

By shrinking the support of all the functions f_w for $w\in S-\{v\}$ we may assume that $\mathcal{S}$ contains only pairs (L,δ) for which δ is conjugate under $G^\circ(F_w)$ to γ for all $w\in S-\{v\}$. By our choice of f_w for $w\notin S$ we may assume that $\mathcal{S}$ contains only pairs (L,δ) such that δ is conjugate under $G^\circ(F_w)$ to an element of $G(\mathfrak{o}_w)$ for all $w\notin S$. By our choice of S we conclude that $\mathcal{S}$ contains only pairs (L,δ) for which δ is conjugate under $G^\circ(F_v)$ to γ.

Now consider the contribution of (L,δ) to (6.3). Since δ is conjugate under $G^\circ(F_v)$ to γ, and γ is elliptic G-regular in $M(F_v)$, the Levi subset L must contain some conjugate of M. Therefore by [Art88a, (2.4)*], and our second induction assumption, $I_L(\delta,f_v)=0$ unless L is conjugate to M. Using [Art88a, (2.4)*], again, plus the fact that δ and γ are conjugate under $G^\circ(F_v)$, we see that if L is conjugate to M, then

$$I_L(\delta,f_v) = I_M(\gamma,f_v).$$

Therefore every non-zero term in (6.3) is a non-negative multiple of $I_M(\gamma,f_v)$. Of course at least one term is actually a positive multiple, namely the one indexed by (M,γ). Therefore $I_M(\gamma,f_v)=0$. ■

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https://doi.org/10.4153/CJM-2000-034-6 Published online by Cambridge University Press