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ON THE SOLVABILITY OF SEMILINEAR DIFFERENTIAL EQUATIONS AT RESONANCE

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Abstract In this paper we use the Leray–Schauder continuation method to study the existence of solutions for semilinear differential equations $Lu + g(x,u) = h$, in which the linear operator L on $L^2(\Omega)$ may be non-self-adjoint, the $L^2(\Omega)$-function h belongs to $N^{\perp}(L)$, the nonlinear term $g(x,u) \in O(|u|^{\alpha})$ as $|u| \to \infty$ for some $0 \leqslant \alpha < 1$ and satisfies

$$\int_{v(x)>0} g_{\beta}^{+}(x)|v(x)|^{1-\beta}\,dx + \int_{v(x)<0} g_{\beta}^{-}(x)|v(x)|^{1-\beta}\,dx > 0,$$

for all $v \in N(L) - \{0\}$, where $\beta \in \mathbb{R}$, $-\alpha \leqslant \beta \leqslant 1$ and $2\alpha + \beta \leqslant 1$, $g_{\beta}^{+}(x) = \liminf_{u\to\infty}(g(x,u)u/|u|^{1-\beta})$ and $g_{\beta}^{-}(x) = \liminf_{u\to-\infty}(g(x,u)u/|u|^{1-\beta})$.

Keywords: Landesman–Lazer condition; Leray–Schauder continuation method

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ be a bounded domain and $H = L^2(\Omega)$ with the inner product $(\cdot,\cdot)_H$, $(u,v)_H = \int_{\Omega} uv$. We consider the following abstract differential equation

$$Lu + g(x,u) = h, \tag{1.1}$$

where $h \in H$ is given, $L : D(L) \subset H \to H$ is a closed, densely defined linear operator satisfying the following conditions:

(L_1) the null space $N(L)$ of L is finite-dimensional;

(L_2) the range $R(L)$ of L is closed;

(L_3) $R(L) = N^{\perp}(L)$;

(L_4) the right inverse $L^{-1} : R(L) \to R(L)$ of L is a compact linear operator;

and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function satisfying

(G_1) there exist constants $a \geqslant 0$, $0 \leqslant \alpha < 1$, and $b \in H$, $b \geqslant 0$ such that for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$

$$|g(x,u)| \leqslant a|u|^{\alpha} + b(x);$$

(G_2) there exist constants $|\beta| \leqslant 1$, $r_0 \geqslant 0$ and $c \in L^{2/(1+\beta)}(\Omega)$ such that for a.e. $x \in \Omega$ and $|u| \geqslant r_0$

$$g(x,u)u \geqslant c(x)|u|^{1-\beta};$$

(G_3)

$$\int_{w(x)>0} g_{\beta}^{+}(x)|w(x)|^{1-\beta}\,\mathrm{d}x + \int_{w(x)<0} g_{\beta}^{-}(x)|w(x)|^{1-\beta}\,\mathrm{d}x > 0,$$

for all $w \in N(L) - \{0\}$;

where $g_{\beta}^{+}(x) = \liminf_{u\to\infty}(g(x,u)u/|u|^{1-\beta})$ and $g_{\beta}^{-}(x) = \liminf_{u\to-\infty}(g(x,u)u/|u|^{1-\beta})$. The solvability of (1.1) has been extensively studied if L (or $-L$) $= A + \lambda$, A may be a non-self-adjoint uniformly elliptic operator with the principal eigenvalue λ and the nonlinearity g may be assumed to grow superlinearly in u as $|u| \to \infty$ (see [**1, 3, 7, 8, 11, 13, 14**]). When A is self-adjoint with a higher eigenvalue λ, and the nonlinearity g has at most linear growth in u as $|u| \to \infty$, existence theorems of (1.1) are proved in [**2, 4–6, 12, 15, 16**] if h satisfies the following Landesman–Lazer condition:

$$\int_{\Omega} h(x)v(x)\,\mathrm{d}x < \int_{v>0} g_0^{+}(x)|v(x)|\,\mathrm{d}x + \int_{v<0} g_0^{-}(x)|v(x)|\,\mathrm{d}x, \tag{1.2}$$

for each $v \in N(L) - \{0\}$.

The purpose of this paper to give several abstract existence theorems of (1.1) by using the Leray–Schauder continuation method (see [**17**]) when $g(x,u) \in O(|u|^{1/2})$ as $|u| \to \infty$, $h \in N^{\perp}(L)$ and (G_3) may be satisfied with $\beta > 0$ and $2\alpha + \beta \leqslant 1$, in which we improve the main results of Ha [**9**], Hess [**10**] and Robinson and Landesman [**18**], where they assume that g is a bounded function that satisfies (G_2) and (G_3) with $c = r_0 = 0$, $\beta = 1$ and $h \in N^{\perp}(L)$. Our results can be applied to many well-known differential operators. For example, let $\tilde{\Omega}$ be a bounded open set in $\mathbb{R}^N (N \geqslant 1)$, and λ_n be the nth eigenvalue of the Laplacian $-\Delta : W^{2,2}(\tilde{\Omega}) \cap H_0^1(\tilde{\Omega}) \to L^2(\tilde{\Omega})$. We first consider the existence of solutions of the problem

$$\text{(i)} \quad \begin{cases} \pm(\Delta u + \lambda_n u) + g(x,u) = h \text{ on a.e. } x = \tilde{x} \in \Omega = \tilde{\Omega}, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$D(L) = \{u \in L^2(\Omega) \mid \Delta u \in L^2(\Omega) \text{ and } u = 0 \text{ on } \partial\Omega\} \quad \text{and} \quad L(u) = \pm(\Delta u + \lambda_n u).$$

In order, we consider the existence of time-periodic solutions of problems

$$\text{(ii)} \quad \begin{cases} \pm[u_t - \Delta u - \lambda_n u] + g(x,u) = h \text{ on a.e. } x = (\tilde{x}, t) \in \Omega = \tilde{\Omega} \times (-\pi, \pi), \\ u = 0 \text{ on } \partial\tilde{\Omega} \times \mathbb{R}, \end{cases} \tag{1.4}$$

where $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$D(L) = \{u \in L^2(\Omega) \mid D_t u, \Delta u \in L^2(\Omega) \text{ and } u = 0 \text{ on } \partial\tilde{\Omega} \times \mathbb{R}\}$$

and $L(u) = \pm(u_t - \Delta u - \lambda_n u)$; and

$$\text{(iii)} \quad \begin{cases} \pm[u_{tt} - \Delta u + \nu u_t - \lambda_n u] + g(x,u) = h \text{ on a.e. } x = (\tilde{x},t) \in \Omega = \tilde{\Omega} \times (-\pi,\pi), \\ u = 0 \text{ on } \partial\tilde{\Omega} \times \mathbb{R}, \end{cases} \tag{1.5}$$

where $\nu \neq 0$, $L : D(L) \subset L^2(\Omega) \to L^2(\Omega)$ is defined by

$$D(L) = \{u \in L^2(\Omega) \mid D_t u, D_{tt} u, \Delta u \in L^2(\Omega) \text{ and } u = 0 \text{ on } \partial\tilde{\Omega} \times \mathbb{R}\}$$

and $L(u) = \pm[u_{tt} - \Delta u + \nu u_t - \lambda_n u]$.

2. Existence theorems

In this section we shall always assume that the linear operator L is closed, densely defined and satisfies (L_1)–(L_4).

Theorem 2.1. *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function satisfying (G_1) and (G_2) with $2\alpha + \beta < 1$. Then, for each $h \in N^{\perp}(L)$, the problem (1.1) is solvable, provided that (G_3) holds.*

Proof. Let P and Q be the orthogonal projections of H on $N(L)$ and $R(L)$, respectively, and let $f : H \to H$ be a continuous function defined by

$$f(u) = \begin{cases} u, & \text{if } \|u\| \leqslant 1, \\ u/\|u\|, & \text{if } \|u\| > 1. \end{cases}$$

We consider the following semilinear equations

$$Lu + (1-t)f(Pu) + tg(x,u) = th, \tag{2.1}$$

for $0 \leqslant t \leqslant 1$. Then the problem (2.1) has only a trivial solution when $t = 0$, and becomes the original problem (1.1) when $t = 1$. To apply the Leray–Schauder continuation method, it suffices to show that there exists $R_0 > 0$ such that $\|u\| < R_0$ for each $0 < t < 1$ and for all possible solutions u to (2.1). Now let u be a possible solution of (2.1) for some $0 < t < 1$. By (L_4) we have

$$\begin{aligned} \|Qu\| &= \|L^{-1}\{(1-t)f(Pu) + tg(x,u) - th\}\| \\ &\leqslant \|L^{-1}\| \, \|(1-t)f(Pu) + tg(x,u) - th\| \\ &\leqslant \|L^{-1}\|((1-t) + a\|u\|^{\alpha} + \|b\| + \|h\|) \\ &\leqslant C_1 + C_2\|u\|^{\alpha}, \end{aligned} \tag{2.2}$$

for some constants $C_1, C_2 \geqslant 0$ independent of u. To show that solutions to (2.1) for $0 < t < 1$ have an *a priori* bound in H, we argue by contradiction, and suppose that there exists a sequence $\{u_n\}$ in H and a corresponding sequence $\{t_n\}$ in $(0,1)$ such that u_n is a solution to (2.1) with $t = t_n$ and $\|u_n\| \geqslant n$ for all n. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$, and, by (2.2), we have, for each $n \in \mathbb{N}$,

$$\|Qv_n\| \leqslant \frac{(C_1 + C_2\|u_n\|^\alpha)}{\|u_n\|}. \tag{2.3}$$

Since $\alpha < 1$, the right-hand side of (2.3) tends to zero in $\mathbb{R}$ as $n \to \infty$, and, since $\{Pv_n\}$ is bounded in H and $N(L)$ is of finite dimension, we may assume, without loss of generality, that $\{v_n\}$ is bounded by an $L^2(\Omega)$-function independent of n, converges to w in H, and is pointwise convergent to w on a.e. $x \in \Omega$. It follows that $u_n(x) \to \infty$ for a.e. $x \in \Omega_w^+ = \{y \in \Omega \mid w(y) > 0\}$, $u_n(x) \to -\infty$ for a.e. $x \in \Omega_w^- = \{y \in \Omega \mid w(y) < 0\}$, and $w \not\equiv 0$ because $\|v_n\| = 1$ for all $n \in \mathbb{N}$. Taking the inner product of (2.1) in H when $u = u_n$ and $t = t_n$ with Pu_n, we obtain from (L_3) that

$$\begin{aligned} t_n \int g(x, u_n)Pu_n &\leqslant (1 - t_n)\int f(Pu_n)Pu_n + t_n \int g(x, u_n)Pu_n \\ &= t_n \int hPu_n. \end{aligned} \tag{2.4}$$

It is clear from the assumption of $h \in N^\perp(L)$ that the right-hand side of the last equality of (2.4) is equal to zero. From (G_1), (2.2) and the assumption of $2\alpha + \beta < 1$ that there exist constants $C_3, C_4 \geqslant 0$ independent of n such that

$$\begin{aligned} \frac{|\int g(x,u_n)Qu_n|}{\|u_n\|^{1-\beta}} &\leqslant \frac{\int (a|u_n|^\alpha + b)|Qu_n|}{\|u_n\|^{1-\beta}} \\ &\leqslant \frac{(C_3\|u_n\|^\alpha + C_4)(C_1 + C_2\|u_n\|^\alpha)}{\|u_n\|^{1-\beta}} \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{2.5}$$

By (G_1), we have, for $0 \neq |u_n(x)| \leqslant r_0$,

$$\begin{aligned} \frac{|g(x,u_n)u_n|}{|u_n|^{1-\beta}}|v_n|^{1-\beta} &\leqslant \frac{|g(x,u_n)|\,|u_n|}{\|u_n\|^{1-\beta}} \\ &\leqslant \frac{[ar_0^\alpha + b(x)]r_0}{\|u_n\|^{1-\beta}}, \end{aligned} \tag{2.6}$$

and, by (G_2) and the assumption of $\beta \leqslant 1$, we also have for $|u_n(x)| > r_0$

$$\frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \geqslant c(x)|v_n|^{1-\beta}. \tag{2.7}$$

It follows from (2.6), (2.7) and the fact that $|v_n|$ is pointwise bounded by an $L^2(\Omega)$-function independent of n, that we have $(g(x,u_n)u_n/|u_n|^{1-\beta})|v_n|^{1-\beta}$ is bounded from below by an $L^1(\Omega)$-function independent of n. Using (2.3), (2.4), (2.6), (2.7), the fact that $t_n \neq 0$ and $h \in N^{\perp}(L)$, we also have

$$\begin{aligned}
\int_{\substack{v_n(x)>0\\ w(x)\neq 0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} &+ \int_{\substack{v_n(x)<0\\ w(x)\neq 0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&= \int_{\substack{v_n(x)\neq 0\\ w(x)\neq 0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&= \int_{\substack{u_n(x)\neq 0\\ w(x)\neq 0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&= \int_{u_n(x)\neq 0} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} - \int_{\substack{u_n(x)\neq 0\\ w(x)=0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&= \frac{1}{\|u_n\|^{1-\beta}} \int g(x,u_n)u_n - \int_{\substack{u_n(x)\neq 0\\ w(x)=0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&\leqslant \frac{1}{\|u_n\|^{1-\beta}} \int g(x,u_n)Qu_n - \int_{\substack{u_n(x)\neq 0\\ w(x)=0}} \frac{g(x,u_n)u_n}{|u_n|^{1-\beta}}|v_n|^{1-\beta} \\
&\leqslant \frac{1}{\|u_n\|^{1-\beta}} \left|\int g(x,u_n)Qu_n\right| + \int_{\substack{|u_n(x)|>r_0\\ w(x)=0}} |c|\,|v_n|^{1-\beta} + \int_{\substack{0<|u_n(x)|\leqslant r_0\\ w(x)=0}} \left[\frac{ar_0^{\alpha}+b}{\|u_n\|^{1-\beta}}\right].
\end{aligned} \tag{2.8}$$

Clearly, from (2.5), the assumption of $2\alpha+\beta<1$, the fact of $v_n(x)\to 0$ for a.e. $x\in\Omega_w^0=\{y\in\Omega \mid w(y)=0\}$ and the Lebesgue bounded convergence theorem that the right-hand side of the last inequality of (2.8) is convergent to zero as n approaches ∞. Applying Fatou's Lemma to the left-hand side of the first equality of (2.8), we have

$$\begin{aligned}
\int_{w(x)>0} g_\beta^+(x)|w(x)|^{1-\beta}\,\mathrm{d}x &+ \int_{w(x)<0} g_\beta^-(x)|w(x)|^{1-\beta}\,\mathrm{d}x \\
&= \int g_\beta^+(x)|w(x)|^{1-\beta}\chi_{\Omega_w^+}\,\mathrm{d}x + \int g_\beta^-(x)|w(x)|^{1-\beta}\chi_{\Omega_w^-}\,\mathrm{d}x \\
&= \int_{w(x)\neq 0} g_\beta^+(x)|w(x)|^{1-\beta}\chi_{\Omega_w^+}\,\mathrm{d}x + \int_{w(x)\neq 0} g_\beta^-(x)|w(x)|^{1-\beta}\chi_{\Omega_w^-}\,\mathrm{d}x \\
&\leqslant \int_{w(x)\neq 0} \liminf_{n\to\infty}\left[\frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}}|v_n(x)|^{1-\beta}\chi_{\Omega_{v_n}^+}\right]\mathrm{d}x \\
&\qquad + \int_{w(x)\neq 0} \liminf_{n\to\infty}\left[\frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}}|v_n(x)|^{1-\beta}\chi_{\Omega_{v_n}^-}\right]\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \int_{w(x)\neq 0} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}}|v_n(x)|^{1-\beta}\chi_{\Omega_{v_n}^+}\,\mathrm{d}x \\
&\qquad + \liminf_{n\to\infty} \int_{w(x)\neq 0} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}}|v_n(x)|^{1-\beta}\chi_{\Omega_{v_n}^-}\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= \liminf_{n\to\infty} \int_{\substack{v_n(x)>0\\ w(x)\neq 0}} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}} |v_n(x)|^{1-\beta}\,\mathrm{d}x \\
&\qquad + \liminf_{n\to\infty} \int_{\substack{v_n(x)<0\\ w(x)\neq 0}} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}} |v_n(x)|^{1-\beta}\,\mathrm{d}x \\
&\leqslant \liminf_{n\to\infty} \bigg[\int_{\substack{v_n(x)>0\\ w(x)\neq 0}} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}} |v_n(x)|^{1-\beta}\,\mathrm{d}x \\
&\qquad + \int_{\substack{v_n(x)<0\\ w(x)\neq 0}} \frac{g(x,u_n(x))u_n(x)}{|u_n(x)|^{1-\beta}} |v_n(x)|^{1-\beta}\,\mathrm{d}x \bigg] \\
&\leqslant 0,
\end{aligned}
$$

which contradicts the inequality (G_3), and the proof is complete. □

By modifying slightly the proof of Theorem 2.1, we can obtain the following theorems in which $2\alpha+\beta$ may be equal to 1.

Theorem 2.2. *Let $g : \Omega\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function satisfying (G_1), (G_2) with $2\alpha+\beta=1$ and $\beta<1$. Then the problem (1.1) is solvable for each $h\in N^{\perp}(L)$, provided that (G_3) holds and for a.e. $x\in\Omega$*

$$
\lim_{|u|\to\infty} \frac{g(x,u)}{|u|^{\alpha}} = 0. \tag{2.9}
$$

Proof. In proving Theorem 2.1, the condition $2\alpha+\beta<1$ is used only to show that the sequence $\{(1/\|u_n\|^{1-\beta})\int g(x,u_n)Qu_n\}$ is convergent to zero in $\mathbb{R}$. Thus we can proceed exactly the same way as in the proof of Theorem 2.1, and it suffices to prove that $\{(1/\|u_n\|^{1-\beta})\int g(x,u_n)Qu_n\}$ is convergent to zero. By the assumption of (G_1), the sequence $\{Lu_n/\|u_n\|^{\alpha}\}$ is bounded in H. Using the compactness of L^{-1} that $\{Qu_n/\|u_n\|^{\alpha}\}$ has a subsequence that is convergent in H. We may assume without loss of generality that $\{Qu_n/\|u_n\|^{\alpha}\}$ is bounded by an $L^2(\Omega)$-function independent of n. Since $2\alpha+\beta=1$ and $\beta<1$, we have $\alpha>0$. It follows from (2.9), the fact that $u_n(x)\to\infty$ for a.e. $x\in\Omega_w^+$, $u_n(x)\to-\infty$ for a.e. $x\in\Omega_w^-$ and the Lebesgue bounded convergence theorem that we have

$$
\begin{aligned}
&\frac{1}{\|u_n\|^{1-\beta}} \left| \int g(x,u_n)Qu_n \right| \\
&\qquad \leqslant \frac{1}{\|u_n\|^{1-\beta}} \bigg[\int_{|u_n(x)|\leqslant r_0} |g(x,u_n)Qu_n| + \int_{\substack{|u_n(x)|>r_0\\ w(x)\neq 0}} |g(x,u_n)Qu_n| \\
&\qquad\qquad + \int_{\substack{|u_n(x)|>r_0\\ w(x)=0}} |g(x,u_n)Qu_n| \bigg]
\end{aligned}
$$

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$$
\begin{aligned}
&\leqslant \frac{1}{\|u_n\|^{1-\beta}} \int_{|u_n(x)|\leqslant r_0} |g(x,u_n)Qu_n| + \int_{\substack{|u_n(x)|>r_0 \\ w(x)\neq 0}} \left[\frac{|g(x,u_n)|}{|u_n|^{\alpha}}|v_n|^{\alpha}\frac{|Qu_n|}{\|u_n\|^{\alpha}}\right] \\
&\qquad + \int_{\substack{|u_n(x)|>r_0 \\ w(x)=0}} \frac{|g(x,u_n)|}{|u_n|^{\alpha}}\left[|v_n|^{\alpha}\frac{|Qu_n|}{\|u_n\|^{1-\alpha-\beta}}\right] \\
&\leqslant \frac{1}{\|u_n\|^{1-\beta}}\|a_{r_0}\|\,\|Qu_n\| + \int_{\substack{|u_n(x)|>r_0 \\ w(x)\neq 0}} \left[\frac{|g(x,u_n)|}{|u_n|^{\alpha}}|v_n|^{\alpha}\right]\frac{|Qu_n|}{\|u_n\|^{\alpha}} \\
&\qquad + \int_{\substack{|u_n(x)|>r_0 \\ w(x)=0}} \frac{|g(x,u_n)|}{|u_n|^{\alpha}}\left[|v_n|^{\alpha}\frac{|Qu_n|}{\|u_n\|^{1-\alpha-\beta}}\right] \\
&\to 0 \text{ as } n\to\infty. \qquad (2.10)
\end{aligned}
$$

□

Theorem 2.3. *Let $g:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function satisfying (G_1), (G_2) with $2\alpha+\beta\leqslant 1$. Then the problem (1.1) is solvable for each $h\in N^{\perp}(L)$, provided that for each $w\in N(L)\setminus\{0\}$,*

$$\int_{w(x)>0} g_{\beta}^{+}(x)|w(x)|^{1-\beta}\,\mathrm{d}x + \int_{w(x)<0} g_{\beta}^{-}(x)|w(x)|^{1-\beta}\,\mathrm{d}x = \infty. \qquad (2.11)$$

Proof. By the assumption of $2\alpha+\beta\leqslant 1$, we find that the left-hand side of the first inequality of (2.5) is bounded by a constant independent of n and (2.8) is satisfied. Clearly, both

$$\int_{\substack{|u_n(x)|>r_0 \\ w(x)=0}} |c(x)|\,|v_n(x)|^{1-\beta}\,\mathrm{d}x \quad\text{and}\quad \int_{\substack{0<|u_n(x)|\leqslant r_0 \\ w(x)=0}} \frac{ar_0^{\alpha}+b(x)}{\|u_n\|^{1-\beta}}\,\mathrm{d}x$$

are bounded by a constant independent of n. Applying Fatou's Lemma to the left-hand side of the first equality of (2.8), we have

$$
\begin{aligned}
&\int_{w(x)>0} g_{\beta}^{+}(x)|w(x)|^{1-\beta}\,\mathrm{d}x + \int_{w(x)<0} g_{\beta}^{-}(x)|w(x)|^{1-\beta}\,\mathrm{d}x \\
&\quad\leqslant \limsup_{n\to\infty}\left[\frac{1}{\|u_n\|^{1-\beta}}\left|\int g(x,u_n)Qu_n\right| + \int_{\substack{|u_n(x)|>r_0 \\ w(x)=0}} |c|\,|v_n|^{1-\beta} + \int_{\substack{0<|u_n(x)|\leqslant r_0 \\ w(x)=0}} \frac{ar_0^{\alpha}+b}{\|u_n\|^{1-\beta}}\right] \\
&\quad< \infty,
\end{aligned}
$$

which contradicts the condition (2.11), and the proof is complete. □

If the null space of L enjoys the unique continuation property, then the assumption of $\beta<1$ in Theorem 2.2 is superfluous, and the following theorem can be proved.

Theorem 2.4. *Under assumptions of Theorem 2.3, the problem (1.1) is solvable for each $h\in N^{\perp}(L)$, provided that $N(L)$ has the unique continuation property and both (2.9) and (G_3) hold.*

Proof. It suffices to prove that the theorem is true when $\beta = 1$ and $\alpha = 0$, and it needs only to be shown that

$$\int g(x, u_n) Q u_n \to 0, \quad \text{as } n \to \infty. \tag{2.12}$$

Indeed, the unique continuation property of $N(L)$ implies that, for a.e. $x \in \Omega$, $|u_n(x)| \to \infty$ as $n \to \infty$. It follows from this, (2.9) and the boundedness of $\{Qu_n\}$ in H that (2.12) is satisfied. Hence the proof is complete. □

If $h = 0$ in $L^2(\Omega)$ and $(Lu, u)_H \geqslant 0$ for all $u \in D(L)$, then the condition (2.9) in Theorem 2.4 is superfluous, and the following theorem can be obtained.

Theorem 2.5. *Under the assumptions of Theorem 2.3. Assume that* $(Lu, u)_H \geqslant 0$ *for all* $u \in D(L)$, *then the problem (1.1) is solvable, provided that* $h = 0$ *in* $L^2(\Omega)$, $N(L)$ *has the unique continuation property and* (G_3) *is satisfied.*

Proof. Taking the inner product of (2.1) in H when $u = u_n$ and $t = t_n$ with u_n, we have

$$\begin{aligned} t_n \int g(x, u_n) u_n &\leqslant (Lu_n, u_n)_H + (1 - t_n) \int f(Pu_n) Pu_n + t_n \int g(x, u_n) u_n \\ &= t_n \int h u_n = 0. \end{aligned}$$

Combining this with (G_3), we obtain

$$\begin{aligned} 0 &< \int_{w(x)>0} g_\beta^+(x) |w(x)|^{1-\beta} \, dx + \int_{w(x)<0} g_\beta^-(x) |w(x)|^{1-\beta} \, dx \\ &\leqslant \liminf_{n\to\infty} \frac{1}{\|u_n\|^{1-\beta}} \int g(x, u_n) u_n \\ &\leqslant 0, \end{aligned}$$

which is a contradiction. □

If $\alpha = 0$, $\beta = 1$ and $\dim N(L) = 1$, then the unique continuation property for $N(L)$ in Theorem 2.4 can be omitted, and the following theorem can be proved.

Theorem 2.6. *Let* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function satisfying* (G_1), (G_2) *with* $\alpha = 0$ *and* $\beta = 1$. *Assume that* $\dim N(L) = 1$, *then, for each* $h \in N^\perp(L)$, *the problem (1.1) is solvable, provided that both* (G_3) *and (2.9) hold.*

Proof. Let $w \in N(L) \setminus \{0\}$ be obtained as in the proof of Theorem 2.1, and let $\Omega_w = \{x | w(x) \neq 0\}$. Then

$$\int_{\Omega_w} g(x, u_n) P u_n = \int g(x, u_n) P u_n \leqslant \int h P u_n = 0.$$

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Therefore, if integrals in (2.4) and (2.5) are taken over Ω_w with $\alpha = 0$ and $\beta = 1$, then we have, analogously,

$$\begin{aligned} 0 &< \int_{w(x)>0} g_1^+(x)\,\mathrm{d}x + \int_{w(x)<0} g_1^-(x)\,\mathrm{d}x \\ &\leqslant \liminf_{n\to\infty} \int_{\Omega_w} g(x, u_n)u_n \\ &\leqslant \liminf_{n\to\infty} \int_{\Omega_w} g(x, u_n)Qu_n \\ &= 0, \end{aligned} \tag{2.13}$$

which has arrived at a contradiction. Hence the proof is complete. □

Remark 2.7. Under the special case $\alpha = 0$, $\beta = 1$ and $c(x) \geqslant c_0 > 0$ for a.e. $x \in \Omega$ and a fixed positive number c_0. Conclusions of Theorems 2.4 and 2.6 have been obtained by Ha [9] and Robinson and Landesman [**18**].

Remark 2.8. By slightly modifying the proofs of Theorems 2.1–2.6. The condition $h \in N^{\perp}(L)$ can be replaced by either (1.2) if $\beta = 0$; or $h \in L^2(\Omega)$ is arbitrary and (G_3) is satisfied if $-\alpha \leqslant \beta < 0$.

Finally, we give an example to show that problems (1.3)–(1.5) are solvable when the nonlinearity $g(x, u)$ has sublinear growth in u as $|u| \to \infty$ and (1.2) may be excluded. Let $\alpha, \beta \in \mathbb{R}$, $0 \leqslant \beta$, $\alpha \leqslant 1$ and $2\alpha + \beta \leqslant 1$, let $c, d \in L^2(\Omega)$ and let $a \in L^\infty(\Omega)$, $a \geqslant 0$. We define

$$g_1(x, u) = a(x)(\operatorname{sgn} u)|\sin u||u|^\alpha, \qquad g_2(x, u) = \begin{cases} \dfrac{c(x)u}{1 + |u|^{1+\beta}}, & \text{if } u \geqslant 0, \\ \dfrac{d(x)u}{1 + |u|^{1+\beta}}, & \text{if } u \leqslant 0, \end{cases}$$

and $g(x, u) = g_1(x, u) + g_2(x, u)$. Then $|g(x, u)| \leqslant \|a\|_\infty |u|^\alpha + |c(x)| + |d(x)|$, $g_\beta^+(x) = c(x)$, $g_\beta^-(x) = d(x)$, and $\liminf_{u\to\infty} g(x, u) = \limsup_{u\to-\infty} g(x, u) = 0$ for $\beta > 0$. Hence one of problems (1.3)–(1.5) is solvable, provided that

$$\int_{v(x)>0} c(x)|v(x)|^{1-\beta}\,\mathrm{d}x + \int_{v(x)<0} d(x)|v(x)|^{1-\beta}\,\mathrm{d}x > \int_\Omega h(x)v(x)\,\mathrm{d}x = 0$$

for all $v \in N(L) - \{0\}$, and either (i) $2\alpha + \beta < 1$; or (ii) (2.9) is satisfied and $2\alpha + \beta = 1$, holds, where $N(L) = N(\Delta + \lambda_n)$.

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