ONE-RELATOR PRODUCTS OF TORSION-FREE GROUPS

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(Received 3 September, 1991)

If A and B are torsion-free groups, and W is a cyclically reduced word of even length in $A*B$, it is generally conjectured that a Freiheitssatz holds, namely that each of A and B are embedded via the natural map into the *one-relator product* group $G=(A*B)/N(W)$, where N denotes normal closure. If W has length 2, then G is a free product of A and B with infinite cyclic amalgamation, and the result is obvious. The purpose of this note is to prove the Freiheitssatz in some special cases.

DEFINITION. Let $\underline{x}=(x_1,\ldots,x_n)$ be a sequence of elements of a group G. We say that x_i is *isolated* in $\underline{x}$ if no x_j belongs to the cyclic subgroup generated by x_i for $j\neq i$.

THEOREM 1. *Let A, B be torsion-free groups, and let $W=a_1b_1\ldots a_kb_k$ be a cyclically reduced word in $A*B$, such that some a_i is isolated in $(a_1,\ldots,a_k)$ and some b_j is isolated in $(b_1,\ldots,b_k)$. Let $G=(A*B)/N(W)$. Then the natural maps $A\to G$ and $B\to G$ are injective.*

A second result concerns the case where one of our two groups, say B, is cyclic. Define the *sign-index* of a cyclically-reduced word

$$W=a_1x^{m_1}\ldots a_kx^{m_k}\in A*\langle x\rangle$$

to be the number of changes of sign in the cyclic sequence $(m_1,\ldots,m_k)$, in other words the number of negative terms in the sequence $(m_1m_2,\ldots,m_{k-1}m_k,m_km_1)$. Clearly this is an even integer between 0 and k.

THEOREM 2. *Let A be a torsion-free group and let*

$$W=a_1x^{m_1}\ldots a_kx^{m_k}\in A*\langle x\rangle$$

be a cyclically reduced word of length at least 2 and sign-index $\sigma\leq 2$. Let $G=(A*\langle x\rangle)/N(W)$. Then the natural map $A\to G$ is injective. If, in addition, one of the following holds:

(i) $\sigma>0$;

(ii) $a_1\ldots a_k\in A\setminus\{1\}$;

(iii) $\alpha=m_k+m_1a_1+m_2(a_1a_2)+\ldots+m_{k-1}(a_1\ldots a_{k-1})$ *is not a unit in* $\mathbb{Q}G$;

(iv) $k\leq 3$;

then the map $\langle x\rangle\to G$ is injective, in other words x has infinite order in G.

REMARKS. (1) The first part of Theorem 2 is not new. It follows from results of Levin [**7**] (for the case $\sigma=0$) and Stallings [**10**] (for the case $\sigma=2$). The second part of the Theorem, the fact that x has infinite order, is we believe, new, and equally important for the application below.

(2) It is conjectured that $\mathbb{Q}A$ has no nontrivial units for any torsion-free group A; in which case (iii) is satisfied whenever $k>1$, so the second part of Theorem 2 holds without the restrictions (i)–(iv). The widest class of groups for which the unit conjecture is known is that of *unique product groups* [**8**, Chapter 13] (for every pair U, V of nonempty finite

† This work was supported by a visiting fellowship under SERC grant GR/F 97355.

subsets, at least one element of UV is *uniquely* expressible as uv with $u \in U$ and $v \in V$). In particular we see that x will always have infinite order in A provided A is a unique-product group. Unfortunately, we do not know how to avoid the unit problem in proving Theorem 2.

Corollary 3. *Let A, B be torsion-free groups, and $W = a_1b_1 \dots a_kb_k$ with $a_i \in A$, $b_i \in B$, $a \neq 1 \neq b$ and $k \leq 3$. Let $G = (A * B)/N(W)$. Then the natural maps $A \to G$ and $B \to G$ are injective.*

Proof. Without loss of generality, we may assume that A is generated by $a_1, \dots, a_k$ and B by $b_1, \dots, b_k$. By Theorem 1 we may assume that one of the sequences $(a_1, \dots, a_k)$, $(b_1, \dots, b_k)$ (say the latter) has no isolated elements. Since $k \leq 3$ it follows easily that B is cyclic, say $B = \langle x \rangle$. Since $k \leq 3$ the sign-index of W is either 0 or 2, and the result follows from Theorem 2.

Before proving the theorems, we first note an interesting consequence for the study of equations over groups. Recall that an equation $W = 1$ in an unknown X over a group G is *nonsingular* if the word $W \in G * \langle X \rangle$ has nonzero exponent sum in X. One version of the Kervaire conjecture asserts that any nonsingular equation over any group G is *soluble over G*, in other words that a solution exists in some overgroup of G. We say that a group G is *good* if it satisfies this conjecture, in other words if every nonsingular equation over G is soluble over G. Known examples of good groups include all locally residually finite groups [**4**] and all locally indicable groups [**2, 5, 9**].

The *syllable length* of the equation $W = 1$ is defined to be the length of W as a (cyclically reduced) word in the free product $G * \langle X \rangle$. In particular, this length is always assumed to be even. A result of Edjvet [**3**] asserts that any equation of the form $aX^mbX^n = 1$ with $m \neq -n$ has a solution over any group G, except possibly if a has order 2 in G and b has order 3, or *vice versa*. In particular, any equation of syllable length 4 over any torsion-free group G has a solution over G.

Corollary 4. *If G is a good, torsion-free group, then any equation over G of syllable length at most* 14 *has a solution over G.*

Proof. The result follows from the definition of good unless the equation is singular. In this case the group $H = (G * \mathbb{Z})/N(W)$ is an HNN-extension of a group $A = (G_1 * \dots * G_k)/N(V)$, where $k \geq 2$, each $G_i \cong G$, and V is a word of length at most 7 in the free product, that contains letters from both G_1 and G_k. (Here the associated subgroups are $G_1 * \dots * G_{k-1}$ and $G_2 * \dots * G_k$, which are naturally embedded in A by Corollary 3.)

Remark. The proofs of the results use the weight test method of Bogley and Pride [**1**]. Recall [**1**] that a *weight function* for a *relative presentation*

$$\langle G, X_1, \dots, X_m \mid W_1, \dots, W_n \rangle$$

of a group H (where G is a group and the W_i are words in $G * \langle X_1, \dots, X_m \rangle$) is a real valued function w on the edges of the star graph of the presentation. A weight function is *aspherical* if it is non-negative; $w(g_1) + \dots + w(g_t) \leq (t-2)\pi$ if $W_i = g_1X^{\pm 1}_{\alpha(1)} \dots g_tX^{\pm 1}_{\alpha(t)}$ for some i; and $w(R) \geq 2\pi$ whenever R is a cyclically reduced closed path in the star graph labelled by a relation in G (among the coefficients g_i). Here $w(R)$ denotes the sum of the weights of the edges in R. A relative presentation with an aspherical weight function is called aspherical, and for such a presentation G always embeds in H via the natural map. (For details, see **[1]**.)

If w is an aspherical weight function with the additional property that $w(R) \geq \pi$ for any reduced non-closed path R in the star graph labelled by a relation among the g_i, then the methods of **[1]** can also be used to show that no nonempty, cyclically reduced word in the X_i represents the identity element of H, in other words the X_i form a basis for a free subgroup of H. We omit the details.

Proof of Theorem 1. Without loss of generality, we may assume that A is generated by $a_1, \ldots, a_k$, and B by $b_1, \ldots, b_k$; and that a_i is isolated in $(a_1, \ldots, a_k)$ and b_j in $(b_1, \ldots, b_k)$. The theorem certainly holds if the equation

$$a_1Xb_1X^{-1} \ldots a_kXb_kX^{-1} = 1$$

is soluble over $A * B$. The proof proceeds by applying the weight test method to this equation, or equivalently to the relative presentation

$$\langle A * B, X \mid a_1Xb_1X^{-1} \ldots a_kXb_kX^{-1} \rangle. \tag{1}$$

The star graph of this relative presentation consists of two disjoint bouquets of circles, with k circles in each. One of these corresponds to the A-letters, and the other to the B-letters. Define a weight function w by $w(a_i) = w(b_j) = 0$, $w(a_m) = \pi$ for $m \neq i$, and $w(b_m) = \pi$ for $m \neq j$. Then certainly w is non-negative, and the sum of the weights is $(2k-2)\pi$. To see that w is aspherical, it remains only to check that no nonempty closed path of weight less than 2π in the star graph represents the identity element of A or of B.

Any such path has the form a_i^t or b_j^t for some $t \neq 0$; or $a_m^{\pm 1} a_i^t$ for some $t \in \mathbb{Z}$, $m \neq i$ or $b_m^{\pm 1} b_j^t$ for some $t \in \mathbb{Z}$, $m \neq j$ (up to cyclic permutation). The first two possibilities are ruled out by the condition that A and B are torsion-free (for example, if $a_i^t = 1$ then $a_i = 1$, contradicting the hypothesis that W is cyclically reduced). The remaining possibilities are ruled out by the hypothesis that a_i and b_j are isolated.

Hence w is aspherical, and so the natural maps $A \to G$, $B \to G$ are injective, as claimed.

Proof of Theorem 2. Suppose first that W has sign-index $\sigma = 2$ (condition (i) of the theorem). Then the star graph of the relative presentation

$$\langle A, x \mid W \rangle \tag{2}$$

consists of two vertices, x and x^{-1}, a loop at each, and a collection (possibly empty) of edges joining x^{-1} to x. Let w be the weight function that assigns the weight 0 to each of the two loops, and π to each of the other edges. Then the only closed paths of weight less than 2π are powers of the loops, which cannot be relations in A since A is torsion-free. Hence this is an aspherical weight function for (2), and (2) is aspherical. In particular, A is naturally embedded in G. Moreover, any path in the star graph from x^{-1} to x has weight at least π, so x has infinite order in G.

Next suppose that W has sign-index $\sigma = 0$. In other words, after replacing x by x^{-1} if necessary, W is a positive word in x. Then certainly A is embedded in G, by Levin's Theorem **[7]**. Indeed, Levin shows that G has a homomorphic image isomorphic to the wreath product of A by a cyclic group of order n, where $n = m_1 + \ldots + m_k$. The cyclic

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group of order n is generated by an element t, conjugation by t permutes the n copies of A in the direct product, and x maps to the element

$$(a_1^{-1}, 1, \ldots, 1, a_2^{-1}, 1, \ldots, 1, a_k^{-1}, 1, \ldots, 1)t.$$

(Here a_1^{-1} is separated from a_2^{-1} by $(m_1 - 1)$ 1's, and so on, so that each a_p^{-1} occurs in the $(m_1 + \ldots + m_{p-1} + 1)$'th place in the vector.)

Hence x^n maps to an element of the direct power of A, each component of which is conjugate to the inverse of the product (in order) of the a_i. Since A is torsion-free, it follows that x has infinite order in G, except possibly if this product $a_1 \ldots a_k$ is the identity in A, in other words if condition (ii) fails. From now on we assume that this is the case.

The injection $A \to G$ is then split, via the map $G \to A$ that is the identity on A and sends x to 1. The kernel of the splitting map has a presentation

$$K = \langle x_a (a \in A) \mid x_{aa_1}^{m_1} x_{aa_1a_2}^{m_2} \ldots x_a^{m_k} (a \in A) \rangle, \tag{3}$$

by Reidemeister-Schreier rewriting, where $x_a = axa^{-1}$. If x has finite order in G, then K is generated by elements of finite order, so

$$H_1(K; \mathbb{Q}) \cong K^{ab} \otimes_{\mathbf{Z}} \mathbb{Q} = 0.$$

But A acts regularly on the generators and relations of (3), and so $H_1(K;\mathbb{Q})$ can be computed explicitly as the cokernel of the Fox derivative $\partial : \mathbb{Q}A \to \mathbb{Q}A$ of W with respect to x. Since $H_1(K;\mathbb{Q}) = 0$, ∂ is surjective (and hence an isomorphism). But ∂ is given by right multiplication by α on $\mathbb{Q}A$, so α must be a unit in $\mathbb{Q}A$. Hence condition (iii) fails.

Suppose, finally, that $k \leq 3$. If $k = 1$ then (ii) holds, so x has infinite order in G. If A is cyclic, then $\mathbb{Q}A$ has no nontrivial units, in particular no units of the form α with $k > 1$, so condition (iii) holds. In particular this is the case if $k = 2$, for then $A = \langle a_1, a_2 \rangle$ with $a_1 a_2 = 1$.

Thus we are reduced to the case where $k = 3$, $A = \langle a_1, a_2 \rangle$ is non-cyclic, and $W = a_1 x^p a_2 x^q (a_1 a_2)^{-1} x^r$ for some $p, q, r \geq 1$. The star graph of (2) consists of $p + q + r$ edges, all joining the vertex x^{-1} to x. Three of these are labelled a_1, a_2, a_3, and the others are labelled by the identity of A. Let w be the weight function that assigns a weight of $\pi/3$ to each of the edges a_1, a_2, a_3, and π to each of the remaining edges, The only non-closed, reduced paths of weight less than π are labelled $a_i^{\pm 1}$ for $i = 1, 2, 3$, none of which is a relation in A. Similarly, the only closed, cyclically reduced paths of weight less than 2π are labelled (up to cyclic permutation and inversion) a_i, $a_i a_j^{-1}$ for $i \neq j$, or $a_i a_j^{-1} a_i a_k^{-1}$ for $j \neq i \neq k$. Given that A is torsion-free, non-cyclic, and generated by a_1, a_2, a_3 with $a_1 a_2 a_3 = 1$, it can readily be verified that only the last of these, with $j \neq k$, are possible relations in A. If no such word is a relation in A then (2) is aspherical. Hence x has infinite order in G, and we are done.

Suppose then that a word $a_i a_j^{-1} a_i a_k^{-1}$ is a relation in A, with i, j, k pairwise distinct. Without loss of generality we may assume that $i = 1$, $j = 2$ and $k = 3$. Then A is a homomorphic image of

$$\hat{A} = \langle a_1, a_2, a_3 \mid a_1 a_2 a_3, a_1 a_2^{-1} a_1 a_3^{-1} \rangle \cong \langle x, y \mid xyx^{-1}y^2 \rangle.$$

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But no proper homomorphic image of $\hat{A}$ is torsion free and noncyclic, so $A \cong \hat{A}$. Now in this case there is an alternative weight function: take $w(a_2) = w(a_3) = 0$, and $w(e) = \pi$ for all other edges. This makes (2) aspherical (compare [**1**, Theorem 3.2, Case 3], and also has the property that no reduced, nonclosed path of weight less than π is labelled by a relation in A. Once again, x has infinite order in G. This completes the proof.

Alternatively, we can complete the proof using the fact that the group $\hat{A}$ is a torsion-free one-relator group, hence locally indicable [**2**], and so $\mathbb{Q}\hat{A}$ has no nontrivial units (see for example [**8**]). Now apply (iii).

Concluding remarks. 1. Our method of proof also shows that the one-relator product group $G = (A * B)/N(W)$ in Corollary 3 is itself torsion-free, except in the case where W is a proper power in $A * B$. Indeed, we showed that either G is (essentially) a one-relator group, or G is (a subgroup of) a group given by an aspherical relative presentation over a torsion-free group. In either case, G can have torsion only if the relator is a proper power. (See [**1**] for the relative presentation case).

2. It is reasonable to expect that the length restrictions on Corollaries 3 and 4 could be relaxed somewhat. More subtle methods of proof would be required however, as one can construct examples of length 8 words in $A * B$ such that neither factor group is cyclic, yet no aspherical weight function exists. One such example is $W = abadcbcd^{-1}$, where $A = \langle a, c \rangle$ and $B = \langle b, d \rangle$ are both free of rank 2. One possible approach might be to use the cycle test of Huck and Rosebrock [**6**] in place of the weight test.

3. Finally, the intrusion of the unit problem into Theorem 2 raises the question of whether it can be avoided. Explicitly, one can ask, given any positive word $W = a_1x \ldots a_kx \in A * \langle x \rangle$, with at least one of the a_i nontrivial in A, whether x has infinite order in the one-relator product $G = (A * \langle x \rangle)/N(W)$. Theorem 2 provides some evidence for an affirmative answer, but not yet a complete proof. It is rather curious that the infinite order part of Theorem 2 should be so much easier for words of sign-index 2 than for positive words.

ACKNOWLEDGEMENTS. We are grateful to the referee for useful comments, in particular for pointing out and filling a gap in our original proof of Theorem 2. We are also grateful to Steve Pride for a number of helpful remarks.

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