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NILPOTENCY IN UNCOUNTABLE GROUPS

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Abstract

The main purpose of this paper is to investigate the behaviour of uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ are nilpotent. It is proved that such a group G is nilpotent, provided that G has no infinite simple homomorphic images and either $\aleph$ has cofinality strictly larger than $\aleph_0$ or the generalized continuum hypothesis is assumed to hold. Furthermore, groups whose proper subgroups of large cardinality are soluble are studied in the last part of the paper.

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1. Introduction

In a long series of papers, it has been shown that the structure of a (generalized) soluble group of infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see, for instance, [2], where a full reference list on this subject can be found). The results in these papers suggest that the behaviour of *small* subgroups in a *large* group can be neglected, at least for the right choice of the definition of largeness and within a suitable universe. This point of view was adopted in the recent paper [6] by considering uncountable groups whose proper uncountable subgroups belong to certain relevant group classes, for example, that of groups with finite conjugacy classes.

The aim of this paper is to give a further contribution to this topic by investigating uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ are nilpotent. The main obstacle here is a relevant result of Shelah [12], who proved (without appeal to the continuum hypothesis) that there exists a group of cardinality $\aleph_1$ whose proper subgroups have cardinality strictly smaller than $\aleph_1$. In order to avoid Shelah's example and other similar obstructions, we will use the additional

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requirement that the group has no simple homomorphic images of large cardinality, a condition which is obviously satisfied in the case of locally soluble groups. In fact, our first main result is the following.

THEOREM A. *Let $\aleph$ be a cardinal number whose cofinality is strictly larger than $\aleph_0$ and let G be a group of cardinality $\aleph$ which has no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are nilpotent, then G itself is nilpotent.*

Of course, any uncountable regular cardinal number has cardinality strictly larger than $\aleph_0$, and so the above theorem holds, in particular, for such cardinals. On the other hand, there exist cardinals with cofinality strictly larger than $\aleph_0$ which are not regular, for example $\aleph_{\aleph_1}$. Observe also that, in Section 2, it will be proved that the assumption on the cofinality of the cardinal number $\aleph$ can be dropped under the assumption of the *generalized continuum hypothesis* (GCH).

As a consequence of Theorem A, it turns out that, under the same hypotheses, if all proper subgroups of cardinality $\aleph$ of G have nilpotency class bounded by a positive integer c, then also G has nilpotency class at most c.

Our second main result deals with uncountable locally graded groups whose proper uncountable subgroups are locally nilpotent. Recall here that a group G is *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index. Thus all locally (soluble-by-finite) groups are locally graded.

THEOREM B. *Let G be an uncountable locally graded group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of cardinality $\aleph$ of G are locally nilpotent, then G itself is locally nilpotent.*

It is known that, in many problems concerning (generalized) supersoluble groups, the main obstacle is the behaviour of the commutator subgroup. For instance, it was proved in [5] that if G is a group of infinite rank whose proper subgroups of infinite rank are locally supersoluble, then G itself is locally supersoluble, provided that its commutator subgroup G' is locally nilpotent. A corresponding result holds for groups of large cardinality.

THEOREM C. *Let G be a group of uncountable cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are locally supersoluble. If the commutator subgroup G' of G is locally nilpotent, then G is locally supersoluble.*

The final part of the paper is dedicated to the study of uncountable groups whose proper subgroups of large cardinality are soluble and, in this case, the following result has been proved.

THEOREM D. *Let G be an uncountable group of cardinality $\aleph$ which has no simple nonabelian homomorphic images. If all proper subgroups of cardinality $\aleph$ are soluble with derived length at most k (where k is a fixed positive integer), then G itself is soluble with derived length at most k.*

We leave as an open question whether a result similar to Theorem A holds when nilpotency is replaced by solubility. An obstacle here is caused by the fact that the structure of insoluble locally soluble groups whose proper subgroups are soluble seems to be unknown; although such groups must be countable, they may occur as homomorphic images of uncountable groups whose proper subgroups of large cardinality are soluble.

Most of our notation is standard. In particular, we shall use the first chapter of the monograph [11] as a general reference for definitions and properties of closure operations on group classes. Moreover, we refer to the monograph [7] for terminology and properties concerning cardinal numbers.

2. Nilpotency

An uncountable group of cardinality $\aleph$ is called a *Jónsson group* if all its proper subgroups have cardinality strictly smaller than $\aleph$. As we mentioned in the introduction, Shelah constructed a Jónsson group of cardinality $\aleph_1$, answering in the negative to a question posed by Kuroš and Černikov in their seminal paper [8]. It is easy to prove that if G is any Jónsson group of cardinality $\aleph$, then $G = G'$ and $G/Z(G)$ is a simple group of cardinality $\aleph$ (see, for instance, [6, Corollary 2.6]).

LEMMA 2.1. *Let $\mathfrak{X}$ be an $\boldsymbol{S}$-closed class of groups, and let G an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathfrak{X}$. If G has no simple homomorphic images of cardinality $\aleph$, then all proper normal subgroups of G belong to $\mathfrak{X}$.*

PROOF. Let N be any normal subgroup of G whose cardinality is strictly smaller than $\aleph$. Then G/N has cardinality $\aleph$, and it cannot be a Jónsson group because G has no simple homomorphic images of cardinality $\aleph$. It follows that G/N contains a proper subgroup X/N of cardinality $\aleph$. As X belongs to the $\boldsymbol{S}$-closed class $\mathfrak{X}$, N also lies in $\mathfrak{X}$. □

Our next lemma shows, in particular, that any uncountable abelian group admits a countable homomorphic image which is not finitely generated.

LEMMA 2.2. *Let G be a nilpotent-by-finite group whose countable homomorphic images are finitely generated. Then G is finitely generated, and so it satisfies the maximal condition.*

PROOF. First, suppose that G is abelian-by-finite. Let A be an abelian subgroup of finite index of G and consider a countable homomorphic image A/H of A. Clearly, H has only a finite number of conjugates in G, so G/H_G is countable and hence finitely generated. Thus A/H is finitely generated, and so all countable homomorphic images of A are finitely generated.

Obviously, A has no divisible nontrivial homomorphic images and, in particular, it is reduced. Suppose first that A is periodic. Since every nontrivial primary component of A has a nontrivial cyclic direct factor, it is clear that A has only a finite number of nontrivial primary components. Let P be any primary component of A and let B be a

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basic subgroup of P. Then P/B is a divisible homomorphic image of A, so it is trivial, and $P = B$ is the direct product of a collection of cyclic subgroups and hence it is finite. It follows that A itself is finite. In the general case, let T be the subgroup consisting of all elements of finite order of A. Since A has no divisible nontrivial homomorphic images, a maximal free abelian subgroup U/T of A/T must be finitely generated. Moreover, it follows, from the first part of the proof, that the periodic group A/U is finite, so that A/T is finitely generated and A splits over T. Thus T is a homomorphic image of A and so it is finite. Therefore A is finitely generated, and hence also G is finitely generated.

Suppose, finally, that G is nilpotent-by-finite and let N be a nilpotent normal subgroup of finite index of G. Then the factor group G/N' is abelian-by-finite and it follows, from the first part of the proof, that G/N' is finitely generated. In particular, N/N' is finitely generated, so that N itself is finitely generated. Therefore G is finitely generated. □

Recall that a group class $\mathfrak{X}$ is $\boldsymbol{N}_0$-closed if, in any group, the product of two normal $\mathfrak{X}$-subgroups is likewise an $\mathfrak{X}$-subgroup. Generalizing this concept, if $\mathfrak{U}$ is a class of groups, we shall say that a group class $\mathfrak{X}$ is *$\boldsymbol{N}_0$-closed* in the universe $\mathfrak{U}$ if, whenever G is an $\mathfrak{U}$-group and X and Y are normal $\mathfrak{X}$-subgroups of G, also the product XY belongs to $\mathfrak{X}$. Of course, a group class is $\boldsymbol{N}_0$-closed in the ordinary sense if and only if it is $\boldsymbol{N}_0$-closed in the universe of all groups.

In our considerations, we will need the well-known facts that the order of any finite minimal nonsupersoluble group is divisible by at most three prime numbers (see [4]) and that a polycyclic group whose finite homomorphic images are supersoluble is likewise supersoluble (see [1]). Combining these two results, it is easy to show that if G is a finitely generated soluble group whose proper subgroups are supersoluble, then G is either finite or supersoluble (see, for instance, [5, Lemma 3.1]).

Theorem 2.3. *Let $\mathfrak{X}$ be an $\boldsymbol{S}$-closed class of locally supersoluble groups which is also $\boldsymbol{N}_0$-closed in a universe $\mathfrak{U}$ and let G be an uncountable $\mathfrak{U}$-group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathfrak{X}$. If G is locally graded and has no simple homomorphic images of cardinality $\aleph$, then G is locally $\mathfrak{X}$.*

Proof. Assume, for a contradiction, that the statement is false, so that G contains a finitely generated subgroup E which is not in $\mathfrak{X}$. By Lemma 2.1, all proper normal subgroups of G belong to $\mathfrak{X}$. As the class $\mathfrak{X}$ is $\boldsymbol{N}_0$-closed in $\mathfrak{U}$, the subgroup E cannot be contained in the product of a finite number of proper normal subgroups of G, and so, in particular, G cannot be the join of its proper normal subgroups. Since G has no simple homomorphic images of cardinality $\aleph$, it follows that G contains a proper normal subgroup of cardinality $\aleph$. If N is any such normal subgroup, all proper subgroups of G containing N belong to $\mathfrak{X}$ and, in particular, they are locally supersoluble. On the other hand, the product EN is a subgroup of cardinality $\aleph$ which is not in $\mathfrak{X}$, so that $EN = G$ and G/N is finitely generated. As N is locally supersoluble, its commutator subgroup N' is locally nilpotent and hence the factor group G/N is locally graded (see [10]). Then G/N has a finite nontrivial homomorphic image, which is soluble

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because all its proper subgroups are supersoluble. Therefore the commutator subgroup G' is properly contained in G.

Since $EN = G$ for each normal subgroup N of G of cardinality $\aleph$, all countable homomorphic images of G are finitely generated and an application of Lemma 2.2 yields that any abelian-by-finite homomorphic image of G is finitely generated. In particular, G/G' is finitely generated. As G cannot be the product of two proper normal subgroups, it follows that G/G' is cyclic of prime-power order, and so G is locally polycyclic because G' is locally supersoluble.

Suppose that $G/G^{(i)}$ is finite for some positive integer i and assume that the next factor $G^{(i)}/G^{(i+1)}$ of the derived series of G is infinite. Since the abelian-by-finite group $G/G^{(i+1)}$ is finitely generated, the subgroup $G^{(i+1)}$ has cardinality $\aleph$, and so all proper subgroups of $G/G^{(i+1)}$ are supersoluble. Therefore $G/G^{(i+1)}$ is an infinite supersoluble group whose commutator subgroup has finite index and hence it admits an infinite dihedral homomorphic image, which is impossible because G/G' has prime-power order. This contradiction shows that the group $G/G^{(n)}$ is finite for each nonnegative integer n: that is, all soluble homomorphic images of G are finite.

Let J be the finite residual of G. As $G = NE$ for each normal subgroup of finite index N of G, the factor group G/J is soluble and so also finite. Moreover, $J = J'$ and J cannot contain proper G-invariant subgroups of cardinality $\aleph$. Let M be the join of all proper G-invariant subgroups of J. If M is properly contained in J, J/M is a chief factor of G, and hence it is abelian, which is a contradiction. Therefore $J = M$ is the join of its proper G-invariant subgroups. As G is locally polycyclic, the subgroup $E \cap J$ is finitely generated, and so its normal closure $(E \cap J)^G$ is a proper subgroup of J. Put $\bar{G} = G/(E \cap J)^G$. Since J is perfect, there exists a proper G-invariant subgroup $\bar{K}$ of $\bar{J}$ containing an element $\bar{x}$ such that $[\bar{x}, \bar{J}] \neq \{1\}$. Obviously, all conjugates of $\bar{x}$ belong to $\bar{K}$, and so the index of $\bar{C} = C_{\bar{G}}(\bar{x})$ in $\bar{G}$ is strictly smaller than $\aleph$. If $\bar{y}$ is any element of the finite subgroup $\bar{E}$, then

$$|\bar{C} : \bar{C} \cap \bar{C}^{\bar{y}}| \leq |\bar{G} : \bar{C}^{\bar{y}}| < \aleph$$

and hence

$$\bar{W} = \bar{J} \cap \Big(\bigcap_{\bar{y} \in \bar{E}} \bar{C}^{\bar{y}} \Big)$$

is a proper $\bar{E}$-invariant subgroup of $\bar{J}$ of cardinality $\aleph$. It follows that the product EW is a proper subgroup of G of cardinality $\aleph$, and this contradiction completes the proof of the theorem. □

Observe that the class of locally nilpotent groups is N_0-closed by the theorem of Hirsch and Plotkin, while it follows easily from a result of Baer that the class of locally supersoluble groups is N_0-closed in the universe of groups with locally nilpotent commutator subgroup (see, for instance, [5, Lemma 2.2]). Therefore both Theorems B and C are special cases of Theorem 2.3.

We can now prove our main result.

Proof of Theorem A. Assume, for a contradiction, that the statement is false. As the class of nilpotent groups is countably recognizable, there exists in G a countable nonnilpotent subgroup X. Moreover, all proper normal subgroups of G are nilpotent by Lemma 2.1, and so X cannot be contained in a proper normal subgroup of G: that is, $X^G = G$.

Let H be a normal subgroup of G of cardinality strictly smaller than $\aleph$ and suppose that G has no proper normal subgroups of cardinality $\aleph$ containing H. Since G has no infinite simple homomorphic images, it follows that G is generated by its proper normal subgroups containing H. Moreover, there clearly exists a sequence $(K_n)_{n\in\mathbb{N}}$ of proper normal subgroups of G containing H such that X lies in

$$\langle K_n \mid n \in \mathbb{N}\rangle,$$

and this latter is a proper normal subgroup of G because $\aleph$ has cofinality strictly larger than $\aleph_0$. This contradiction proves that H is contained in a proper normal subgroup of G of cardinality $\aleph$ and, in particular, that G has proper normal subgroups of cardinality $\aleph$. If N is any such normal subgroup, the product NX is not nilpotent, so that $NX = G$ and hence G/N is countable. Moreover, N/N', likewise, has cardinality $\aleph$, and then the commutator subgroup N' of N must have cardinality strictly smaller than $\aleph$.

Let a be any element of G such that $\langle a\rangle^G \neq G$. The above argument shows that a belongs to a proper normal subgroup N of G of cardinality $\aleph$. As N' has cardinality $\aleph' < \aleph$, the element a has less than $\aleph$ conjugates in N, and so also in G. Then the normal closure $\langle a\rangle^G$ has cardinality strictly smaller than $\aleph$. On the other hand, X is countable and $X^G = G$ so that, by the cofinality assumption on $\aleph$, there exists an element x of X such that $\langle x\rangle^G = G$. It follows, in particular, that G properly contains the join M of all its proper normal subgroups. Then M is nilpotent and G/M is finite.

The group G is locally nilpotent, by Theorem B, and so its elements of finite order form a subgroup T. Suppose that T is properly contained in G. As G is nilpotent-by-finite, it follows, from the theory of isolators in torsion-free locally nilpotent groups, that the factor group G/T is nilpotent (see, for instance, [9, Section 2.3]); on the other hand, G cannot be the product of two proper normal subgroups, and so it has no torsion-free abelian nontrivial homomorphic images. Then $G = T$ is a periodic group, and hence it is the direct product of its Sylow subgroups. But all proper normal subgroups of G are nilpotent, and so it follows that G is a p-group for some prime number p.

As $\bar{M} = M/M'$ has cardinality $\aleph$, also its socle $\bar{S}$ has cardinality $\aleph$, and so G/S is countable. Let $\bar{U}$ be a G-invariant subgroup of $\bar{S}$ which is maximal with respect to the condition of being the direct product of a collection of finite G-invariant subgroups and assume that $\bar{U}$ has cardinality strictly smaller than $\aleph$. Then

$$\bar{S} = \bar{U} \times \bar{V}$$

for a suitable subgroup $\bar{V}$ and the index of $|\bar{G} : \bar{V}|$ is strictly smaller than $\aleph$. On the other hand, $\bar{V}$ has a finite number of conjugates in $\bar{G}$, so that also the index $|\bar{G} : \bar{V}_{\bar{G}}|$

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is strictly smaller than $\aleph$, and hence the core $\bar{V}_{\bar{G}}$ of $\bar{V}$ in G has cardinality $\aleph$. If $\bar{y}$ is a nontrivial element of $\bar{V}_{\bar{G}}$, then the normal closure $\langle \bar{y}\rangle^{\bar{G}}$ is finite and

$$\langle \bar{U}, \langle \bar{y}\rangle^{\bar{G}}\rangle = \bar{U} \times \langle \bar{y}\rangle^{\bar{G}}.$$

This contradiction shows that $\bar{U}$ has cardinality $\aleph$, and so $\bar{U} = \bar{U}_1 \times \bar{U}_2$, where both $\bar{U}_1 = U_1/M'$ and $\bar{U}_2 = U_2/M'$ have cardinality $\aleph$. It follows that XU_1 is a proper subgroup of G of cardinality $\aleph$ and this last contradiction completes the proof of the theorem. □

Our next result deals with groups whose proper subgroups of large cardinality are nilpotent, but under the assumption of the GCH.

THEOREM A′. *Assume that the GCH holds and let G be an uncountable group of cardinality $\aleph$ which has no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are nilpotent, then G itself is nilpotent.*

PROOF. Assume, for a contradiction, that the statement is false, so that G contains a countable nonnilpotent subgroup X and Theorem A yields that $\aleph > \aleph_1$. By Lemma 2.1, all proper normal subgroups of G are nilpotent, so that, in particular, G cannot be the product of two proper normal subgroups. Then G/G' is a locally cyclic p-group for some prime number p, and hence all nilpotent homomorphic images of G are countable. It follows that if K is any normal subgroup of G of cardinality strictly smaller than $\aleph$, then the factor group G/K is not nilpotent, and so it is a counterexample to the statement.

Suppose that all proper normal subgroups of G have cardinality strictly smaller than $\aleph$ and let N be a proper normal subgroup of G containing a noncentral element x. The factor group $G/C_G(\langle x\rangle^G)$ embeds into the automorphism group of $\langle x\rangle^G$ and so it has cardinality at most $2^{\aleph'}$, where $\aleph' < \aleph$ is the cardinality of N. On the other hand, also the cardinality of $C_G(\langle x\rangle^G)$ is strictly smaller than $\aleph$, so that $G/C_G(\langle x\rangle^G)$ has cardinality $\aleph$, and hence $2^{\aleph'} = \aleph$, by the GCH. It follows that each proper normal subgroup of G has cardinality at most $\aleph'$. Moreover, G cannot contain proper subgroups of finite index, so that it has no simple homomorphic images, and hence, by Zorn's lemma, G can be decomposed as the set-theoretic union of a chain $(N_\lambda)_{\lambda\in\Lambda}$ of proper normal subgroups. Let Λ_0 be a countable subset of Λ such that X is contained in the normal subgroups

$$W = \bigcup_{\lambda\in\Lambda_0} N_\lambda,$$

which, of course, has cardinality at most $\aleph'$. This is a contradiction, because all proper normal subgroups of G are nilpotent. Therefore G contains a proper normal subgroup M of cardinality $\aleph$.

As M is nilpotent, the group M/M', likewise, has cardinality $\aleph$. Then the product XM' is a proper subgroup of G, so that M' has cardinality strictly smaller than $\aleph$ and hence G/M' is not nilpotent. Thus G may be replaced by G/M', and so, without loss

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of generality, it can be assumed that M is abelian. Clearly, $XM = G$, so that $X \cap M$ is a normal subgroup of G and the factor group $G/X \cap M$ is not nilpotent. A further replacement of G by $G/X \cap M$ allows us now to suppose that $X \cap M = \{1\}$. Clearly, M has a countable nontrivial homomorphic image M/L and the subgroup L has only a countable number of conjugates in G. Therefore there exists a countable subset Y of G such that the factor group M/L_G embeds into the Cartesian product of the collection of groups $(M/L^y)_{y\in Y}$, and hence M/L_G has cardinality at most $\aleph_1$. But $\aleph > \aleph_1$, and so the normal subgroup L_G has cardinality $\aleph$. Then $XL_G = G$ and hence

$$M = XL_G \cap M = L_G(X \cap M) = L_G.$$

This last contradiction completes the proof of the theorem. □

Corollary 2.4. *Let G be an uncountable group of cardinality $\aleph$ which has no infinite simple homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are nilpotent with class at most c (where c is a fixed positive integer), then G itself is nilpotent with class at most c, provided that either the cofinality of $\aleph$ is strictly larger than $\aleph_0$ or the GCH is assumed to hold.*

Proof. The group G is nilpotent, either by Theorem A or by Theorem A′, and so the factor group G/G' has cardinality $\aleph$. It follows, from Lemma 2.2, that G contains a normal subgroup N such that G/N is a countable abelian group which is not finitely generated. Let E be any finitely generated subgroup of G. Then EN is a proper subgroup of G of cardinality $\aleph$, and hence it has class at most c. Therefore also G has nilpotency class at most c. □

3. Solubility

The first result of this section shows that if G is an uncountable group whose proper subgroups of large cardinality belong to a group class $\mathfrak{X}$, then G contains a large normal $\mathfrak{X}$-subgroup, under suitable closure conditions on the class $\mathfrak{X}$.

Lemma 3.1. *Let $\mathfrak{X}$ be a class of groups which is $\boldsymbol{S}$ and $\boldsymbol{L}$-closed, and let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of G of cardinality $\aleph$ belong to $\mathfrak{X}$. Then either $G/Z(G)$ is a simple group of cardinality $\aleph$ or G contains a normal $\mathfrak{X}$-subgroup N such that the factor group G/N is simple.*

Proof. Assume, for a contradiction, that the group G does not contain any normal $\mathfrak{X}$-subgroup N such that G/N is simple. Then it follows from Zorn's lemma and from the $\boldsymbol{L}$-closure of the class $\mathfrak{X}$ that G contains proper normal subgroups which are not in $\mathfrak{X}$. Let K be any such subgroup and let $E = \langle x_1, \dots, x_t\rangle$ be any finitely generated subgroup of K which is not in $\mathfrak{X}$. It follows from the $\boldsymbol{S}$-closure of $\mathfrak{X}$ that E cannot be contained in a proper subgroup of G of cardinality $\aleph$. The conjugacy class of any element of K in G has cardinality strictly smaller than $\aleph$ and so, in particular, $|G : C_G(x_i)| < \aleph$ for each $i = 1, \dots, t$. Thus $|G : C_G(E)| < \aleph$, and so the centralizer $C_G(E)$ has cardinality $\aleph$. On the other hand, the product $EC_G(E)$ cannot belong to $\mathfrak{X}$, so that $G = EC_G(E)$ and the

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subgroup $C_G(E)$ is normal in G. If $C_G(E) \neq G$, the centralizer $C_G(E)$ is contained in a maximal normal subgroup M of G, which, of course, has cardinality $\aleph$ and so belongs to $\mathscr{X}$, contrary to our assumptions. Therefore E is contained in $Z(G)$, so K also lies in $Z(G)$. The factor group G/E has cardinality $\aleph$, while all its proper subgroups have cardinality strictly smaller than $\aleph$: that is, G/E is a Jónsson group, and so G/C is simple of cardinality $\aleph$, where $C/E = Z(G/E)$. Moreover, the normal subgroup C of G is not in $\mathscr{X}$, and hence the same argument used above for K yields that C is contained in $Z(G)$. Therefore $C = Z(G)$ and $G/Z(G)$ is a simple group of cardinality $\aleph$. □

The consideration of Jónsson groups shows that, in the above results, the group G can be far from being in $\mathscr{X}$. However, something more can be said if the group has no large simple homomorphic images.

Corollary 3.2. *Let $\mathscr{X}$ be a class of groups which is **S** and **L**-closed and contains all abelian groups. If G is an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ belong to $\mathscr{X}$, then G contains a normal $\mathscr{X}$-subgroup N such that the factor group G/N is simple.*

Corollary 3.3. *Let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ locally satisfy the maximal condition. If G has no infinite simple homomorphic images, then it locally satisfies the maximal condition.*

Proof. It follows, from Corollary 3.2, that G contains a normal subgroup N locally satisfying the maximal condition such that G/N is simple. Then G/N is finite, so G locally satisfies the maximal condition. □

Corollary 3.4. *Let $\mathscr{X}$ be a class of groups which is **S** and **L**-closed and let G be an uncountable group of cardinality $\aleph$ which has no simple homomorphic images of cardinality $\aleph$. If every proper subgroup of cardinality $\aleph$ of G belongs to $\mathscr{X}$, then G contains a normal $\mathscr{X}$-subgroup N such that G/N is a finitely generated simple group.*

Proof. The statement is obvious when G lies in $\mathscr{X}$, so we may suppose that G is not an $\mathscr{X}$-group, and hence it contains a finitely generated subgroup E which is not in $\mathscr{X}$. Since G has no simple homomorphic images of cardinality $\aleph$, it follows, from Lemma 3.1, that there exists a normal $\mathscr{X}$-subgroup N of G such that the factor group G/N is simple and has cardinality strictly smaller than $\aleph$. Then N has cardinality $\aleph$, so $EN = G$ and hence G/N is finitely generated. □

Corollary 3.5. *Let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are locally soluble. If G has no simple nonabelian homomorphic images, then it is locally soluble.*

Corollary 3.6. *Let G be an uncountable group of cardinality $\aleph$ which has no simple nonabelian homomorphic images. If all proper subgroups of G of cardinality $\aleph$ are locally polycyclic, then G itself is locally polycyclic.*

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If, in the statements of Lemma 3.1 and Corollary 3.4, we choose for $\mathscr{X}$ the class of soluble groups with derived length at most k (where k is a fixed positive integer), then we obtain the following consequence.

Corollary 3.7. *Let G be an uncountable group of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ are soluble with derived length at most k (where k is a fixed positive integer). Then G contains a soluble normal subgroup N of derived length at most k such that the factor group G/N is simple. Moreover, if G is locally graded and has no simple homomorphic images of cardinality $\aleph$, then G/N is finite and, in particular, G is soluble-by-finite.*

Proof. The first part of the statement is just a special case of Corollary 3.2. Suppose now, in addition, that G is locally graded and has no simple homomorphic images of cardinality $\aleph$. Then the simple factor group G/N is finitely generated, by Corollary 3.4. Moreover, as N is soluble, G/N is locally graded (see [10]) and hence it must be finite. □

We can finally prove our last main result.

Proof of Theorem D. The group G is soluble by Corollaries 3.5 and 3.7. Assume, for a contradiction, that G has derived length $n > k$ and consider a finitely generated subgroup E of G with derived length n. Let i be the largest nonnegative integer such that the term $G^{(i)}$ of the derived series of G has cardinality $\aleph$. In particular, the subgroup $G^{(i)}E$ has cardinality $\aleph$, and so $G = G^{(i)}E$. Suppose that $E/E \cap G^{(i)}$ is infinite. Then $E \cap G^{(i)}$ is properly contained in a proper subgroup X of E with derived length n (see [3, Theorem 1]), so that $G^{(i)}X = G$ and hence

$$E = E \cap G^{(i)}X = (E \cap G^{(i)})X = X.$$

This contradiction shows that $E/E \cap G^{(i)}$ must be finite, so that also $G/G^{(i)}$ is finite. On the other hand, the subgroup $G^{(i+1)}$ has cardinality strictly smaller than $\aleph$, so that the abelian group $G^{(i)}/(EG^{(i+1)} \cap G^{(i)})$ is infinite and hence it contains a proper subgroup $U/(EG^{(i+1)} \cap G^{(i)})$ such that $G^{(i)}/U$ is countable. Then $G^{(i)}/U_G$ is likewise countable, and so U_G has cardinality $\aleph$. Therefore $U_GE = G$, so that

$$G^{(i)} = U_GE \cap G^{(i)} = U_G(E \cap G^{(i)}) \leq U$$

and this last contradiction completes the proof. □

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