A RECURSION FORMULA FOR THE COEFFICIENTS IN AN ASYMPTOTIC EXPANSION

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(Received 19th August, 1958)

Many authors have proved results deducing an asymptotic expansion of

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n$$

for large $|z|$ from the behaviour of $f(t)$, when $f(t)$ is regular in an appropriate part of the complex t-plane. For example, if, for some $\kappa > 0$ and some A_m, α_m,

$$f(t) = \sum_{m=1}^{M} \frac{\kappa A_m}{\Gamma(\kappa t + \alpha_m)} + O\left(\frac{1}{\Gamma(\kappa t + \alpha_{m+1})}\right) \quad \text{.................................(1)}$$

for all large $|t|$ such that $\mathscr{R}(t) > C$, then, as $|z| \to \infty$ in a suitable sector in the z-plane, we have

$$F(z) = Ze^Z \left\{ \sum_{m=1}^{M} A_m Z^{-\alpha_m} + O(Z^{-\alpha_{M+1}}) \right\}, \quad \text{.................................(2)}$$

where Z is an appropriate value of $z^{1/\kappa}$.

Of course, the expansion (1) could be replaced by one of another form, but (1) has the merit of displaying the actual coefficients which occur in (2). So far as I am aware, the first notice of this phenomenon in special cases occurred in [1] and [6]; the general result was found in [7] and, independently, in [2]. See also [8].

A particular case of the generalised hypergeometric function studied in [6] is

$$G(x) = \sum_{n=0}^{\infty} g(n) x^n,$$

where

$$g(t) = \prod_{r=1}^{p} \Gamma(t+\beta_r) \Big/ \prod_{r=0}^{q} \Gamma(t+\gamma_r)$$

and $q \geqslant p \geqslant 0$. If we write $\kappa = q+1-p$ and

$$\vartheta = \sum_{r=1}^{p} \beta_r - \sum_{r=0}^{q} \gamma_r + \tfrac{1}{2}(\kappa+1),$$

we can deduce from the well-known asymptotic expansion of the Γ-function that

$$g(t) = a(\kappa^\kappa)^t \left\{ \sum_{m=0}^{M} \frac{\kappa c_m}{\Gamma(\kappa t - \vartheta + m + 1)} + O\left(\frac{1}{\Gamma(\kappa t - \vartheta + M + 2)}\right) \right\}$$

for large $|t|$ and $|\arg t| < \pi - \epsilon$, where $\epsilon > 0$, $c_0 = 1$ and $a = (2\pi)^{\frac{1}{2}-\frac{1}{2}\kappa}\kappa^{-\frac{1}{2}-\vartheta}$. It then follows that, in a suitable sector of the x-plane enclosing the positive half of the real axis, we have

$$G(x) = aX^{\vartheta} e^X \left\{ \sum_{m=0}^{M} c_m X^{-m} + O(X^{-M-1}) \right\}, \quad \text{.............................(3)}$$

where $X = \kappa x^{1/\kappa}$.

Recently Riney [4, 5] has found two linear recurrence relations satisfied by the c_m. In the one, c_m is given in terms of all the preceding c_n; in the other, in terms of the preceding $q+1$ terms of the sequence. His method in each case depends on fairly elaborate manipulations of $g(t)$. My purpose here is to point out that $G(x)$ and its asymptotic expansion alike satisfy a simple differential equation and that from this a finite recurrence formula for the c_m can be deduced fairly simply.

We write $\theta = x\,d/dx$ and

$$P(t) = \prod_{r=1}^{p}(t+\beta_r), \quad Q(t) = \prod_{r=0}^{q}(t+\gamma_r)$$

and note that

$$Q(t)\,g(t+1) = P(t)\,g(t).$$

Again

$$\begin{aligned} xP(\theta)G(n) &= \sum_{n=0}^{\infty} P(n)g(n)x^{n+1} \\ &= \sum_{n=0}^{\infty} Q(n)g(n+1)x^{n+1} \\ &= Q(\theta-1)\sum_{n=0}^{\infty} g(n+1)x^{n+1} \\ &= Q(\theta-1)\{G(x)-g(0)\}. \end{aligned}$$

Hence, if $R(t) = Q(t-1) - xP(t)$, we have

$$R(\theta)\,G(x) = Q(-1)\,g(0). \qquad \text{(4)}$$

This is the linear differential equation of the $(q+1)$-th order satisfied by $G(x)$.

We need not appeal to the general theory of asymptotic solutions of differential equations to see that (4) is satisfied asymptotically for (say) large positive X, if the right-hand side of (3) is substituted for $G(x)$. For $\theta G(x)$ is a function of the same form as $G(x)$ and so has a similar asymptotic expansion.

Let us write $\phi = X(d/dX) = \kappa\theta$,

$$T(t) = \prod_{r=0}^{q}(t-\kappa+\kappa\gamma_r+\vartheta), \quad U(t) = \prod_{r=1}^{p}(t+\kappa\beta_r+\vartheta)$$

and

$$S(t) = T(t-\vartheta) - X^{q+1-p}U(t-\vartheta) = \kappa^{q+1}R(t/\kappa). \qquad \text{(5)}$$

By (4) and (5),

$$S(\phi)e^X \sum_{m=0}^{M} c_m X^{\vartheta-m} = O(X^{\vartheta+q-M}e^X), \qquad \text{(6)}$$

for any positive M. Now

$$\phi\, X^j e^X = X^j(\phi+j)e^X$$

and so

$$S(\phi)e^X X^{\vartheta-m} = X^{\vartheta-m}S(\phi+\vartheta-m)e^X. \qquad \text{(7)}$$

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Since $T(t)$ is a polynomial in t of degree $q+1$, we have

$$T(t-m) = \sum_{s=0}^{q+1} T_s(-m)\, t(t-1)\dots(t-s+1),$$

where

$$T_s(-m) = \sum_{r=0}^{s} \frac{(-1)^{s-r} T(r-m)}{r!\,(s-r)!} = \frac{\Delta^s T(-m)}{s!}$$

in the usual notation of the difference calculus. Hence

$$T(\phi - m) = \sum_{s=0}^{q+1} T_s(-m) X^s (d/dX)^s$$

and

$$e^{-X} T(\phi - m) e^X = \sum_{s=0}^{q+1} T_s(-m) X^s.$$

Similarly

$$e^{-X} U(\phi - m) e^X = \sum_{s=0}^{p} U_s(-m) X^s.$$

Hence

$$e^{-X} S(\phi + \vartheta - m) e^X = X^{q+1} \left\{ \sum_{s=0}^{q+1} T_{q+1-s}(-m) X^{-s} - \sum_{s=0}^{p} U_{p-s}(-m)^{-} X^s \right\}$$

and so, by (6) and (7),

$$\sum_{s=0}^{q+1} T_{q+1-s}(s-m) c_{m-s} - \sum_{s=0}^{p} U_{p-s}(s-m) c_{m-s} = 0, \qquad \text{.......................(8)}$$

where $c_n = 0$, when $n < 0$.

We shall see later that

$$T_{q+1}(-m) - U_p(-m) = 0, \qquad \text{.......................................(9)}$$

$$T_q(-m) - U_{p-1}(-m) = -\kappa m. \qquad \text{.................................(10)}$$

Hence, if we replace m by $m+1$ and s by $s+1$ in (8), we have

$$\kappa m c_m = \sum_{s=1}^{q} T_{q-s}(s-m) c_{m-s} - \sum_{s=1}^{p-1} U_{p-s-1}(s-m) c_{m-s}, \qquad \text{....................(11)}$$

where the second sum is empty if $p = 0$ or 1. This is the recurrence relation required.

If the largest m for which we wish to calculate c_m is of about the size of q, the coefficients in (11) can be most easily calculated by evaluating $T(t)$ for $t = q-1, q-2, \dots, -m$ and then differencing these values up to $(q-1)$ times. If m is large compared with q, we remark that

$$T_{q-s}(s-m) = \sum_{r=0}^{s+1} (-1)^{s+1-r} T_{q+1-r}(0)\,(q+1-r)!\,(m-r)! \,/\, \{(m-s-1)!\,(q-s)!\,(s+1-r)!\},$$

so that we need only calculate $T_s(0)$ (by differencing) for $s = 0, \dots, q+1$. Similarly

$$U_{p-s-1}(s-m) = \sum_{r=0}^{s+1} (-1)^{s+1-r} U_{p-r}(0)(p-r)!\,(m-r)! \,/\, \{(m-s-1)!\,(p-s-1)!\,(s+1-r)!\}.$$

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If the largest m is small compared with q, these methods are not very efficient. In this case, let

$$T(t) = \sum_{r=0}^{q+1} A_r t^{q+1-r}, \quad U(t) = \sum_{r=0}^{p} B_r t^{p-r},$$

so that $A_0 = B_0 = 1$ and

$$A_1 = (\vartheta - \kappa)(q+1) + \kappa \sum_{r=0}^{q} \gamma_r, \quad B_1 = p\vartheta + \kappa \sum_{r=1}^{p} \beta_r. \qquad \text{(12)}$$

We have

$$T(t-m) = \sum_{r=0}^{q+1} A_r \sum_{l=0}^{q+1-r} (-m)^l \binom{q+1-r}{l} t^{q+1-r-l}. \qquad \text{(13)}$$

With the notation of Jordan [3], let us write $\mathfrak{S}_n^s$ for the Stirling number of the second kind, so that

$$\mathfrak{S}_n^s = [\Delta^s t^n / s!]_{t=0}, \quad \mathfrak{S}_n^n = 1, \quad \mathfrak{S}_n^{n-1} = \tfrac{1}{2}n(n-1). \qquad \text{(14)}$$

By (13), we have

$$T_{q+1-s}(-m) = \sum_{r=0}^{s} A_r \sum_{l=0}^{s-r} (-m)^l \binom{q+1-r}{l} \mathfrak{S}_{q+1-r-l}^{q+1-s}$$

and similarly

$$U_{p-s}(-m) = \sum_{r=0}^{s} B_r \sum_{l=0}^{s-r} (-m)^l \binom{p-r}{l} \mathfrak{S}_{p-r-l}^{p-s}.$$

If m is small, these formulae provide a convenient method of calculating the coefficients in (11) for $s \leqslant m$; no others are required. In particular, we can verify (9) and (10) very easily, using (12) and (14).

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