# AN APPLICATION OF A THEOREM OF J. CZIPSZER AND G. FREUD TO A PROBLEM OF SIMULTANEOUS APPROXIMATION

A.K. Varma*

[Dedicated in memory of J. Czipszer]

(received May 20, 1969)

1. Introduction. In an earlier work [12] we considered the case of $(0, 2, 3)$ interpolation by trigonometric polynomials at the points $x_i = \frac{2i\pi}{n}$, $i = 0, 1, \ldots, n-1$. By $(0, 2, 3)$ interpolation we mean the problem of finding a trigonometric polynomial of suitable order whose values, second and third derivatives are prescribed at some given points. An interesting distinction between the $(0, 2)$ interpolation studied by the Hungarian mathematician O. Kiš [8] and $(0, 2, 3)$ case studied by us is that the sequence of interpolatory polynomials $R_n(x)$ in our case converges uniformly to the given function if $f(x)$ is periodic and $f(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$, whereas Kiš [8] requires the Zygmund condition. For another generalization of the results of Kiš we refer to our other work [11]. As a matter of fact a systematic study of $(0, 2)$ interpolation by power polynomials is due to Professor P. Turán and his associates (see e.g. [1], [2]). We remark that the problem of $(0, 2, 3)$ interpolation when the nodes are taken to be zeros of ultraspherical polynomials is still open.

The main objective of this paper is to obtain sufficient conditions for the convergence of the $p^{th}$ derivative of $R_n(f;x)$ to $f^{(p)}(x)$ on the real line. The sufficient conditions turn out to be $f^{(p)}(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$. For details see Theorem 1.1, 1.2 and 1.3. For the case $p = 0$ we get results of our earlier work [12]. As the proof of these theorems will indicate, an essential role is played by a theorem due to J. Czipszer and G. Freud [3] as well as the estimation of the fundamental polynomials. The above-quoted authors' theorem is included in the preliminaries. Probably our condition can be improved so that $f(x)$ is supposed to be $2\pi$-periodic and $f^{(p)}(x)$ continuous on the real line.

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*The author acknowledges financial support from the Post Doctoral Fellowship, Department of Mathematics, University of Alberta, Edmonton, and the Summer Research Institute of the Canadian Mathematical Congress. The author is also grateful to the referee for his valuable suggestions.

There are many results analogous to this problem. It is well known that Bernstein polynomials provide simultaneous approximation of the function and their derivatives (see e.g. [4, p. 112]). Another important contribution of simultaneous approximation of function and its derivatives by means of interpolatory power polynomials is due to G. Freud [5]. He investigated in his work the sufficient conditions under which the $p^{th}$ differentiated sequence of Lagrange interpolation polynomials (associated with the fundamental point system $\{x_{in}\}$ which are the zeros of certain orthogonal polynomials) converges uniformly to $f^{(p)}(x)$ in $[a,b] \subset (-1,+1)$. It turns out as he proved the sufficient conditions are $f^{(p)}(x) \in \text{Lip}\,\alpha$, $\alpha > \frac{1}{2}$. Thus in our case sufficient conditions are much better than in Freud's case. Another important contribution to this aspect of simultaneous approximation theory is due to Russian mathematician Professor A.F. Timan [9]. He considered the problem of simultaneous approximation of arbitrary differentiable functions and their derivatives on the whole real axis by means of entire functions of exponential type. Further generalizations of S.N. Bernstein's approximation theorem for functions that are bounded and uniformly continuous on $(-\infty, \infty)$ and the best approximation to derivatives on the whole real axis were also obtained. For other interesting results, see [6] and [13]. More precisely the following theorems will be proved.

THEOREM 1.1. (Case 1, $p = 1$; notations as in [12]). Let $f(x)$ be a $2\pi$-periodic function such that $f'(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$. Consider

$$(1.1) \qquad R_n(f;x) = \sum_{i=0}^{n-1} f(x_{in})\, U(x-x_{in}) + \sum_{i=0}^{n-1} b_{in}\, V(x-x_{in}) + \sum_{i=0}^{n-1} c_{in}\, W(x-x_{in})$$

with

$$(1.2) \qquad b_{in} = o\left(\frac{n}{\log n}\right), \qquad c_{in} = o\left(\frac{n^2}{\log n}\right)$$

then $R_n'(f;x)$ converges uniformly to $f'(x)$ on the real line.

THEOREM 1.2. (Case 2, $p = 2$). Let $f(x)$ be $2\pi$-periodic and such that $f''(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$; let further $q = 0, 1$ or $2$. Consider

$$(1.3) \qquad S_n(f;x) = \sum_{i=0}^{n-1} f(x_{in})\, U(x-x_{in}) + \sum_{i=0}^{n-1} f''(x_{in})\, V(x-x_{in}) + \sum_{i=0}^{n-1} c_{in}\, W(x-x_{in})$$

with

$$(1.4) \qquad c_{in} = o\left(\frac{n^{3-q}}{\log n}\right),$$

then $S_n^{(q)}(f;x)$ will converge uniformly to $f^{(q)}(x)$ on the real line.

THEOREM 1.3. (Case 3, $p \geq 3$). Let $f(x)$ be a $2\pi$-periodic function satisfying $f^{(p)}(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$. Consider

$$(1.5) \qquad P_n(f,x) = \sum_{i=0}^{n-1} f(x_{in})\, U(x-x_{in}) + \sum_{i=0}^{n-1} f''(x_{in})\, V(x-x_{in}) + \sum_{i=0}^{n-1} f'''(x_{in})\, W(x-x_{in})$$

then $P_n^{(q)}(f;x)$; $(q = 0, 1, \ldots, p)$ will converge uniformly to $f^{(q)}(x)$ on the real line.

For the expressions of the fundamental polynomials see preliminaries.

2. Preliminaries. The paper of J. Czipszer and G. Freud [3] is concerned with the following problem. Let $f(x)$ be a $2\pi$-periodic function having $p^{th}$ derivative continuous $(p = 1, 2, \ldots)$ and $Q_n(x)$ is a trigonometric polynomial of order $n$ which approximates $f(x)$ within $\epsilon$;

$$(2.1) \qquad \max_{0 \leq x \leq 2\pi} |f(x) - Q_n(x)| \leq \epsilon \;;$$

then they asked: what upper estimate can be given to the quantity $\max |f^{(p)}(x) - Q_n^{(p)}(x)|$ ? They answered the question by proving the following inequality:

$$(2.2) \qquad \max_{0 \leq x \leq 2\pi} |f^{(p)}(x) - Q_n^{(p)}(x)| \leq 3\; 2^p\, n^p\, \epsilon + 4E_n(f^{(p)}).$$

We shall need some details of our earlier work [12]. There we were interested in obtaining a trigonometric polynomial $R_n(x)$ of suitable order such that

$$(2.3)\quad R_n(x_i) = a_i,\ R''_n(x_i) = b_i,\ R'''_n(x_i) = c_i,\ x_i = \frac{2i\pi}{n}\ (i = 0, 1, \ldots, n-1);$$

when $n$ is even $(= 2m)$, we require the trigonometric polynomial $R_n(x)$ of the form

$$(2.4)\qquad d_0 + \sum_{j=1}^{3m-1} (d_j \cos jx + e_j \sin jx) + d_{3m} \cos 3mx ,$$

but when $n$ is odd $(= 2m+1)$ we require it to be of the form

$$(2.5)\qquad d_0 + \sum_{j=1}^{3m+1} (d_j \cos jx + e_j \sin jx) .$$

The explicit representation of $R_n(x)$ ($n$ even) satisfying (2.3) and (2.4) has the form

$$(2.6)\qquad R_n(x) = \sum_{i=0}^{n-1} a_i U(x-x_i) + \sum_{i=0}^{n-1} b_i V(x-x_i) + \sum_{i=0}^{n-1} c_i W(x-x_i) ,$$

where

$$(2.7)\qquad W(x) = \sum_{j=1}^{3m-1} \alpha_j \sin jx,\quad a_j = -\frac{4j}{n^3(n^2-3j^2)},\qquad j = 1, 2, \ldots, m,$$

$$= \frac{-(3n-2j)}{n^3(n^2-3(n-j)^2)},\qquad j = m+1, \ldots, 3m-1,$$

$$(2.8)\qquad V(x) = \frac{1}{n^3}\Bigg[1 + 2\sum_{j=1}^{m-1} \frac{n^2+3j^2}{n^2-3j^2} \cos jx + \frac{1}{2}(\cos mx - \cos 3mx)$$

$$- \frac{1}{4}\sum_{j=m+1}^{3m-1} \frac{n^2+3(3n-2j)^2}{n^2-3(n-j)^2} \cos jx\Bigg],$$

and finally



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$$(2.9)\qquad U(x) = \frac{1}{n}\left[1 + \frac{2}{n^2}\sum_{j=1}^{m-1}\frac{(n^2-j^2)^2}{n^2-3j^2}\cos jx - \frac{1}{n^2}\sum_{j=m+1}^{3m-1}\frac{(n-j)^2(2n-j)^2}{n^2-3(n-j)^2}\cos jx - \frac{1}{8}(\cos x - 9\cos 3mx)\right].$$

The following estimates of these fundamental polynomials were also obtained in [12]:

$$(2.10)\qquad \sum_{i=0}^{n-1} | W(x-x_{in}) | \le c_1 n^{-3} \log n ,$$

$$(2.11)\qquad \sum_{i=0}^{n-1} | V(x-x_{in}) | \le c_2 n^{-2} \log n ,$$

and

$$(2.12)\qquad \sum_{i=0}^{n-1} | U(x-x_{in}) | \le c_3 \log n .$$

$c_1, c_2, c_3$ are positive constants independent of $n$ and $x$.

3. Approximation polynomials. In order to prove the theorems mentioned in Section 1 we need a lemma on approximating polynomials.

LEMMA 3.1. *Let* $f(x)$ *be a* $2\pi$-*periodic function satisfying* $f^{(p)}(x) \in \text{Lip}\,\alpha$, $0 < \alpha < 1$; *then there exists a trigonometric polynomial* $T_n(x)$ *of order* $\le n-1$ *such that*

$$(3.1)\qquad | f(x) - T_n(x) | = O(n^{-p-\alpha}) ;$$

$$(3.2)\qquad | T_n^{(r)}(x) | = O(n^{-p-\alpha+r}) \text{ for } p = 0 \text{ or } 1,\ r \ge 2 \text{ and } p=2,\ r \ge 3 ;$$

*and for* $p \ge 3$ *we have*

$$(3.3)\qquad |f^{(r)}(x) - T_n^{(r)}(x) | = O(n^{-p-\alpha+r}) .$$



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Proof. Existence of $T_n(x)$ satisfying (3.1) is due to D. Jackson [7] and is well known. (3.3) follows from (3.1) on applying the theorem J. Czipszer and G. Freud mentioned in preliminaries. It remains to prove (3.2). (The referee has very kindly pointed out that proof of (3.2) is due to S. Bernstein; see Approximation of functions by G.G. Lorentz: formula 6 on page 60. For the sake of completeness we give here the conventional proof.) Proof of (3.2) is similar to that of Lemma 9 of O. Kiś [8]. From the condition of the Lemma there exists a trigonometric polynomial $U_n(x)$ of order $\leq n-1$ for which

$$(3.4) \qquad | f(x) - U_n(x) | = O(n^{-p-\alpha}) .$$

Following the approach suggested by Professor P. Turán, one can set $T_n(x) = U_{2^k}(x)$ for $2^k \leq n \leq 2^{k+1}$. Then

$$(3.5) \qquad T_n(x) = \sum_{j=1}^{k} \left[ U_{2^j}(x) - U_{2^{j-1}}(x) \right] + U_1(x) \quad ;$$

owing to (3.4) we have

$$(3.6) \qquad U_{2^j}(x) - U_{2^{j-1}}(x) = O(2^{-j(p+\alpha)}) .$$

On applying the Bernstein inequality for trigonometric polynomials we have

$$(3.7) \qquad | U^{(r)}_{2^j}(x) - U^{(r)}_{2^{j-1}}(x) | = O(2^{-j(p+\alpha-r)}) .$$

Now differentiating (3.5) r times and applying (3.7) we obtain, on summing the geometric series, the result stated in (3.2). This proves the lemma.

4. Proof of Theorem 1.1. Since $p = 1$, we have, on using Lemma 3.1, that there exists a trigonometric polynomial $T_n(x)$ of order $\leq n$ such that

$$(4.1) \qquad |f(x) - T_n(x)| = O(n^{-1-\alpha}), \qquad 0 < \alpha < 1 ,$$

and

$$(4.2) \qquad |T_n''(x)| = O(n^{1-\alpha}), \qquad |T_n'''(x)| = O(n^{2-\alpha}) .$$

But we know that

$$(4.3) \qquad f(x) - R_n(x) = f(x) - T_n(x) + T_n(x) - R_n(x) .$$

Therefore

$$(4.4) \qquad |f(x) - R_n(x)| = O(n^{-1-\alpha}) + |T_n(x) - R_n(x)| .$$

On using the uniqueness theorem [12] we have

$$(4.5) \qquad T_n(x) = \sum_{i=0}^{n-1} T_n(x_{in})U(x-x_{in}) + \sum_{i=0}^{n-1} T_n''(x_{in})V(x-x_{in}) + \sum_{i=0}^{n-1} T'''(x_{in})W(x-x_{in}).$$

On using (1.1) and (4.5) we have

$$T_n(x) - R_n(f;x) = \sum_{i=0}^{n-1} [T_n(x_{in}) - f(x_{in})]U(x-x_{in})$$

$$+ \sum_{i=0}^{n-1} [T_n''(x_{in})-b_{in}] V(x-x_{in}) + \sum_{i=0}^{n-1} [T_n'''(x_{in})-c_{in}] W(x-x_{in}) .$$

From (4.1), (1.2) and (2.10)-(2.12) we have

$$(4.6) \qquad |T_n(x) - R_n(f;x)| = O(n^{-1-\alpha}) \log n + o(n^{-1}) = o(n^{-1}) .$$

Now (4.4) and (4.6) gives at once

$$|f(x) - R_n(x,f)| = o(n^{-1}).$$

On applying the theorem of J. Czipszer and G. Freud once more, we have

$$|f'(x) - R'_n(x,f)| \leq 3\cdot 2\cdot n\cdot\ o(n^{-1}) + 4E_n(f^{(1)})$$

$$= o(1) + O(n^{\alpha}) = o(1)$$

which proves Theorem 1.1. Proof of Theorem 1.2 is similar and so we omit the details.

Proof of Theorem 1.3. Here we assume $p \geq 3$. From the conditions of the theorem, and on applying Lemma 3.1 there exists a trigonometric polynomial of order $\leq n$ such that

$$(4.7) \qquad |f(x) - T_n(x)| = O(n^{-p-\alpha}), \qquad 0 < \alpha < 1,$$

$$(4.8) \qquad |f''(x) - T''_n(x)| = O(n^{2-p-\alpha}).$$

and

$$(4.9) \qquad |f'''(x) - T'''_n(x)| = O(n^{3-p-\alpha}).$$

On having the uniqueness theorem we have similar representation of $T_n(x)$ as given in (4.5) satisfying (4.7)-(4.9). From (1.5) and (4.5) we obtain

$$P_n(f;x) - T_n(x) = \sum_{i=0}^{n-1} [f(x_{in}) - T_n(x_{in})]\, U(x-x_{in})$$

$$+ \sum_{i=0}^{n-1} [f''(x_{in}) - T''_n(x_{in})]\, V(x-x_{in}) + \sum_{i=0}^{n-1} [T'''(x_{in}) - T'''_n(x_{in})]\, W(x-x_{in}).$$

Applying (4.7)-(4.9) and (2.10)-(2.12) we get

(4.10) $$|T_n(x) - P_n(f;x)| = O(n^{-p-\alpha}) \log n .$$

But we know that

$$f(x) - P_n(f;x) = f(x) - T_n(x) + T_n(x) - P_n(f;x) .$$

On using (4.7) and (4.10) we have

(4.11) $$|f(x) - P_n(f;x)| = O(n^{-p-\alpha}) + O(n^{-p-\alpha}) \log n$$

$$= O(n^{-p-\alpha}) \log n ;$$

using once more the theorem of J. Czipszer and G. Freud, we have

$$|f^{(q)}(x) - P_n^{(q)}(f;x)| \leq 3\, 2^q\, n^q\, O(n^{-p-\alpha}) \log n + 4E_n(f^{(q)})$$

$$= O(n^{q-p-\alpha}) = o(1) \text{ as } q = 0, 1, \ldots, p \text{ and } 0 < \alpha < 1 .$$

This proves the theorem.

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University of Florida
Gainesville
Florida



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