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Properties of the Invariants of Solvable Lie Algebras

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Abstract. We generalize to a field of characteristic zero certain properties of the invariant functions of the coadjoint representation of solvable Lie algebras with abelian nilradicals, previously obtained over the base field $\mathbb{C}$ of complex numbers. In particular we determine their number and the restricted type of variables on which they depend. We also determine an upper bound on the maximal number of functionally independent invariants for certain families of solvable Lie algebras with arbitrary nilradicals.

1 Introduction

The invariants of the coadjoint representation of a solvable Lie algebra are still not completely determined, although these invariants are well known for semisimple Lie algebras. In particular, for semisimple Lie algebras, they can all be chosen to be homogeneous polynomial and their number is equal to the dimension of the Cartan subalgebra [14], [15], [6]. Neither the number nor the type of functions in terms of which the invariants can be expressed is known in the case of a general solvable Lie algebra. One of the difficulties in this determination is due to the fact that contrary to the semisimple case, no general classification method is available for solvable Lie algebras. The problem of the determination of these invariant functions has been treated for low dimension Lie algebras in [11] and for certain families of solvable and other Lie algebras in [1], [8], [12], [13].

In this paper we generalize to a solvable Lie algebra L over a field $\mathbb{K}$ of characteristic zero some results obtained in [10], [8] over $\mathbb{C}$ about the number of invariant functions of the coadjoint representation and about the type of functions in terms of which they can be expressed. We determine an upper bound for the number $\mathcal{N}$ of elements in any fundamental system of invariant functions for a large family of solvable Lie algebras including solvable Lie algebras with an abelian nilradical. Some consequences of these results are discussed. We also analyze the algebraic structure of L and derive a structure theorem useful for the classification of solvable Lie algebras and particularly for the study of the invariant functions in which we are interested.

2 Algebraic structure

Suppose that the solvable Lie algebra L of finite dimension n over the field $\mathbb{K}$ of characteristic zero admits a vector space decomposition of the form

$$L = M \dotplus E \tag{2.1}$$

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where M is the nilradical of L which we suppose to be of dimension r and E is a complement of M in L. Consider a basis of L of the form

$$\mathcal{B}_L = \{N_1, \dots, N_r; X_1, \dots X_k\} \tag{2.2}$$

where $\{N_1, \dots, N_r\}$ is a basis of M and $\{X_1, \dots, X_k\}$ is a basis of E with $k = \dim E$. For each basis element X_u of E, let $\mathrm{ad}_M X_u$ denote the restriction to the nilradical M of the adjoint representation $\mathrm{ad}\, X_u$ of L defined by the element X_u. The following Lemma is proven in [8].

Lemma 1

(a) The set of operators $\mathrm{ad}_M X_u$ $(u = 1, \dots, k)$ *is linearly nil-independent.*
(b) If the nilradical is abelian, then the linear operators $\mathrm{ad}_M X_u$ $(u = 1, \dots, k)$ *are pairwise commutative.*

The basis elements $X_1, \dots, X_k$ of E are also said to be linearly nil-independent, since no non-trivial linear combination of them can be ad-nilpotent. Let A^u denotes the matrix of the linear operator $\mathrm{ad}_M X_u$ for $u = 1, \dots, k$. We are now interested in the effect of a change of basis in L on the system of matrices $(A^1, \dots, A^k)$. It is well known (see for example [17]), that the effect of the more general change of basis in L on the system of k matrices $(A^1, \dots, A^k)$ when the nilradical is abelian is given by the transformation

$$(A^1, \dots, A^k) \mapsto \Big(\sum_{j=1}^{k} \beta_{1j}(T^{-1}A^jT), \dots, \sum_{j=1}^{k} \beta_{kj}(T^{-1}A^jT)\Big) \tag{2.3}$$

where T is the matrix of an invertible linear operator on M and $B = (\beta_{uj})_{u,j=1}^{k}$ is an element of $\mathrm{GL}(k, \mathbb{K})$, the group of all invertible matrices of order k over $\mathbb{K}$. Furthermore, it is easy to see that the transformation (2.3) defined on the set of k matrices $(A^1, \dots, A^k)$ of the algebra $\mathrm{gl}(r, \mathbb{K})$ of all square matrices of order r over $\mathbb{K}$ determines an equivalence relation in $[\mathrm{gl}(r, \mathbb{K})]^k$ which we denote by $\sim$. This makes meaningful the classification of such systems of matrices. In particular, the classification of solvable Lie algebras with a given abelian nilradical is reduced to the determination of all classes with respect to $\sim$ in $[\mathrm{gl}(r, \mathbb{K})]^k$ and the determination of the corresponding commutation relations of the type $[E, E]$.

In order to deal with the condition of linear nil-independence of the system of matrices $(A^1, \dots, A^k)$ we shall need the following lemma proved in [8].

Lemma 2 *Let*

$$B^u = B^u_s + B^u_n, \quad (u = 1, \dots, m)$$

denote the Jordan-Chevalley decomposition of a system $(B^1, \dots, B^m)$ *of matrices of* $\mathrm{gl}(r, \mathbb{K})$, *where* B^u_s *and* B^u_n *represent the semisimple part and the nilpotent part of* B^u *respectively. Then* $B^1, \dots, B^m$ *are linearly nil-independent if and only if the semisimple parts* B^u_s $(u = 1, \dots, m)$ *are linearly independent.*

For the proof of our structure theorem, we shall need the following result demonstrated in [16]. We denote by $D = [D_1, \dots, D_m]$ a block-diagonal matrix with diagonal blocks $D_1, \dots, D_m$.

Proposition 1 *Let $\mathbb{P}$ be an arbitrary field and Σ a set of pairwise commutative operators of* $\mathrm{gl}(r, \mathbb{P})$*. Then the vector space $\mathbb{P}^r$ of dimension r over $\mathbb{P}$ can be represented as a direct sum of subspaces Q_j, $(j = 1, \ldots, s)$, invariant with respect to Σ and such that for all $f \in \Sigma$, the minimal polynomial of the restriction $f_{|Q_j}$ is a power of an irreducible polynomial over $\mathbb{P}$.*

Theorem 1 *Let $\Sigma = (B^1, \ldots, B^m)$ be a system of linearly nil-independent and pairwise commutative operators in $\mathrm{gl}(r, \mathbb{C})$ with $m \leq r$. Suppose furthermore that Σ represents an equivalence class with respect to $\sim$ in $[\mathrm{gl}(r, \mathbb{C})]^m$. Then all the matrices B^u can simultaneously be decomposed into quasi-diagonal matrices of the form*

$$B^u = [B_{u1}, \ldots, B_{us}], \quad (u = 1, \ldots, m) \tag{2.4}$$

where

(a) The number s of blocks satisfies $s \geq m$.
(b) Each block B_{ut} is triangular of order n_t independent of u and for $u = 1, \ldots, m$ and $t \leq m$ the semisimple part S_{ut} of B_{ut} is $\delta_u^t I_{n_t}$, where I_{n_t} is the unit matrix of order n_t.

Proof Every irreducible polynomial over $\mathbb{C}$ being linear, we know from the preceding proposition that one can simultaneously decompose the B^u's into the quasi-diagonal form

$$B^u = [B_{u1}, \ldots, B_{us}], \quad (u = 1, \ldots, m)$$

where B_{ut} is a square matrix of order n_t independent of u (for $t = 1, \ldots, s$) with $\sum_{t=1}^{s} n_t = r$, and has a semisimple part of the form $S_{ut} = \lambda_{ut} I_{n_t}$, $\lambda ut \in \mathbb{C}$. Again by the argument of commutativity, all the matrices B_{ut} can be chosen triangular. Using the argument of linear nil-independence and the equivalence relation defined by (2.3) on the system $(B^u)_{u=1}^m$, we show by induction on m that $s \geq m$ and that we can choose the diagonal elements λ_{ut} in such a way that they satisfy $\lambda_{ut} = \delta_u^t$ $(t \leq m)$. If $m = 1$, it is trivial that $s \geq m$. Now, B^1 being triangular and non-nilpotent, we may assume that $\lambda_{11} \neq 0$, which amounts to supposing that $\lambda_{11} = 1$ (by using (2.3)). This proves the theorem for $m = 1$. If this result is true for $m - 1 \geq 1$, the system of matrices $(B^u)_{u=1}^m$ can be simultaneously decomposed according to Proposition 1 into a quasi-diagonal form of the type

$$B^u = [B_{u1}, \ldots, B_{us}], \quad (u = 1, \ldots, m),$$

where B_{ut} is triangular with a semisimple part $S_{ut} = \lambda_{ut} I_{n_t}$ $(t = 1, \ldots, s)$, $\lambda_{ut} \in \mathbb{C}$. By the induction hypothesis on the system $(B^1, \ldots, B^{m-1})$ we obtain that $s \geq m - 1$ and $S_{ut} = \delta_u^t I_{n_t}$ (for $u = 1, \ldots, m - 1$; $t \leq m - 1$). By the equivalence relation $\sim$, we can put B^m into a form where $S_{mt} = 0$ for $t = 1, \ldots, m - 1$. If $s = m - 1$, the semisimple part of B^m would be zero, which is a contradiction to Lemma 2. Thus $s > m - 1$ and since the semisimple part of B^m is not zero and $\lambda_{mt} = 0$ for $t = 1, \ldots, m - 1$, we can suppose that $\lambda_{mm} = 1$. With this last equality we can reduce λ_{um} to zero for $u = 1, \ldots, m-1$. We finally obtain $\lambda_{ut} = \delta_u^t$ for $u = 1, \ldots, m$ and $t \leq m$. This completes our proof. ∎

We suppose for the moment that the nilradical M of L is abelian. Therefore by Lemma 1, the matrices A^u of the linear operators $\mathrm{ad}_M X_u$, $u = 1, \ldots, k$ obviously yield the decomposition (2.4) of Theorem 1. This theorem deeply determines the structure of L when the

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nilradical is abelian. It can be applied to the classification of solvable Lie algebras since for a given nilradical this problem amounts to classifying the system $(A^1, \ldots, A^k)$ with respect to $\sim$ and to determine the commutation relations of the $[E, E]$ type. The fact that solvable Lie algebras are classified so far up to just the dimension 7 (see [17], [2]) give more importance to this theorem since it can be applied to speed up this classification at least up to the dimension 10, and particularly when n is large but the quantity $r - k$ is relatively small. In this respect, Theorem 1 is used to prove in [7] that any solvable Lie algebra L over $\mathbb{C}$ with an abelian nilradical and for which $r = k$, *i.e.*, $n = 2r$, is a direct sum of subalgebras

$$L = \bigoplus_r \mathcal{A}_2,$$

where $\mathcal{A}_2$ is the solvable Lie algebra of dimension 2 with commutation relations $[X, N] = N$.

3 Characterization of the Invariants of the Coadjoint Representation

Let G be the connected Lie group generated by the Lie algebra L.

Definition 1 A function $F \in C^\infty(L^*)$, the set of all differentiable functions on the dual space L^* of the Lie algebra L is said to be an invariant of the coadjoint representation of L if

$$F(\mathrm{Ad}_g^* \cdot f) = F(f), \quad \forall g \in G, f \in L^*,$$

where $\mathrm{Ad}^* \colon G \mapsto \mathrm{GL}(L^*) \colon (\mathrm{Ad}_g^* f)(x) = f(\mathrm{Ad}_{g^{-1}} x)$ ($\forall g \in G$, $f \in L^*$, $x \in L$), is the coadjoint representation of the Lie group G.

Several methods exist for the determination of these invariants. M. Perroud uses in [12] a method called the "method of orbits" which is suitable for polynomial invariants. The method that we shall use here is more general and standard, and consists in obtaining the invariants as the solutions of a system of partial differential equations. Indeed, let $\{e_i\}$ be a basis of L and (x_i) a coordinate system associated with the dual basis $\{\varepsilon_i\}$ of $\{e_i\}$ in L^*. Then the invariants of L are the solutions of the system of partial differential equations

$$\tilde{e}_i \cdot F = 0, \quad i = 1, \ldots, n \quad (n = \dim L) \tag{3.1}$$

where $\tilde{e}_i$ is the infinitesimal generator of Ad^* associated with the basis element e_i of L and has the form

$$\tilde{e}_i = -\sum_{j,k}^{n} c_{ij}^k x_k \frac{\partial}{\partial_{x_j}} \quad i = 1, \ldots, n, \tag{3.2}$$

where c_{ij}^k, $i, j, k = 1, \ldots, n$ are the structure constants of L defining the commutation relations

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k. \tag{3.3}$$

More details on the proof of (3.1) can be found in [8], [4].

Denote by $S(L^*)$ and $S(L)$ the symmetric algebras of L^* and L respectively. They are both isomorphic to the ring $\mathbb{K}[X_1, \dots, X_n]$ of polynomials in n indeterminates over $\mathbb{K}$, and thus $S(L^*) \simeq S(L)$. As a consequence, a polynomial invariant can be viewed as a function on L, and by extension invariants of the coadjoint representation are all seen as functions on L and called generalized Casimir invariants. This means that in the expression (3.2) of the infinitesimal generators we shall consider $(x_1, \dots, x_n)$ as a coordinate system on L. If $(y_1, \dots, y_n)$ is any coordinate system associated with the basis $\{e_i\}$ of L, then by abuse of notation we shall write

$$[y_i, y_j] = \sum_{p=1}^{n} c_{ij}^{p} y_p. \tag{3.4}$$

We impose the following restriction on L: L is not nilpotent (*i.e.*, $k \geq 1$). We recall that for now the nilradical M is abelian, and the base field $\mathbb{K}$ is of characteristic zero. Since M is abelian, by using the notation (3.4), the infinitesimal generator determined by a basis element N_i of M has the form

$$\tilde{N}_i = -\sum_{j=1}^{k} [n_i, x_j] \partial_{x_j}, \quad i = 1, \dots, r$$

where

$$\mathcal{S} = (n_1, \dots, n_r; x_1, \dots, x_k) \tag{3.5}$$

is a coordinate system associated with the basis of L given by (2.2). If we set $n_{ij} = [x_j, n_i]$, then we obtain the system of equations

$$\tilde{N}_i = \sum_{j=1}^{k} n_{ij} \partial_{x_j}, \quad i = 1, \dots, r. \tag{3.6}$$

Equation (3.6) can be seen as an equality of the form

$$AX = B \tag{3.7}$$

where $A = (n_{ij})_{\substack{i=1,\dots,r\\ j=1,\dots,k}}$, $X = (\partial_{x_1}, \dots, \partial_{x_k})^T$, and $B = (\tilde{N}_1, \dots, \tilde{N}_r)^T$. Furthermore the entries n_{ij} can be considered as linear polynomials in the r variables $n_1, \dots, n_r$, *i.e.*, as elements of the field $\mathcal{R} = \mathbb{K}(n_1, \dots, n_r)$ of all rational functions in r variables over $\mathbb{K}$.

Proposition 2 *The rank of the matrix $A = (n_{ij})_{\substack{i=1,\dots,r\\ j=1,\dots,k}}$ of equation (3.7) where the entries n_{ij} are polynomial in the r variables $n_1, \dots, n_r$ is independent of any change of basis in L.*

Proof The matrix A is completely determined in a given basis of L by the system of matrices $(A^1, \dots, A^k)$ and the effect of a general change of basis in L on the system is given by (2.3). In equation (2.3), the matrix $B = (\beta_{uj})_{u,j=1}^k$ representing a change of basis in E has no effect on the rank of A since changing the basis in E is just equivalent to performing column $\mathbb{K}$-operations on A. Consequently, we just need to look at the effect of a change of basis of the type $(A^1, \dots, A^k) \mapsto (T^{-1}A^1T, \dots, T^{-1}A^kT)$ on the rank of A, where T represents the matrix of a change of basis in M. Note that if $A^j = (a_{ui}^j)_{u,i=1,\dots,r}$, then $n_{ij} = [x_j, n_i] = \sum_{u=1}^r a_{ui}^j n_u$. Now let $B = (w_{ij})_{\substack{i=1,\dots,r\\ j=1,\dots,k}}$ be the expression of A in the new basis $\{W_1, \dots, W_r\}$ of M. If the new coordinates in this basis are $(w_1, \dots, w_r)$, then we have $w_{ij} = [x_j, w_i] = \sum_{u=1}^r a_{ui}^j w_u$. Comparing this expression with that of n_{ij}, we see that the effect of the given change of basis just amounts to changing coordinates and thus does not modify the rank of the matrix. This completes our proof. ■

Theorem 2 *Suppose that the nilradical M of L is abelian and that the base field $\mathbb{K}$ has characteristic zero. Then the columns of A are $\mathcal{R}$-linearly independent; that is, A is of maximal rank.*

Proof It suffices to prove the result for the case of an algebraically closed field. For $\mathbb{K}$ is a subfield of an algebraically closed extension field $\mathbb{K}_c$, for which the corresponding field of rational functions $\mathcal{R}$ is denoted $\mathcal{R}_c$. If the columns of $A = (C_1, \dots, C_k)$ are $\mathcal{R}_c$-linearly independent, then the inclusion $\mathcal{R} \subset \mathcal{R}_c$ implies that these columns are also $\mathcal{R}$-linearly independent. According to Theorem 1, when the base field is algebraically closed, the matrices A^u of the linear operators $\operatorname{ad}_M X_u$ can simultaneously be put into the quasi-diagonal form

$$A^u = [A_{u1}, \dots, A_{us}], \quad (u = 1, \dots, k \text{ and } s \geq k)$$

where each diagonal block A_{ut}, $1 \leq t \leq k$ is an upper triangular matrix such that all the elements on its diagonal are equal to δ_u^t. It follows that we can find k basis elements N_u^0 in M such that $[X_j, N_u^0] = \delta_u^j N_u^0$ $(u, j = 1, \dots, k)$. This means that in terms of coordinates we have $[x_j, n_u^0] = \delta_u^j n_u^0$, where n_u^0 is the coordinate associated with the basis vector N_u^0. By rearranging the elements in the basis of M in such a way that the first k elements are precisely $N_1^0, \dots, N_k^0$ in this order, we obtain the matrix A in the form

$$A = \begin{pmatrix} n_1^0 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & n_1^0 \\ \hline & X & \end{pmatrix},$$

where X is a $(r-k) \times k$ matrix over $\mathcal{R}$. This proves that A is of maximal rank and since this rank is basis-independent according to Proposition 2, our proof is complete. ■

We now extend the second part of Theorem 1 of [10] to a field of characteristic zero.

Theorem 3 *Suppose that the nilradical M is abelian and that the base field $\mathbb{K}$ has characteristic zero. Then*

$$\partial_{x_i} \cdot F = 0, \quad i = 1, \dots, k$$

for any invariant F of L.

Proof Since for every invariant F of L and for every infinitesimal generator $\tilde{N}_i$ of the coadjoint representation we have $\tilde{N}_i \cdot F = 0$, according to equation (3.6) it suffices to show that the matrix A of (3.7) is of maximal rank. But this is true according to Theorem 2. ■

Definition 2 A fundamental set of invariants is a set consisting of a maximal number of functionally independent invariants.

As a consequence of Theorem 3, we obtain the following result.

Corollary 1 *Suppose that the nilradical M is abelian and that the base field $\mathbb{K}$ has characteristic zero. Then the invariants of L are completely determined by the operators $\operatorname{ad}_M X_u$, $(u = 1, \dots, k)$. In particular, this determination does not depend on the commutation relations $[X_i, X_j]$ $(i, j = 1, \dots, k)$.*

The proof of this corollary is similar to that given in [8].

Remark As a consequence of this corollary, any two solvable Lie algebras with abelian nilradicals which are not necessarily isomorphic but for which the systems of matrices $(A^u)_{u=1}^k$ are equivalent with respect to $\sim$ have the same fundamental system of invariants.

Note that according to Theorem 3, the infinitesimal generator $\tilde{N}_i$ associated with the basis element N_i of L vanishes for $i = 1, \dots, r$ and the remaining infinitesimal generators determined by the basis elements X_j of E have the form

$$\tilde{X}_j = -\sum_{i=1}^{r} n_{ij} \partial_{n_i}, \quad j = 1, \dots, k. \tag{3.8}$$

In particular, the original system of determining equations (3.1) is reduced from n to k equations, that is at least by half since we always have as proven in [8] that $r \geq n/2$.

4 The Case of an Arbitrary Nilradical

Our aim in this section is to generalize Theorem 2 and certain results of [8], namely Theorem 4 of that paper to solvable Lie algebras with arbitrary nilradical and without restrictions on the dimensions r and k of M and E respectively. However, we require that the matrices A^u of the linear operators $\operatorname{ad}_M X_u$, $(u = 1, \dots, k)$ be simultaneously triangularizable in a basis of L. Although, the nilradical is not abelian, the transformation (2.3) will still determine the effect of a particular change of basis in L on the matrices (A^u).

Let

$$\Gamma = (C_1, \dots, C_k) \tag{4.1}$$

be a square matrix of order k over an arbitrary field $\mathbb{P}$ and whose columns are denoted by $C_1, \dots, C_k$. Suppose that each of these columns is the sum of r other column vectors C^u_j, *i.e.*,

$$C_j = \sum_{u=1}^{r} C^u_j, \quad (j = 1, \dots, k; r \geq k), \tag{4.2}$$

where each C^u_j has the form

$$C^u_j = (C^u_{ij})^T_{i=1,\dots,k}, \quad C^u_{ij} \in \mathbb{P} \quad (j = 1, \dots, k; u = 1, \dots, r). \tag{4.3}$$

Denote by Δ the Cartesian product $\{1, 2, \dots, r\} \times \cdots \times \{1, 2, \dots, r\}$ (k times) and let $D_{u_1\cdots u_k}$ represent the determinant of the matrix $(C^{u_1}_1, \dots, C^{u_k}_k)$, where $(u_1, \dots, u_k) \in \Delta$.

Lemma 3 *The determinant of* Γ *is given by the formula*

$$\det \Gamma = \sum_{(u_1,\dots,u_k)\in\Delta} D_{u_1\cdots u_k}.$$

Proof Assume that Γ has the general form $\Gamma = (\Gamma_{ij})_{i,j=1,\dots,k}$. Then $\Gamma_{ij} = \sum_{u=1}^{r} C^u_{ij}$, and if S_k represents the group of permutations of the set of k elements $\{1, \dots, k\}$ then

$$\begin{aligned}
\det \Gamma &= \sum_{\sigma\in S_k} \Gamma_{\sigma(1)1} \cdots \Gamma_{\sigma(k)k} \\
&= \sum_{\sigma\in S_k} \Big(\sum_{u=1}^{r} C^u_{\sigma(1)1}\Big) \cdots \Big(\sum_{u=1}^{r} C^u_{\sigma(k)k}\Big) \\
&= \sum_{\sigma\in S_k} \sum_{(u_1,\dots,u_k)\in\Delta} C^{u_1}_{\sigma(1)1} \cdots C^{u_k}_{\sigma(k)k} \\
&= \sum_{(u_1,\dots,u_k)\in\Delta} \sum_{\sigma\in S_k} C^{u_1}_{\sigma(1)1} \cdots C^{u_k}_{\sigma(k)k}
\end{aligned}$$

and the result follows since $\sum_{\sigma\in S_k} C^{u_1}_{\sigma(1)1} \cdots C^{u_k}_{\sigma(k)k}$ is just $D_{u_1\cdots u_k}$. ■

Suppose that each matrix A^u corresponding to the linear operator $\operatorname{ad}_M X_u$ has the form

$$A^u = (a^u_{ij})_{i,j=1,\dots,r} \quad u = 1, \dots, k \tag{4.4}$$

and that these matrices are all (lower) triangular. Let $(n_1, \dots, n_r)$ represent the coordinate system associated with a fixed basis of the nilradical given by equation (2.2). By means of the k matrices A^u and the r independent variables $n_1, \dots, n_r$, construct the matrix $A = (n_{ij})_{\substack{i=1,\dots,r \\ j=1,\dots,k}}$ of equation (3.7) with polynomial entries $n_{ij} = \sum_{\nu\geq i}^{r} a^j_{\nu i} n_\nu$. The u-th column $C(u, A)$ of A has the form

$$C(u, A) = \begin{pmatrix} a^u_{11}n_1 + a^u_{21}n_2 + \cdots + a^u_{r-1,1}n_{r-1} + a^u_{r1}n_r \\ a^u_{22}n_2 + a^u_{32}n_3 + \cdots + a^u_{r2}n_r \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a^u_{rr}n_r \end{pmatrix}. \tag{4.5}$$

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The submatrix B of A formed by the first k lines of A is clearly a matrix of the form Γ defined by the equations (4.1)–(4.3). That is, it has the form $B = (C_1, \ldots, C_k)$ and according to (4.5) the corresponding coefficients C^u_{ij} (for $u, j = 1, \ldots, k$ and $i = 1, \ldots, r$) of equation (4.3) are given by

$$C^u_{ij} = \alpha^u_{ij} n_{i+u-1}, \quad i + u - 1 \le r \tag{4.6}$$

where $\alpha^u_{ij} = a^j_{i+u-1,i}$ with $a^j_{pi} = 0$ if $p > r$, or equivalently, $\alpha^u_{ij} = 0$ if $u + i > r + 1$.

Lemma 4 *Let $B = (C_1, \ldots, C_k)$ be a matrix of the form (4.1)–(4.3) for which the entries are polynomial functions in the r variables $n_1, \ldots, n_r$ and such that the corresponding coefficients C^u_{ij} verify as in (4.6) the condition*

$$C^u_{ij} = \alpha^u_{ij} \cdot n_{i+u-1}, \quad i, j = 1, \ldots, k; u = 1, \ldots, r$$

where $\alpha^u_{ij} \in \mathbb{P}$ and $n_{i+u-1} = 0$ if $i + u - 1 > r$. Let $d = \det \alpha^1$, where $\alpha^1 = (\alpha^1_{ij})_{i,j=1,\ldots,k}$. Then

(a) $D_{1\cdots 1} = d \cdot n_1 n_2 \cdots n_k$.
(b) The quantity $\det B - D_{1\cdots 1}$ *is independent of $n_1 n_2 \cdots n_k$. That is, the monomial in the variable $n_1 n_2 \cdots n_k$ in* $\det B$ *viewed as a polynomial function is precisely $D_{1\cdots 1}$.*

Proof By definition and according to the notation used in (4.1)–(4.3), $D_{1\cdots 1}$ is the determinant of the matrix $(C^u_{ij})_{i,j=1,\ldots,k}$ corresponding to $u = 1$. The equality $C^1_{ij} = \alpha^1_{ij} n_i$ readily shows that $D_{1\cdots 1} = d \cdot n_1 n_2 \cdots n_k$. All that remains to prove is that in any determinant $D_{u_1 \cdots u_k}$, the term $n_1 n_2 \cdots n_k$ is a factor of a term of the form $C^{u_1}_{\sigma(1)1} \cdots C^{u_j}_{\sigma(j)j} \cdots C^{u_k}_{\sigma(k)k} = T(\sigma; u_1, \ldots, u_k)$ if and only if $u_j = 1$ (for all $j = 1, \ldots, k$). Now, since each entry $C^{u_j}_{\sigma(j)j}$ has the form

$$C^{u_j}_{\sigma(j)j} = \alpha^{u_j}_{\sigma(j)j} \cdot n_{\sigma(j)+u_j-1}, \quad (j = 1, \ldots, k; \sigma \in S_k),$$

we see that $T(\sigma; u_1, \ldots, u_k)$ depends on the variable $n_1 \cdots n_k$ if and only if the mapping $\tau \colon \{1, \ldots, k\} \longrightarrow \{1, \ldots, r + k - 1\} \colon j \mapsto \tau(j) = \sigma(j) + u_j - 1$ is an element of S_k. Since for all $\sigma \in S_k$ we have $\sum_{j=1}^k \sigma(j) = k(k+1)/2$, we see that $\sum_{j=1}^k \tau(j) = k(k+1)/2 + \sum_{j=1}^k (u_j - 1)$, and this shows that $\tau \in S_k$ if and only if $u_j = 1$ for all $j = 1, \ldots, k$. This completes our proof. ∎

Recall that the condition of the abelian nilradical on L is now dropped and replaced by the requirement that the matrices A^u are simultaneously triangularizable in the same basis of L.

Theorem 4 *Let $A^j = (a^j_{ui})_{u,i=1,\ldots,r}$ represent the matrix of the linear operators $\mathrm{ad}_M X_j$, $j = 1, \ldots, k$, and suppose that these matrices are simultaneously triangularizable in a given basis of L. Let $A = (n_{ij})$ be the matrix of the form (3.7) associated with these matrices, with*

$$n_{ij} = \sum_{i \le v \le r} a^j_{vi} n_v.$$

Then A has the maximal rank k.

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Proof By the remark made in the proof of Theorem 2, we may assume that the base field $\mathbb{K}$ is algebraically closed. The matrices A^j, $(j = 1, \dots, k)$ being linearly nil-independent, it follows from Lemma 2 that their semisimple parts represented by their diagonals $\operatorname{diag}(a_{11}^j, \dots, a_{rr}^j)$ are linearly independent. Thus, even if it may mean to perform some column operations on the matrix $F = (a_{ii}^j)_{\substack{i=1,\dots,r\\ j=1,\dots,k}}$, which merely corresponds to a change of basis in E (that conserves the triangular form of the matrices A^j), we may assume that the triangular matrices A^j are such that the square submatrix $G = (a_{ii}^j)_{i,j=1,\dots,k}$ formed by the first k lines of the matrix F is non-singular. The square matrix W with polynomial entries formed by the first k lines of the matrix A is then a matrix of the form (4.1)–(4.3) with the coefficients C_{ij}^u as in equation (4.6). According to Lemma 4, the monomial in $n_1 n_2 \cdots n_k$ in the determinant of W is $d\, n_1 n_2 \cdots n_k$, where d is the determinant of the matrix $(\alpha_{ij}^1)_{i,j=1,\dots,k}$ with the notation of (4.6). But this last matrix is just the matrix G. This means that $d \neq 0$ and W is non-singular. Thus A is of rank k, which is maximal. ■

Theorem 5 *The system of partial differential equations*

$$\sum_{j=1}^{q} f_{ij} \frac{\partial F}{\partial_{y_j}} = 0 \quad (i = 1, \dots, p)$$

where $f_{ij} = f_{ij}(y_1, \dots, y_q)$ is a function on $\mathbb{K}^q$ has exactly $p - r(S)$ functionally independent solutions, where $r(S)$ is the rank of the matrix $S = (f_{ij})_{\substack{i=1,\dots,p\\ j=1,\dots,q}}$.

The proof of this theorem can be found in [3]. Let $\tilde{N}_i$ and $\tilde{X}_j$ represent as usual the infinitesimal generators respectively associated with the basis elements N_i and X_j of L (for $i = 1, \dots, r$ and $j = 1, \dots, k$). The invariants of L are determined by the system of partial differential equations

$$\begin{cases} \tilde{N}_i \cdot F = 0, & i = 1, \dots, r \\ \tilde{X}_j \cdot F = 0, & j = 1, \dots, k. \end{cases} \tag{4.7}$$

The matrix S of Theorem 5 associated with this system has the form $S = \left(\begin{smallmatrix} S_1 & S_2 \\ S_3 & S_4 \end{smallmatrix}\right)$ where S_1, S_2, S_3 and S_4 are submatrices defined with the notation of equation (3.4) by $S_1 = ([n_i, n_j])_{i,j=1,\dots,r}$, $S_2 = ([n_i, x_j])_{\substack{i=1,\dots,r\\ j=1,\dots,k}}$, $S_3 = -S_2^T$ and $S_4 = ([x_i, x_j])_{i,j=1,\dots,k}$. In fact, S is just the matrix of the commutation relations of L.

Theorem 6 *Assume that the non-nilpotent solvable Lie algebra L of finite dimension n over the field $\mathbb{K}$ of characteristic zero has a basis of the form* (2.2), *where r is the dimension of its nilradical M.*

(a) If the linear operators $\operatorname{ad}_M X_u$ $(u = 1, \dots, k)$ are simultaneously triangularizable, then $r(S) \geq k$ and the maximal number $\mathcal{N}$ of functionally independent invariants of L satisfies $\mathcal{N} \leq r$.

(b) If the nilradical is abelian, then $\mathcal{N} = 2r - n$.

Proof The submatrix S_2 of S is just the matrix A of Theorem 4 with a change of sign. Since the linear operators $\mathrm{ad}_M X_u$, $(u = 1, \dots, k)$ are simultaneously triangularizable, it follows by Theorem 4 that S_2 is of maximal rank k. Consequently $r(S) \geq k$ and Theorem 5 together with the equality $n = r+k$ leads to $\mathcal{N} \leq r$, and this proves part (a) of the theorem. To prove part (b), we first notice that when the nilradical is abelian the matrix S_2 is of maximal rank k by Theorem 2 and so is the matrix $S_3 = -S_2^T$. By rearranging if necessary, we can put the matrix S into the partitioned form $S = \left(\begin{smallmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & R \end{smallmatrix}\right)$ where $R = \left(\begin{smallmatrix} \mathbf{0} & R_1 \\ R_2 & S_4 \end{smallmatrix}\right)$ is a square matrix of order $2k$ and R_1 and R_2 are non-singular matrices of order k. This shows that $r(S) = 2k$, and the result follows by Theorem 5. ∎

5 Applications

We give in this section a number of examples illustrating the results of Theorem 6. In [11] invariants have been determined for all real Lie algebras of dimension less than or equal to five and a classification of solvable Lie algebras of dimension six having nilradicals of dimension 4 is available in [17]. We show by a couple of examples how the maximal number of functionally independent invariants of these Lie algebras agree with part (a) of Theorem 6. The notation of the generators of a Lie algebra are those given by (2.2) and the corresponding coordinate system is that given by (3.5).

A) Consider the four dimensional non-nilpotent algebra $A_{4,8}$ of [11] with commutation relations $[N_2, N_3] = N_1$, $[N_2, X_1] = N_2$, $[N_3, X_1] = -N_3$. The dimension r of the nilradical is three 3 and the matrix of $\mathrm{ad}_M X_1$ is the triangular matrix $A^1 = \mathrm{diag}\{0, -1, 1\}$. This Lie algebra has two functionally independent invariants $F_1 = n_1$ and $F_2 = n_2 n_3 - n_1 x_1$. Thus $\mathcal{N} \leq r$, as predicted by part (a) of Theorem 6.

B) The Lie algebra $N_{6,31}$ of [17] with commutation relations

$$[X_1, N_2] = N_2, \quad [X_1, N_3] = -N_3, \quad [N_2, N_3] = N_1$$

$$[X_2, N_3] = N_3, \quad [X_2, N_4] = N_1 + N_4, \quad [X_2, N_2] = N_2$$

is a solvable Lie algebra of dimension six with a non-abelian nilradical of dimension 4. Since the linear operators $\mathrm{ad}_M X_1$ and $\mathrm{ad}_M X_2$ have the upper triangular matrices $A^1 = \mathrm{diag}\{0, 1, -1, 0\}$ and $A^2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ & 0 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$ respectively, we are in the conditions of part (a) of Theorem 6. The infinitesimal generators of the coadjoint representation are

$$\tilde{N}_1 = -n_1 \partial x_2, \quad \tilde{N}_2 = n_1 \partial n_3 - n_2 \partial x_1$$

$$\tilde{N}_3 = -n_1 \partial n_2 - n_3 \partial x_2 + n_3 \partial x_1, \quad \tilde{N}_4 = -(n_1 + n_4) \partial x_2$$

$$\tilde{X}_1 = n_2 \partial n_2 - n_3 \partial n_3, \quad \tilde{X}_2 = n_1 \partial n_1 + n_3 \partial n_3 + (n_1 + n_4) \partial n_4$$

solving the corresponding system $\tilde{e}_i \cdot F = 0$, $(i = 1, \dots, 6)$ of equation (3.1) we find the two functionally independent invariants $F_1 = x_1 + (n_2 n_3)/n_1$ and $F_2 = n_4/n_1 - \log n_1$. Consequently, as stipulated by Theorem 6, the condition $\mathcal{N} \leq r$ is satisfied.

C) The algebra $N_{6,33}$ of [17] with commutation relations

$$[X_1, N_1] = N_1, \quad [X_1, N_2] = N_2, \quad [N_2, N_3] = N_1$$

$$[X_2, N_1] = N_1, \quad [X_2, N_3] = N_3 + N_4, \quad [X_2, N_4] = N_4$$

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has dimension six and a non-abelian nilradical of dimension four. It is readily verified that the matrices of the linear operators $\mathrm{ad}_M X_u$ $(u = 1, 2)$ are all triangular. The infinitesimal generators of the solvable Lie algebra $N_{6,33}$ are

$$\tilde{N}_1 = -n_1\partial x_1 - n_1\partial x_2, \quad \tilde{N}_2 = n_1\partial n_3 - n_2\partial x_1$$
$$\tilde{N}_3 = -n_1\partial n_2 - (n_3 + n_4)\partial x_2, \quad \tilde{N}_4 = n_4\partial x_2$$
$$\tilde{X}_1 = n_1\partial n_1 + n_2\partial n_2, \quad \tilde{X}_2 = n_1\partial n_1 + (n_3 + n_4)\partial n_3 + n_4\partial n_4.$$

A simple examination of these vector fields starting with $\tilde{N}_4$ shows that the Lie algebra $N_{6,33}$ has no non-trivial invariant. Thus $0 = \mathcal{N} \leq r$, which is again in accordance with Theorem 6.

It is tempting to believe that for given values of n and r, a Lie algebra has more invariants when its nilradical is abelian. This would mean according to part (b) of Theorem 6, that $\mathcal{N} \leq 2r - n$ in general. But this is not true according to the following counter-example. Let $L(m, k)$ represent the solvable Lie algebra having as nilradical the $(2m + 1)$-dimensional Heisenberg algebra $H(m)$, and k linearly nil-independent elements in the complement subspace E of the nilradical. The nilradical $M = H(m)$ has standard basis $\{P_1, \dots, P_m; B_1, \dots, B_m; H\}$ with commutation relations

$$[P_i, B_k] = \delta_{ik}H, \quad [P_i, H] = [B_i, H] = 0.$$

A basis of E is given by $\{X_1, \dots, X_k\}$ with the X_j satisfying

$$[X_i, X_j] = r_{ij}H, \quad i, j = 1, \dots, k, \quad r_{ij} \in \mathbb{K}.$$

When $k = 3$, and $r_{ij} = 0$ (for $i, j = 1, \dots, k$), $L(m, 3)$ has 4 invariants [13], and $2r - n = 2m - 2$. And for $m \leq 2$ we see that $2r - n < 4$, thus showing that the number of invariants is not bounded by $2r - n$ in this case.

An upper bound for the dimension of the center of complex Lie algebras is determined in [9]. We now make use of part (a) of Theorem 6 to give an analogue of this result for the dimension of the center of the universal enveloping algebra $\mathfrak{A}(L)$ of the Lie algebra L over the field $\mathbb{K}$ of characteristic zero.

Corollary 2 *Suppose that the matrices A^j of the linear operators $\mathrm{ad}_M X_j$ $(j = 1, \dots, k)$ are all triangularizable in a basis of the solvable Lie algebra L over $\mathbb{K}$. Then the dimension of the center of the universal enveloping algebra $\mathfrak{A}(L)$ of L admits $r = \dim M$ as upper bound.*

Proof By a well known result of Gel'fand [5], there exits a one-to-one correspondence between the elements of the center of $\mathfrak{A}(L)$, that is, the Casimir operators of L, and the polynomial invariants of the coadjoint representation of L. Since the maximal number of functionally independent polynomial invariants is not greater than the maximal number of functionally independent invariants, it then follows by part (a) of Theorem 6 that the dimension of the center of $\mathfrak{A}(L)$ is bounded by r. ∎

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