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A characterization of numerical ranges for antilinear operators *

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Abstract. **This paper aims to study the problem of determining the numerical ranges of antilinear operators on complex Hilbert spaces. First, we provide a concrete description of the numerical range $W(R)$ for every bounded antilinear operator R on a complex Hilbert space $\mathcal{H}$, solving the preceding problem. Second, given a bounded linear operator T on $\mathcal{H}$, we determine the possible value of the numerical radius $w(CT)$ of CT when C ranges over the collection of all conjugations on $\mathcal{H}$.**

1 Introduction

The chief aim of this paper is to study the problem of determining the numerical ranges of antilinear operators on complex Hilbert spaces which was started in [7, 15, 17]. By connecting the preceding problem to the theory of complex symmetric operators (as well as their relatives), we shall give a concrete description of the numerical ranges of antilinear operators on complex Hilbert spaces (see Theorem 1.1). To proceed, we first recall some definitions and basic facts.

Throughout this paper, $\mathcal{H}$ denotes an infinite dimensional, complex Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$, $\mathcal{B}(\mathcal{H})$ denotes the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{B}_a(\mathcal{H})$ denotes the collection of all bounded antilinear operators on $\mathcal{H}$. The *adjoint* of an operator $T\in\mathcal{B}(\mathcal{H})$ is denoted by T^*. For $R\in\mathcal{B}_a(\mathcal{H})$, the *antilinear adjoint* of R is the unique operator $R^\#\in\mathcal{B}_a(\mathcal{H})$ satisfying $\langle R^\# x,y\rangle=\langle Ry,x\rangle$ for all $x,y\in\mathcal{H}$.

We remark that antilinear operators are closely related to those linear ones. In fact, if C is a conjugation on $\mathcal{H}$, then

$$\mathcal{B}_a(\mathcal{H})=\{CX : X\in\mathcal{B}(\mathcal{H})\}\quad\text{and}\quad\mathcal{B}(\mathcal{H})=\{CY : Y\in\mathcal{B}_a(\mathcal{H})\}.$$

Recall that a *conjugation* on $\mathcal{H}$ is a bijective operator C in $\mathcal{B}_a(\mathcal{H})$ with $C^{-1}=C$ and $\langle Cx,Cy\rangle=\langle y,x\rangle$ for all $x,y\in\mathcal{H}$. We denote by $\mathcal{B}_c(\mathcal{H})$ the collection of all conjugations on $\mathcal{H}$. Conjugations are closely related to the study of several special classes of operators such as complex symmetric operators [3, 5, 10, 11, 12, 14, 20, 21, 24, 26, 29, 30, 31], skew symmetric operators [4, 6, 18, 27, 28] and conjugate normal operators [1, 13, 16, 19, 23, 25]. Recall that an operator $T\in\mathcal{B}(\mathcal{H})$ is said to be *complex symmetric*

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if $T = CT^*C$ for some $C \in \mathcal{B}_c(\mathcal{H})$, and T is said to be *skew symmetric* if $T = -CT^*C$ for some $C \in \mathcal{B}_c(\mathcal{H})$. Recall that $T \in \mathcal{B}(\mathcal{H})$ is said to be *conjugate normal* if $|T| = C|T^*|C$ for some $C \in \mathcal{B}_c(\mathcal{H})$. Recently, these special classes of operators have received much attention.

The *numerical range* of a linear (or antilinear) operator T on $\mathcal{H}$ is defined by

$$W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H}, \|x\| = 1\}.$$

The *numerical radius* of T is $w(T) := \sup\{|z| : z \in W(T)\}$. The notion of numerical range, which arose from the study of quadratic forms, was initially defined for linear operators on Hilbert spaces and plays important roles in operator theory. In particular, the numerical radius $w(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$, which is equivalent to the usual operator norm.

The numerical ranges of antilinear operators on Hilbert spaces and Banach spaces were introduced and studied by M. Chō, I. Hur and J. E. Lee in [7, 15]. In particular, it was proved that the numerical range $W(R)$ of an antilinear operator R is always circularly symmetric and connected. D. Kołaczek and V. Müller [17] improved the result by showing that $W(R)$ is in fact a disc provided R acts on an at least two-dimensional Banach space. Also it was completely characterized when an antilinear operator R on a Hilbert space has a trivial numerical range, that is, $W(R) = \{0\}$.

This paper concentrates on the basic problem of determining the numerical ranges of antilinear operators on complex Hilbert spaces. For $R \in \mathcal{B}_a(\mathcal{H})$, D. Kołaczek and V. Müller recently proved that $W(R) = W(\frac{R+R^\#}{2})$ (see [17, Corollary 2.8]), reducing the problem of determining $W(R)$ to that of determining $W(\frac{R+R^\#}{2})$.

The main result of this paper is the following theorem, which gives a concrete description of $W(R)$ for $R \in \mathcal{B}_a(\mathcal{H})$.

***Theorem* 1.1** *If $R \in \mathcal{B}_a(\mathcal{H})$, then either*

$$W(R) = \left\{z \in \mathbb{C} : |z| < \frac{\|R + R^\#\|}{2}\right\}$$

or

$$W(R) = \left\{z \in \mathbb{C} : |z| \le \frac{\|R + R^\#\|}{2}\right\},$$

and the latter holds if and only if $R + R^\#$ is norm-attaining (that is, $\|(R + R^\#)x\| = \|R + R^\#\|$ for some unit vector $x \in \mathcal{H}$).

The preceding result improves that obtained by D. Kołaczek and V. Müller in [17]. We do not know whether the result has an analogue for antilinear operators R on Banach spaces, since in this case R and $R^\#$ act on different spaces.

The proof for Theorem 1.1 is inspired by the decomposition of antilinear operators into the product of conjugations and linear operators. The reader will see that Theorem 1.1 is a direct consequence of the following theorem.

***Theorem* 1.2** *Let $T \in \mathcal{B}(\mathcal{H})$, $C \in \mathcal{B}_c(\mathcal{H})$ and $A = \frac{T+CT^*C}{2}$. Then either $W(CT) = \{z \in \mathbb{C} : |z| < \|A\|\}$ or $W(CT) = \{z \in \mathbb{C} : |z| \le \|A\|\}$, and the latter holds if and only if A is norm-attaining.*

Remark 1.3 (i) In Theorem 1.2, one can see that $W(CT) = \{0\}$ if and only if $T + CT^*C = 0$, that is, T is skew symmetric relative to C.

(ii) The result of Theorem 1.2 can be viewed as an application of the theory of complex symmetric operators. In fact, our proof for Theorem 1.2 depends on a result of S. R. Garcia concerning approximate antilinear eigenvalues of complex symmetric operators ([9, Theorem 2]).

As applications of Theorem 1.2, we shall describe in Section 2 the numerical ranges of several concrete antilinear operators (see Proposition 2.3, Examples 2.4 and 2.5).

For $T \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}_c(\mathcal{H})$, the result of Theorem 1.2 illustrates how $w(CT)$ depends on both T and C. In fact, if T is complex symmetric relative to C, that is, $T = CT^*C$, then $w(CT) = \|T\|$. Then does the converse hold? More generally, we are interested in the following natural question: *For fixed $T \in \mathcal{B}(\mathcal{H})$, can one characterize all possible values of $w(CT)$ when C ranges over $\mathcal{B}_c(\mathcal{H})$?*

The second result of this paper answers the preceding question.

***Theorem* 1.4** *Let $T \in \mathcal{B}(\mathcal{H})$ and $\kappa(T) = \inf\{\|T - X\| : X \in SSO\}$, where SSO denotes the collection of all skew symmetric operators on $\mathcal{H}$. Then*

$$(\kappa(T), \|T\|] \subset \{w(CT) : C \in \mathcal{B}_c(\mathcal{H})\} \subset [\kappa(T), \|T\|]. \tag{1.1}$$

Remark 1.5 (i) The result of Theorem 1.4 exhibits the connections between skew symmetric operators and the numerical radius of antilinear operators. In Theorem 1.4, one can see that $\kappa(T) = 0$ if and only if $T \in \overline{SSO}$, that is, T is a norm limit of skew symmetric operators. In Propositions 3.4 and 3.7, we provide concrete descriptions of those normal operators T satisfying $\kappa(T) = 0$ or $\kappa(T) = \|T\|$. It is usually difficult to calculate $\kappa(T)$ for general $T \in \mathcal{B}(\mathcal{H})$.

(ii) In Theorem 1.2, $w(CT) = \|T\|$ does not imply $T = CT^*C$. In fact, given an operator $T \in \mathcal{B}(\mathcal{H})$ that is not complex symmetric, by Theorem 1.4, we can find $C \in \mathcal{B}_c(\mathcal{H})$ such that $w(CT) = \|T\|$, that is, $\frac{\|T+CT^*C\|}{2} = \|T\|$. However, since T is not complex symmetric, we have $T \neq CT^*C$.

(iii) We remark that each inclusion relation in (1.1) might be proper. In fact, if $T = 0$, then one can see that the first inclusion relation in (1.1) is proper. On the other hand, by Example 3.5, we can find $T \in \overline{SSO} \setminus SSO$. Then $\kappa(T) = 0$ and $T \neq -CT^*C$ for any $C \in \mathcal{B}_c(\mathcal{H})$. By Theorem 1.2, the latter implies that $w(CT) \neq 0$ for any $C \in \mathcal{B}_c(\mathcal{H})$. Thus $\{w(CT) : C \in \mathcal{B}_c(\mathcal{H})\} \subsetneq [\kappa(T), \|T\|]$, that is, the second inclusion relation in (1.1) is proper.

The rest of this paper is organised as follows. In Section 2, we shall give the proofs of Theorems 1.1 and 1.2; also we shall describe the numerical ranges of several concrete antilinear operators. Section 3 is devoted to the proof of Theorem 1.4. In addition, we shall provide some results useful to estimate $\kappa(T)$ for $T \in \mathcal{B}(\mathcal{H})$ (see Propositions 3.4 and 3.7).

2 Proofs of Theorems 1.1 and 1.2

To prove Theorem 1.2, we need an extra result concerning complex symmetric operators.

For $A \in \mathcal{B}(\mathcal{H})$, we denote by $|A|$ the unique positive square root of A^*A.

***Lemma* 2.1** *([9, Theorem 2]) Let $A \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Denote $\lambda = \|A\|$. If $CAC = A^*$, then*

(i) there exist a sequence of unit vectors f_n such that $\|Af_n - \lambda Cf_n\| \to 0$.
(ii) λ is an eigenvalue of $|A|$ if and only if $Af = \lambda Cf$ for some nonzero $f \in \mathcal{H}$.

Proof The proof is divided into two steps.

Step 1. $w(CT) = \|A\|$.

We denote $B = \frac{T - CT^*C}{2}$. One can check that $T = A + B$, $CAC = A^*$ and $CBC = -B^*$. For each $x \in \mathcal{H}$, we have

$$\langle CBx, x\rangle = \langle Cx, Bx\rangle = \langle B^*Cx, x\rangle = -\langle CBx, x\rangle,$$

which implies that $\langle CBx, x\rangle = 0$. Hence

$$\langle CTx, x\rangle = \langle CAx, x\rangle + \langle CBx, x\rangle = \langle CAx, x\rangle,$$

which implies that $W(CT) = W(CA)$ and $w(CT) = w(CA)$. Denote $\lambda = \|A\|$. By Lemma 2.1, we can find a sequence of unit vectors f_n such that $\|Af_n - \lambda Cf_n\| \to 0$. Thus

$$\begin{aligned} &|\langle CAf_n, f_n\rangle| = |\langle Cf_n, Af_n\rangle| \\ =&|\langle Cf_n, \lambda Cf_n\rangle + \langle Cf_n, (A - \lambda C)f_n\rangle| \to \lambda = \|A\|, \end{aligned}$$

which implies that $w(CA) \geq \|A\|$. Since the converse is obvious, we conclude that $w(CT) = w(CA) = \|A\|$.

Note that $W(CT)$ is circularly symmetric. It follows from $w(CT) = \|A\|$ that either $W(CT) = \{z \in \mathbb{C} : |z| < \|A\|\}$ or $W(CT) = \{z \in \mathbb{C} : |z| \leq \|A\|\}$.

Step 2. $W(CT) = \{z \in \mathbb{C} : |z| \leq \|A\|\}$ if and only if A is norm-attaining.

"$\Longrightarrow$". By Step 1, we have $W(CT) = W(CA)$. It follows that $\|A\| \in W(CA)$. Then there exists a unit vector $x \in \mathcal{H}$ such that

$$\langle Cx, Ax\rangle = \langle CAx, x\rangle = \|A\|.$$

Thus $\|A\| = \langle Cx, Ax\rangle \leq \|Ax\|$, which implies $\|A\| = \|Ax\|$.

"$\Longleftarrow$". We assume that $\|Ax\| = \|A\|$ for some unit vector $x \in \mathcal{H}$. This implies that $\||A|x\| = \|A\| = \||A|\|$. Thus $\|A\|$ is an eigenvalue of $|A|$. By Lemma 2.1, there exists a unit vector $f \in \mathcal{H}$ such that $Af = \|A\|Cf$. Thus

$$\langle CAf, f\rangle = \langle Cf, Af\rangle = \langle Cf, \|A\|Cf\rangle = \|A\|,$$

which implies that $\|A\| \in W(CA) = W(CT)$. Since $W(CT)$ is circularly symmetric, we conclude that $W(CT) = \{z \in \mathbb{C} : |z| \leq \|A\|\}$. This completes the proof. ∎

Theorem 1.1 follows readily from Theorem 1.2.

Proof Choose a conjugation C on $\mathcal{H}$ and set $T = CR$. Then $T \in \mathcal{B}(\mathcal{H})$ and $R = CT$. By Theorem 1.1, we have either

$$W(R) = \left\{ z \in \mathbb{C} : |z| < \frac{\|T + CT^*C\|}{2} \right\}$$

or

$$W(R) = \left\{ z \in \mathbb{C} : |z| \leq \frac{\|T + CT^*C\|}{2} \right\},$$

and the latter holds if and only if $T + CT^*C$ is norm-attaining. Note that $R = CT$ and $R^\# = T^*C$. Thus

$$\frac{T + CT^*C}{2} = \frac{C(CT + T^*C)}{2} = \frac{C(R + R^\#)}{2}.$$

This implies that $T + CT^*C$ is norm-attaining if and only if so is $R + R^\#$. Thus the desired result follows readily. ∎

Next, as applications of Theorem 1.2, we shall calculate the numerical ranges of several concrete operators.

In the following, $\mathbb{T}$ denotes the unit circle, that is, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We let H^2 denote the classical Hardy space, a closed subspace of $L^2(\mathbb{T}, \mu)$ spanned by $\{z^n : n = 0, 1, 2, \cdots\}$. Here μ is the usual arc length measure on $\mathbb{T}$. Let P be the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. For $\varphi \in L^\infty(\mathbb{T})$, the Toeplitz operator T_φ on H^2 is defined by $T_\varphi(f) = P(\varphi f)$ for $f \in H^2$. We call φ the symbol of T_φ.

***Lemma* 2.2** *([2, Theorem 4.1]) Let $\theta \in L^\infty(\mathbb{T}, \mu)$ with $\|\theta\|_\infty = 1$. Then T_θ is norm-attaining if and only if $\theta = \theta_1\overline{\theta_2}$, where θ_1, θ_2 are inner functions; moreover,*

$$\{x \in H^2 : \|T_\theta x\| = \|x\|\} = \theta_2 H^2.$$

Proposition 2.3. *Let $\varphi \in L^\infty(\mathbb{T}, \mu)$ and C be the canonical conjugation on H^2 defined by $Cz^n = z^n$ for $n = 0, 1, 2, \cdots$. Denote $\psi(z) = \varphi(\overline{z})$. Then*

(i) $w(CT_\varphi) = \frac{\|\varphi+\psi\|_\infty}{2}$;
(ii) $W(CT_\varphi)$ is closed if and only if there exists an inner function ζ and a complex number β such that $\frac{\varphi+\psi}{2} = \beta\zeta(z)\zeta(\overline{z})$.

Proof (i) Assume that $\varphi(z) = \sum_{k=-\infty}^{\infty} a_k z^k$ and denote $\theta = \frac{\varphi+\psi}{2}$. For $i, j \geq 0$, compute to see that $\langle T_\psi z^i, z^j \rangle = \left\langle \sum_k a_k z^{-k+i}, z^j \right\rangle = a_{i-j}$ and

$$\langle CT_\varphi^* C z^i, z^j \rangle = \langle CT_\varphi^* z^i, z^j \rangle = \langle Cz^j, T_\varphi^* z^i \rangle = \langle z^j, T_\varphi^* z^i \rangle = \langle T_\varphi z^j, z^i \rangle = a_{i-j}.$$

It follows that $CT_\varphi^* C = T_\psi$. Then

$$\frac{T_\varphi + CT_\varphi^* C}{2} = \frac{T_\varphi + T_\psi}{2} = T_\theta$$

and, by Theorem 1.2, we have $w(CT_\varphi) = \frac{\|T_\varphi + CT_\varphi^* C\|}{2} = \|T_\theta\| = \|\theta\|_\infty$. Moreover, $W(CT_\varphi)$ is closed if and only if T_θ is norm-attaining.

(ii) Still we denote $\theta = \frac{\varphi+\psi}{2}$.

"$\Longleftarrow$". If $\theta(z) = \beta\zeta(z)\zeta(\overline{z})$ for some inner function ζ and some complex number β, then it is easy to check that $\|\theta\|_\infty = |\beta|$, $\overline{\zeta(\overline{z})} \in H^2$ with norm one and

$$\|T_\theta(\overline{\zeta(\overline{z})})\| = \|\beta\zeta\| = |\beta| = \|\theta\|_\infty = \|T_\theta\|.$$

That is, T_θ is norm-attaining, which implies $W(CT_\varphi)$ is closed.

"$\Longrightarrow$". Assume that $W(CT_\varphi)$ is closed. By Theorem 1.2, we deduce that T_θ is norm-attaining. Without loss of generality, we may assume that $\|T_\theta\| = 1$. By Lemma 2.2, there are inner functions θ_1, θ_2 such that $\theta = \theta_1\overline{\theta_2}$ and

$$\{x \in H^2 : \|T_\theta x\| = \|x\|\} = \theta_2 H^2.$$

Denote $\widetilde{\theta}_1(z) = \overline{\theta_1(\overline{z})}$ and $\widetilde{\theta}_2(z) = \overline{\theta_2(\overline{z})}$. Then

$$\theta(\overline{z}) = \theta_1(\overline{z})\overline{\theta_2(\overline{z})} = \overline{\widetilde{\theta}_1(z)}\widetilde{\theta}_2(z).$$

Denote $\widetilde{\theta}(z) = \theta(\overline{z})$. Then, using Lemma 2.2 again, we obtain

$$\{x \in H^2 : \|T_{\widetilde{\theta}}x\| = \|x\|\} = \widetilde{\theta}_1 H^2.$$

On the other hand, since $\theta(z) = \frac{\varphi(z)+\psi(z)}{2} = \frac{\varphi(z)+\varphi(\overline{z})}{2} = \theta(\overline{z}) = \widetilde{\theta}(z)$, we deduce that $\theta_2 H^2 = \widetilde{\theta}_1 H^2$. Thus we can find a unimodular number β such that

$$\theta_2(z) = \beta\widetilde{\theta}_1(z) = \beta\overline{\theta_1(\overline{z})}.$$

It follows that $\frac{\varphi(z)+\varphi(\overline{z})}{2} = \theta(z) = \overline{\beta}\theta_1(z)\theta_1(\overline{z})$. ■

Example 2.4 Let $\varphi \in L^\infty(\mathbb{T}, \mu)$ with $\varphi(z) = 1 + \sum_{n=1}^{\infty} a_n(z^n - z^{-n})$. Then $\frac{\varphi(z)+\varphi(\overline{z})}{2} = 1 = z\overline{z}$. By the preceding result, $W(CT_\varphi) = \{z \in \mathbb{C} : |z| \le 1\}$ is closed, where C is the canonical conjugation on H^2 satisfying $Cz^n = z^n, n = 1, 2, \cdots$.

Example 2.5 Let $\phi \in L^\infty([0, 1], m)$, where m denotes the Lebesgue measure. Define two conjugations C_1 and C_2 on $L^2([0, 1], m)$ as $C_1 : f(t) \longmapsto \overline{f(t)}$ and $C_2 : f(t) \longmapsto \overline{f(1-t)}$. Denote $\widetilde{\phi}(t) = \frac{\phi(t)+\phi(1-t)}{2}$. One can check that

$$\frac{M_\phi + C_1 M_\phi^* C_1}{2} = M_\phi \quad \text{and} \quad \frac{M_\phi + C_2 M_\phi^* C_2}{2} = M_{\widetilde{\phi}}.$$

Then, by Theorem 1.2,

$$w(C_1 M_\phi) = \frac{\|M_\phi + C_1 M_\phi^* C_1\|}{2} = \|M_\phi\| = \|\phi\|_\infty$$

and

$$w(C_2 M_\phi) = \frac{\|M_\phi + C_2 M_\phi^* C_2\|}{2} = \|M_{\widetilde{\phi}}\| = \|\widetilde{\phi}\|_\infty.$$

Note that M_ϕ is norm-attaining if and only if $|\phi| = \alpha$ almost everywhere for some $\alpha \in [0, \infty)$. Then $W(C_1 M_\phi)$ is closed if and only if $|\phi| = \alpha$ almost everywhere for some $\alpha \in [0, \infty)$. Likewise, we deduce that $W(C_2 M_\phi)$ is closed if and only if $|\widetilde{\phi}| = \beta$ almost everywhere for some $\beta \in [0, \infty)$.

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If $\phi(t) = t$ for $t \in [0,1]$, then $|\phi(t)| = t$ and $|\widetilde{\phi}(t)| \equiv \frac{1}{2}$. It follows immediately that $W(C_1 M_\phi) = \{z \in \mathbb{C} : |z| < 1\}$ and $W(C_2 M_\phi) = \{z \in \mathbb{C} : |z| \leq \frac{1}{2}\}$.

3 Proof of Theorem 1.4

To prove Theorem 1.4, we need to make some preparations.

***Lemma* 3.1** *([30, Theorem 2.1]) Given two unit vectors $x, y \in \mathcal{H}$, there exists a conjugation C on $\mathcal{H}$ such that $Cx = y$.*

For $\delta \geq 0$ and $\lambda \in \mathbb{C}$, we denote $B(\lambda, \delta) = \{z \in \mathbb{C} : |z - \lambda| < \delta\}$. For $A \in \mathcal{B}(\mathcal{H})$, we denote by $\sigma(A)$ and $\sigma_p(A)$ the spectrum of A and the point spectrum of A, respectively. The essential spectrum of A is denoted by $\sigma_e(A)$. We let $\ker A$ and $\operatorname{ran} A$ denote the kernel of A and the range of A, respectively.

***Lemma* 3.2** *Let $T \in \mathcal{B}(\mathcal{H})$. Then*

(i) there exists $C \in \mathcal{B}_c(\mathcal{H})$ such that $w(CT) = \|T\|$;
(ii) there exists $C \in \mathcal{B}_c(\mathcal{H})$ such that $W(CT) = \{z \in \mathbb{C} : |z| \leq \|T\|\}$ if and only if T is norm-attaining.

Proof (i) Let $T = UP$ be the polar decomposition of T, where $P = |T|$ and U is a partial isometry. Denote $r = \|T\|$.

Case 1. $r \in \sigma_p(P)$.

Choose a unit vector $x \in \ker(P - r)$ and set $y = Ux$. Then $\|y\| = 1$. By Lemma 3.1, we can construct a conjugation C on $\mathcal{H}$ such that $Cx = y$. Hence

$$\langle CTx, x\rangle = \langle Cx, Tx\rangle = \langle Cx, UPx\rangle = \langle y, UPx\rangle = r\langle y, Ux\rangle = r\langle y, y\rangle = r = \|T\|,$$

which means that $w(CT) = \|T\|$.

Case 2. $r \notin \sigma_p(P)$.

Since $r = \|P\|$ and $r \in \sigma(P)$, we deduce that r is an accumulation point of $\sigma(P)$. Denote by $E_P(\cdot)$ the projection-valued spectral measure associated with P. Then we can find pairwise disjoint nonempty subsets $\Delta_1, \Delta_2, \Delta_3, \cdots$ of $\sigma(P) \setminus \{r\}$ such that $\mathcal{H}_i := \operatorname{ran} E_P(\Delta_i) \neq \{0\}$ and $\sup \Delta_i \leq \inf \Delta_{i+1} \to r$ for all $i \geq 1$. Clearly, $\mathcal{H} = \oplus_{i=1}^{\infty} \mathcal{H}_i$. For each i, choose a unit vector $e_i \in \mathcal{H}_i$ and denote by P_i the projection of $\mathcal{H}$ onto $\oplus_{j=1}^{i} \mathcal{H}_j$. Then one can check that $\|Pe_i\| \to r$ and $P_i \longrightarrow I$ in the strong operator topology.

Set $n_1 = 1$. Denote by Q_1 the projection of $\mathcal{H}$ onto $\vee\{e_1, Ue_1\}$. Here $\vee$ denotes the closed linear span.

Since both $\{e_i\}_{i=2}^{\infty}$ and $\{Ue_i\}_{i=2}^{\infty}$ are orthonormal, they converge to 0 in the weak topology. Note that Q_1 is of finite rank. Then we deduce that $\lim_i Q_1 e_i = 0$ and $\lim_i Q_1 U e_i = 0$. We can choose $n_2 > n_1$ such that $\|Q_1 e_{n_2}\| + \|Q_1 U e_{n_2}\| < \frac{1}{2}$.

Denote by Q_2 the projection of $\mathcal{H}$ onto $\vee\{e_i, Ue_i : i = n_1, n_2\}$. Clearly, Q_2 is of finite rank. Then we can choose $n_3 > n_2$ such that $\|Q_2 e_{n_3}\| + \|Q_2 U e_{n_3}\| < \frac{1}{2^2}$.

Recursively, we can choose a subsequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that $\|Q_i e_{n_{i+1}}\| + \|Q_i U e_{n_{i+1}}\| < \frac{1}{2^i}$ for $i = 1, 2, \cdots$, where Q_i the projection of $\mathcal{H}$ onto $\vee\{e_j, Ue_j : j = n_1, n_2, \cdots, n_i\}$.

Denote $\mathcal{K}_1 = \operatorname{ran} Q_1$ and $\mathcal{K}_i = \operatorname{ran} Q_i \ominus \operatorname{ran} Q_{i-1}$ for each $i \geq 2$. Moreover, denote $\mathcal{K}_0 = \mathcal{H} \ominus (\oplus_{i\geq 1}\mathcal{K}_i)$. Denote

$$x_1 = e_{n_1},\ \ y_1 = Ue_{n_1};\ \ x_i = \frac{(I-Q_{i-1})e_{n_i}}{\|(I-Q_{i-1})e_{n_i}\|},\ \ y_i = \frac{(I-Q_{i-1})Ue_{n_i}}{\|(I-Q_{i-1})Ue_{n_i}\|},\ \ i \geq 2.$$

For each $i \geq 1$, define a conjugation C_i on $\mathcal{K}_i$ satisfying $C_i x_i = y_i$. Arbitrarily choose a conjugation C_0 on $\mathcal{K}_0$ and set $C = \oplus_{i=0}^{\infty} C_i$. Then C is a conjugation on $\mathcal{H}$. Now it remains to check that $w(CT) \geq \|T\|$.

Compute to see that

$$\langle CTx_i, x_i\rangle = \langle Cx_i, Tx_i\rangle = \langle Cx_i, UPx_i\rangle = \langle y_i, UPx_i\rangle = \langle U^* y_i, Px_i\rangle.$$

Note that $\|x_i - e_{n_i}\| + \|y_i - Ue_{n_i}\| \to 0$. Hence $\lim_i |\langle U^* y_i, Px_i\rangle - \langle U^*Ue_{n_i}, Pe_{n_i}\rangle| = 0$. Since it is obvious that $\langle U^*Ue_{n_i}, Pe_{n_i}\rangle = \langle e_{n_i}, Pe_{n_i}\rangle \to \|P\| = r$. We conclude that

$$\langle CTx_i, x_i\rangle = \langle U^* y_i, Px_i\rangle \to r.$$

This shows that $w(CT) \geq \|T\|$, which completes the proof.

(ii) If $T = 0$, then the result is clear. In the sequel, we only consider the case that $T \neq 0$.

"$\Longrightarrow$". Assume that C is a conjugation on $\mathcal{H}$ such that $W(CT) = B(0, \|T\|)^-$. Then there exists a unit vector $x \in \mathcal{H}$ such that $\langle CTx, x\rangle = \|T\|$. Noting that $|\langle CTx, x\rangle| = |\langle Cx, Tx\rangle| \leq \|Tx\|$, we obtain $\|T\| \leq \|Tx\|$, which implies $\|T\| = \|Tx\|$.

"$\Longleftarrow$". We assume that $\|Tx\| = \|T\|$ for some unit vector $x \in \mathcal{H}$. Set $y = \frac{Tx}{\|Tx\|}$. By Lemma 3.1, we can find a conjugation C on $\mathcal{H}$ such that $Cx = y$. Then

$$\langle CTx, x\rangle = \langle Cx, Tx\rangle = \langle \frac{Tx}{\|Tx\|}, Tx\rangle = \|Tx\| = \|T\|.$$

It follows that $\|T\| \in W(CT)$ and $w(CT) = \|T\|$. Since $W(CT)$ is circularly symmetric, we conclude that $W(CT) = B(0, \|T\|)^-$. ∎

For $T \in \mathcal{B}(\mathcal{H})$, we denote $\operatorname{dist}(T, SSO) = \inf\{\|T - X\| : X \in SSO\}$.

Proof Without loss of generality, we assume that $T \neq 0$. Denote $\Gamma = \{w(CT) : C \in \mathcal{B}_c(\mathcal{H})\}$. The proof is reduced to proving several claims.

Claim 1. $\max \Gamma = \|T\|$.

For each unit vector $x \in \mathcal{H}$, we have $|\langle CTx, x\rangle| \leq \|CTx\| \leq \|T\|$, which implies $\sup\Gamma \leq \|T\|$. From Lemma 3.2 (i), one can see that $\|T\| \in \Gamma$ and $\sup\Gamma \geq \|T\|$. Hence we deduce that $\max\Gamma = \|T\|$.

Claim 2. $\inf\Gamma = \kappa(T)$.

For each $C \in \mathcal{B}_c(\mathcal{H})$, note that $C(\frac{T-CT^*C}{2})C = \frac{CTC-T^*}{2} = -(\frac{T-CT^*C}{2})^*$, that is, $\frac{T-CT^*C}{2} \in SSO$. Then, by Theorem 1.2, we have

$$w(CT) = \frac{\|T + CT^*C\|}{2} = \left\| T - \frac{T - CT^*C}{2} \right\| \geq \operatorname{dist}(T, SSO).$$

Since C is arbitrary, it follows that $\inf\Gamma \geq \operatorname{dist}(T, SSO) = \kappa(T)$.

On the other hand, we choose an operator $A \in SSO$. Thus $A = -CA^*C$ for some $C \in \mathcal{B}_c(\mathcal{H})$. For $x \in \mathcal{H}$, we have

$$\langle CAx, x\rangle = \langle Cx, Ax\rangle = \langle A^*Cx, x\rangle = -\langle CAx, x\rangle,$$

which implies that $\langle CAx, x\rangle = 0$ and hence

$$\langle C(T-A)x, x\rangle = \langle CTx, x\rangle - \langle CAx, x\rangle = \langle CTx, x\rangle.$$

It follows that

$$\inf \Gamma \leq w(CT) = w(C(T-A)) \leq \|T-A\|.$$

Since A was arbitrarily chosen in SSO, we conclude that $\inf \Gamma \leq \operatorname{dist}(T, SSO) = \kappa(T)$. This proves Claim 2.

Claim 3. $\mathcal{B}_c(\mathcal{H})$ is arcwise connected.

Choose a conjugation C_0 on $\mathcal{H}$. Then, by [10, Lemma 1], we can find an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ such that $C_0 e_n = e_n$ for all n. If $C \in \mathcal{B}_c(\mathcal{H})$ and $\{f_n\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$ such that $Cf_n = f_n$ for all n, then we can find a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $Uf_n = e_n$ for all n. Then one can verify that $U^*C_0U = C$. This shows that $\mathcal{B}_c(\mathcal{H}) = \{U^*C_0U : U \in \mathcal{B}(\mathcal{H}) \text{ is unitary}\}$. Since all unitary operators on $\mathcal{H}$ constitute an arcwise connected subset of $\mathcal{B}(\mathcal{H})$, it follows that $\mathcal{B}_c(\mathcal{H})$ is arcwise connected.

Claim 4. $(\kappa(T), \|T\|] \subset \Gamma$.

We define $\Phi : \mathcal{B}_c(\mathcal{H}) \longrightarrow [0, \infty)$ as $\Phi(C) = w(CT)$ for $C \in \mathcal{B}_c(\mathcal{H})$. By Theorem 1.2, we have $\Phi(C) = \frac{\|T+CT^*C\|}{2}$ for $C \in \mathcal{B}_c(\mathcal{H})$. This shows that Φ is continuous. Since $\mathcal{B}_c(\mathcal{H})$ is connected, it follows that

$$\Phi(\mathcal{B}_c(\mathcal{H})) = \{w(CT) : C \in \mathcal{B}_c(\mathcal{H})\} \subset [0, \infty)$$

is connected. In view of Claims 1 and 2, we conclude that Claim 4 holds. This completes the proof. ∎

The rest of this section is devoted to the estimation of $\kappa(T)$ for $T \in \mathcal{B}(\mathcal{H})$. The following result provides a lower bound for $\kappa(T)$.

***Lemma* 3.3** *Let $T \in \mathcal{B}(\mathcal{H})$. Then $\kappa(T) \geq \inf\{|z| : z \in W(T)\}$. If, in addition, T is positive, then $\kappa(T) \geq \inf \sigma(T)$.*

Proof Arbitrarily choose a conjugation C on $\mathcal{H}$. Then we can find a unit vector x such that $Cx = x$. Thus

$$|\langle CTx, x\rangle| = |\langle Cx, Tx\rangle| = |\langle x, Tx\rangle| \geq \inf\{|z| : z \in W(T)\}.$$

Since C is arbitrary, we deduce that $\kappa(T) \geq \inf\{|z| : z \in W(T)\}$.

If T is positive, then it is well known that $\overline{W(T)} = [\inf \sigma(T), \|T\|]$. Then the desired result follows readily. ∎

Given a subset Γ of $\mathbb{C}$, we denote $-\Gamma = \{-z : z \in \Gamma\}$ and denote by $\operatorname{iso} \Gamma$ the set of all isolated points of Γ.

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Proposition 3.4. *If $N \in \mathcal{B}(\mathcal{H})$ is normal, then $\kappa(N) = 0$ if and only if $\sigma(N) = -\sigma(N)$ and* $\dim\ker(N - \lambda) = \dim\ker(N + \lambda)$ *for each $\lambda \in \operatorname{iso}\sigma(N)$.*

Proof "$\Longrightarrow$". Since $\kappa(N) = 0$, that is, $N \in \overline{SSO}$, we can choose $\{T_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$ and $\{C_n\}_{n=1}^{\infty} \subset \mathcal{B}_c(\mathcal{H})$ such that $T_n + C_nT_n^*C_n = 0$ for all n and $T_n \to N$. It follows immediately that $N + C_nN^*C_n \to 0$. For each $\lambda \in \mathbb{C}$, we have

$$C_n(N+\lambda)^*C_n \to -(N-\lambda), \quad -C_n(N-\lambda)C_n \to (N+\lambda)^*, \tag{3.1}$$

which implies that $N - \lambda$ is invertible if and only if $N + \lambda$ is invertible. Hence $\sigma(N) = -\sigma(N)$; moreover, $\lambda \in \operatorname{iso}\sigma(N)$ if and only if $-\lambda \in \operatorname{iso}\sigma(N)$.

For each function f on $\sigma(N)$, we define $\widetilde{f}(z) = \overline{f(-z)}$ for $z \in \sigma(N)$.

Claim. If f is a continuous function on $\sigma(N)$, then $\lim_n C_nf(N)C_n = \widetilde{f}(N)$.

Note that if $g(z) = \alpha z^i\overline{z}^j$, then

$$C_ng(N)C_n = C_n(\alpha N^iN^{*j})C_n \to \overline{\alpha}(-N^*)^i(-N)^j = \widetilde{g}(N).$$

Then one can see that $C_nf(N)C_n \to \widetilde{f}(N)$ for each continuous function f on $\sigma(N)$.

Choose a point $\lambda \in \operatorname{iso}\sigma(N)$. Define a continuous function g on $\sigma(N)$ as

$$h(z) = \begin{cases} 1, & z = \lambda, \\ 0, & z \in \sigma(N) \setminus \{\lambda\}. \end{cases}$$

Then

$$\widetilde{h}(z) = \begin{cases} 1, & z = -\lambda, \\ 0, & z \in \sigma(N) \setminus \{-\lambda\}. \end{cases}$$

Then, by Claim, we deduce that $\lim_n C_nh(N)C_n = \widetilde{h}(N)$. Since $h(N)$ and $\widetilde{h}(N)$ are orthogonal projections, it follows from [22, Lemma 6.2.1] that $\operatorname{rank}\widetilde{h}(N) = \operatorname{rank}h(N)$. Noting that $\operatorname{ran}h(N) = \ker(N - \lambda)$ and $\operatorname{ran}\widetilde{h}(N) = \ker(N + \lambda)$, we conclude that $\dim\ker(N - \lambda) = \dim\ker(N + \lambda)$. This proves the necessity.

"$\Longleftarrow$". We first consider the case that $\ker N = \{0\}$.

By the hypothesis, we can find a sequence $\{\lambda_i\}_{i=1}^{\infty} \subset \sigma(N)$ satisfying

(a) $\{\lambda_i : i \geq 1\}$ is dense in $\sigma(N)$;
(b) if $i \geq 1$ and $\lambda_i \in \operatorname{iso}\sigma(N)$, then $\operatorname{card}\{j \geq 1 : \lambda_j = \lambda_i\} = \dim\ker(N - \lambda_i)$, where $\operatorname{card}(\cdot)$ denotes cardinality;
(c) if $i \geq 1$ and $\lambda_i \notin \operatorname{iso}\sigma(N)$, then $\operatorname{card}\{j \geq 1 : \lambda_j = \lambda_i\} = \infty$;
(d) $\operatorname{card}\{j \geq 1 : \lambda_j = \lambda_i\} = \operatorname{card}\{j \geq 1 : \lambda_j = -\lambda_i\}$ for every $i \geq 1$.

Choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$ and define $T \in \mathcal{B}(\mathcal{H})$ as

$$Te_i = \lambda_ie_i, \quad i = 1, 2, 3, \cdots.$$

One can check that $\sigma(T) = \sigma(N)$ and $\dim\ker(T - z) = \dim\ker(N - z)$ for each $z \in \operatorname{iso}\sigma(N)$. Then, by [8, Proposition 39.10], we deduce that T is approximately unitarily equivalent to N, that is, there are unitary operators $U_n : \mathcal{H} \to \mathcal{H}$ such that $U_nTU_n^* \to N$.

Next we shall show that T is skew symmetric (which implies $\kappa(N) = \operatorname{dist}(N, SSO) = 0$). In fact, in view of (a)-(d), we can find a bijective map τ on the set $\mathbb{N}$ of positive integers such that $\tau^{-1} = \tau$ and $\lambda_{\tau(i)} = -\lambda_i$ for every $i = 1, 2, 3, \cdots$. We define a conjugation C on $\mathcal{H}$ as $C(\sum_i a_i e_i) = \sum_i \overline{a_i} e_{\tau(i)}$. Then one can check that

$$CTCe_i = CTe_{\tau(i)} = C(\lambda_{\tau(i)} e_{\tau(i)}) = \overline{\lambda_{\tau(i)}} e_i = -\overline{\lambda_i} e_i = -T^* e_i, \quad i = 1, 2, \cdots.$$

Hence T is skew symmetric relative to C.

Now we consider the case that $\ker N \neq \{0\}$. Assume that

$$N = \begin{array}{c} \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{N} \end{bmatrix} \end{array} \begin{array}{l} \ker N \\ (\ker N)^{\perp} \end{array}.$$

Thus $\ker \widetilde{N} = \{0\}$, $\sigma(\widetilde{N}) = -\sigma(\widetilde{N})$ and

$$\dim \ker(\widetilde{N} - z) = \dim \ker(\widetilde{N} + z)$$

for every $z \in \operatorname{iso} \sigma(\widetilde{N})$.

By the proof in the case that $\ker N = \{0\}$, one can see that $\widetilde{N}$ is a norm limit of skew symmetric operators. Then so is N. Hence we conclude that $\kappa(N) = \operatorname{dist}(N, SSO) = 0$. This completes the proof. ∎

Next we provide a concrete operator $T \in \overline{SSO} \setminus SSO$.

***Example* 3.5** We define a function h on $[0, 1]$ as

$$h(t) = \begin{cases} e^{4\pi i t}, & t \in [0, \frac{1}{2}], \\ 1, & t \in (\frac{1}{2}, 1]. \end{cases}$$

Define $T \in \mathcal{B}(L^2[0, 1])$ as $T(f) = hf$ for $f \in L^2[0, 1]$. Then T is normal and it is easy to check that

$$\sigma(T) = \{z \in \mathbb{C} : |z| = 1\}, \quad \sigma_p(T) = \{1\}.$$

By Proposition 3.4, we have $T \in \overline{SSO}$. However, $T \notin SSO$. In fact, if not, then $T = -CT^*C$, which implies $(T - 1) = -C(T + 1)^*C$. Since $1 \in \sigma_p(T)$, it follows that $-1 \in \sigma_p(T^*)$ and $-1 \in \sigma_p(T)$, contradicting $\sigma_p(T) = \{1\}$.

***Lemma* 3.6** *If $A, B \in \mathcal{B}(\mathcal{H})$ are approximately unitarily equivalent, then $\kappa(A) = \kappa(B)$.*

Proof Assume that $\{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$ are unitary operators with $U_n A U_n^* \to B$. Note that SSO is unitarily invariant. For any $X \in SSO$, we have

$$\|B - X\| = \lim_n \|U_n A U_n^* - X\| = \lim_n \|A - U_n^* X U_n\| \geq \operatorname{dist}(A, SSO).$$

It follows that $\operatorname{dist}(B, SSO) \geq \operatorname{dist}(A, SSO)$. By the symmetry, one can see that $\operatorname{dist}(B, SSO) \leq \operatorname{dist}(A, SSO)$, which implies $\kappa(A) = \kappa(B)$. ∎

The following result characterizes those normal operators T with $\kappa(T) = \|T\|$.

Proposition 3.7. *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator with $\|N\| = 1$. Then $w(CN) = 1$ for every conjugation C if and only if $\sigma_e(N) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$.*

Proof "$\Longleftarrow$". Without loss of generality, we assume that $\sigma_e(N) = \{1\}$. Since N is normal, then, by the BDF Theorem, we can find a compact operator K such that $N = I + K$.

Fix a conjugation C on $\mathcal{H}$. We shall show that $w(CN) = 1$. By [10, Lemma 1], we can find an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ such that $Ce_i = e_i$ for all i. Then $\{e_i\}_{i=1}^{\infty}$ converges in the weak topology to 0, that is, $\langle e_i, x\rangle \to 0$ for every $x \in \mathcal{H}$. Since K is compact, we deduce that $\lim_i \|Ke_i\| = 0$. Hence

$$\langle CNe_i, e_i\rangle = \langle Ce_i, Ne_i\rangle = \langle e_i, Ne_i\rangle = \langle e_i, (I+K)e_i\rangle = 1 + \langle e_i, Ke_i\rangle \to 1.$$

It follows that $w(CN) \geq 1$. Noting that $\|CN\| = 1$, we obtain $w(CN) = 1$.

"$\Longrightarrow$". Since $w(CN) = 1$ for every conjugation C, we have $\inf_C w(CN) = 1$. By Lemma 3.6, we may directly assume that N is a diagonal operator, say $N = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \cdots\}$ relative to an orthonormal basis $\{e_i\}$ of $\mathcal{H}$. For convenience, we denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Claim 1. $\sigma_e(N) \nsubseteq \mathbb{D}$.

Otherwise, we have $r < 1$, where $r := \sup\{|z| : z \in \sigma_e(N)\}$. So $\mathrm{card}\{i \geq 1 : |\lambda_i| > \frac{1+r}{2}\} < \infty$. Without loss of generality, we assume that $|\lambda_i| > \frac{1+r}{2}$ for $i = 1, 2, \cdots, s$ and $|\lambda_i| \leq \frac{1+r}{2}$ for $i > s$.

We define a conjugation C on $\mathcal{H}$ satisfying

$$Ce_i = \begin{cases} e_{i+s}, & 1 \leq i \leq s, \\ e_{i-s}, & s+1 \leq i \leq 2s, \\ Ce_i = e_i, & i \geq 2s+1. \end{cases}$$

Then for any unit vector $x \in \mathcal{H}$ with $x = \sum_i a_i e_i$, we have

$$Cx = \sum_{i=1}^{s} \overline{a_i} e_{s+i} + \sum_{i=s+1}^{2s} \overline{a_i} e_{i-s} + \sum_{i=2s+1}^{\infty} \overline{a_i} e_i$$

and

$$\begin{aligned} |\langle CNx, x\rangle| &= |\langle Cx, Nx\rangle| = |\langle Cx, \sum_i a_i\lambda_i e_i\rangle| \\ &= \left| \sum_{i=1}^{s} \overline{a_i a_{i+s} \lambda_{i+s}} + \sum_{i=s+1}^{2s} \overline{a_i a_{i-s} \lambda_{i-s}} + \sum_{i=2s+1}^{\infty} \overline{a_i}^2 \overline{\lambda_i} \right| \\ &\leq \sum_{i=1}^{s} |a_i a_{i+s} \lambda_{i+s}| + \sum_{j=1}^{s} \left| a_{j+s} a_j \lambda_j \right| + \sum_{i=2s+1}^{\infty} |a_i|^2 |\lambda_i| \\ &\leq \frac{1+r}{2} \sum_{i=1}^{s} |a_i a_{i+s}| + \sum_{i=1}^{s} |a_{i+s} a_i| + \frac{1+r}{2} \sum_{i=2s+1}^{\infty} |a_i|^2 \\ &\leq (1 + \frac{1+r}{2}) \sum_{i=1}^{s} |a_i a_{i+s}| + \frac{1+r}{2} \sum_{i=2s+1}^{\infty} |a_i|^2 \\ &\leq \frac{1}{2}(1 + \frac{1+r}{2}) \sum_{i=1}^{s} (|a_i|^2 + |a_{i+s}|^2) + \frac{1+r}{2} \sum_{i=2s+1}^{\infty} |a_i|^2 \end{aligned}$$

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$$= \frac{3+r}{4}\sum_{i=1}^{2s}|a_i|^2 + \frac{1+r}{2}\sum_{i=2s+1}^{\infty}|a_i|^2 \le \frac{3+r}{4} < 1.$$

Since x is a unit vector arbitrarily chosen in $\mathcal{H}$, this shows that $w(CN) < 1$, a contradiction. This proves Claim 1.

Claim 2. $\sigma_e(N) \cap \mathbb{D} = \emptyset$.

In fact, if not, then there exists $r \in (0,1)$ such that $\dim \operatorname{ran} E_N(B(0,r)) = \infty$, where $E_N(\cdot)$ is the projection-valued spectral measure associated with N and $B(0,r) = \{z \in \mathbb{C} : |z| < r\}$. By Claim 1, we also have $\dim \mathcal{H} \ominus \operatorname{ran} E_N(B(0,r)) = \infty$. Denote $M = \operatorname{ran} E_N(B(0,r))$. Then, relative to the decomposition $\mathcal{H} = M \oplus M^\perp$, N can be written as $N = N_1 \oplus N_2$, where N_1 and N_2 are normal. Clearly, $\sigma(N_1) \subset \overline{B(0,r)}$ and $\sigma(N_2) \subset \{z \in \mathbb{C} : r \le |z| \le 1\}$.

Choose an antiunitary operator $D : M \to M^\perp$ and define a conjugation C on $\mathcal{H}$ as

$$C = \begin{bmatrix} 0 & D^{-1} \\ D & 0 \end{bmatrix} \begin{matrix} M \\ M^\perp \end{matrix}.$$

For any $x \in M$ and $y \in M^\perp$ with $\|x\|^2 + \|y\|^2 = 1$, we have

$$\begin{aligned} |\langle CN(x+y), x+y\rangle| &= |\langle C(x+y), N(x+y)\rangle| \\ &= \left|\langle Dx + D^{-1}y, N_1x + N_2y\rangle\right| \\ &= \left|\langle Dx, N_2y\rangle + \langle D^{-1}y, N_1x\rangle\right| \\ &\le \|N_2\|\|x\|\|y\| + \|N_1\|\|x\|\|y\| \\ &\le \|x\|\|y\| + r\|x\|\|y\| \le (1+r)\|x\|\|y\| \\ &\le \frac{1+r}{2}(\|x\|^2 + \|y\|^2) = \frac{1+r}{2} < 1. \end{aligned}$$

This shows that $w(CN) < 1$, a contradiction. This proves Claim 2.

Since $\|N\| = 1$, by Claims 1 and 2, we have $\sigma_e(N) \subset \{z \in \mathbb{C} : |z| = 1\}$.

Claim 3. $\operatorname{card} \sigma_e(N) < \infty$.

Otherwise, we can find $\theta_1, \theta_2, \theta_3, \theta_4 \in [0, 2\pi)$ with $\theta_1 < \theta_2 < \theta_3 < \theta_4$ such that $\sigma_e(N) \cap \Delta_i \ne \emptyset, i = 1,2,3,4$, where $\Delta_1 = \{e^{\mathrm{i}\theta} : \theta_4 - 2\pi < \theta < \theta_1\}$ and

$$\Delta_i = \{e^{\mathrm{i}\theta} : \theta_{i-1} < \theta < \theta_i\}, \quad i = 2,3,4.$$

Set $\widetilde{\Delta}_1 = \{re^{\mathrm{i}\theta} : \theta_4 - 2\pi < \theta \le \theta_1, r \in [0,1]\}$ and

$$\widetilde{\Delta}_i = \{re^{\mathrm{i}\theta} : \theta_{i-1} < \theta \le \theta_i, r \in (0,1]\}, \quad i = 2,3,4.$$

Then $\{\widetilde{\Delta}_i\}_{i=1}^4$ is a partition of the closed unit disc $\overline{\mathbb{D}}$; moreover, one can see that $\dim \operatorname{ran} E_N(\Delta_i) = \infty, i = 1,2,3,4$. Hence

$$\Gamma_i := \{j \ge 1 : \lambda_j \in \widetilde{\Delta}_i\}$$

is infinite for each $1 \le i \le 4$. We choose a bijective map τ on the set $\mathbb{N}$ of positive integers satisfying $\tau^{-1} = \tau$, $\tau(\Gamma_1) = \Gamma_3$ and $\tau(\Gamma_2) = \Gamma_4$. It is not difficult to verify that

$$\delta := \sup\left\{\frac{|\lambda_i + \lambda_{\tau(i)}|}{2} : i \ge 1\right\} < 1.$$

For each $x = \sum_i a_i e_i \in \mathcal{H}$, we define $Cx = \sum_i \overline{a_i} e_{\tau(i)}$. Then one can verify that C is a conjugation on $\mathcal{H}$. Then for any unit vector $x \in \mathcal{H}$ with $x = \sum_i a_i e_i$, we have

$$\begin{aligned}
|\langle CNx, x\rangle| = |\langle Cx, Nx\rangle| &= \left|\left\langle \sum_i \overline{a_i} e_{\tau(i)}, \sum_i a_i\lambda_i e_i \right\rangle\right| \\
&= \left|\left\langle \sum_i \overline{a_{\tau(i)}} e_i, \sum_i a_i\lambda_i e_i \right\rangle\right| = \left|\sum_i \overline{\lambda_i a_i a_{\tau(i)}}\right| \\
&\le \left|\sum_{i\in\Gamma_1\cup\Gamma_3} \lambda_i a_i a_{\tau(i)}\right| + \left|\sum_{i\in\Gamma_2\cup\Gamma_4} \lambda_i a_i a_{\tau(i)}\right| \\
&= \left|\sum_{i\in\Gamma_1} (\lambda_i + \lambda_{\tau(i)}) a_i a_{\tau(i)}\right| + \left|\sum_{i\in\Gamma_2} (\lambda_i + \lambda_{\tau(i)}) a_i a_{\tau(i)}\right| \\
&\le \sum_{i\in\Gamma_1} |\lambda_i + \lambda_{\tau(i)}||a_i a_{\tau(i)}| + \sum_{i\in\Gamma_2} |\lambda_i + \lambda_{\tau(i)}||a_i a_{\tau(i)}| \\
&\le \sum_{i\in\Gamma_1} \frac{|\lambda_i + \lambda_{\tau(i)}|}{2}(|a_i|^2 + |a_{\tau(i)}|^2) + \sum_{i\in\Gamma_2} \frac{|\lambda_i + \lambda_{\tau(i)}|}{2}(|a_i|^2 + |a_{\tau(i)}|^2) \\
&\le \sum_{i\in\Gamma_1} \delta(|a_i|^2 + |a_{\tau(i)}|^2) + \sum_{i\in\Gamma_2} \delta(|a_i|^2 + |a_{\tau(i)}|^2) = \delta < 1.
\end{aligned}$$

This shows that $w(CN) < 1$, a contradiction. This proves Claim 3.

Now we shall conclude the proof for the necessity by proving card $\sigma_e(N) = 1$. For a proof by contradiction, we assume that card $\sigma_e(N) > 1$. Then we can choose $\theta_1, \theta_2 \in [0, 2\pi)$ with $\theta_1 < \theta_2$ such that $e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2} \in \sigma_e(N)$. Without loss of generality, we may assume that $\theta_1 = 0$.

By Claim 3, card $\sigma_e(N) < \infty$. Thus we can find real numbers $\theta_3, \theta_4, \theta_5, \theta_6$ with $\theta_1 < \theta_3 < \theta_4 < \theta_2 < \theta_5 < \theta_6 < 2\pi$ such that

$$\sigma(N) \cap \{re^{\mathrm{i}\theta} : \theta \in (\theta_3, \theta_4) \cup (\theta_5, \theta_6), r \in (0, 1]\} = \emptyset.$$

Denote

$$\Omega_1 = \{re^{\mathrm{i}\theta} : \theta \in [\theta_6 - 2\pi, \theta_3], r \in [0, 1]\}$$

and

$$\Omega_2 = \{re^{\mathrm{i}\theta} : \theta \in [\theta_4, \theta_5], r \in (0, 1]\}.$$

Then $\Omega_1 \cap \Omega_2 = \emptyset$, $\sigma(N) \subset [\Omega_1 \cup \Omega_2]$ and one can see that $\dim \operatorname{ran} E_N(\Omega_i) = \infty$, since $e^{\mathrm{i}\theta_i} \in \sigma_e(N) \cap \Omega_i$ for $i = 1, 2$. Hence

$$\Lambda_i := \{j \ge 1 : \lambda_j \in \Omega_i\}$$

is infinite for each $i = 1, 2$. We choose a bijective map τ on the set $\mathbb{N}$ of positive integers satisfying $\tau^{-1} = \tau$, $\tau(\Lambda_1) = \Lambda_2$. It is not difficult to verify that

$$\gamma := \sup\left\{\frac{|\lambda_i + \lambda_{\tau(i)}|}{2} : i \ge 1\right\} < 1.$$

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For each $x = \sum_i a_i e_i \in \mathcal{H}$, we define $Cx = \sum_i \overline{a_i} e_{\tau(i)}$. Then one can verify that C is a conjugation on $\mathcal{H}$. Then for any unit vector $x \in \mathcal{H}$ with $x = \sum_i a_i e_i$, we have

$$\begin{aligned}
|\langle CNx, x\rangle| = |\langle Cx, Nx\rangle| &= \left|\left\langle \sum_i \overline{a_i} e_{\tau(i)}, \sum_i a_i \lambda_i e_i \right\rangle\right| \\
&= \left|\left\langle \sum_i \overline{a_{\tau(i)}} e_i, \sum_i a_i \lambda_i e_i \right\rangle\right| = \left|\sum_i \overline{\lambda_i a_i a_{\tau(i)}}\right| \\
&= \left|\sum_{i\in\Lambda_1} \lambda_i a_i a_{\tau(i)} + \sum_{i\in\Lambda_2} \lambda_i a_i a_{\tau(i)}\right| \\
&= \left|\sum_{i\in\Lambda_1} \lambda_i a_i a_{\tau(i)} + \sum_{i\in\Lambda_1} \lambda_{\tau(i)} a_{\tau(i)} a_i\right| \\
&= \left|\sum_{i\in\Lambda_1} (\lambda_i + \lambda_{\tau(i)}) a_i a_{\tau(i)}\right| \\
&\le \sum_{i\in\Lambda_1} \frac{|\lambda_i + \lambda_{\tau(i)}|}{2} (|a_i|^2 + |a_{\tau(i)}|^2) \\
&\le \sum_{i\in\Lambda_1} \gamma(|a_i|^2 + |a_{\tau(i)}|^2) = \gamma < 1.
\end{aligned}$$

This shows that $w(CN) < 1$, a contradiction. This completes the proof. ∎

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