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A NOTE ON THE ADJOINT OF THE PRODUCT OF OPERATORS

BY
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1. Cordes and Labrousse ([2] p. 697), and Kaniel and Schechter ([6] p. 429) showed that if S and T are domain-dense closed linear operators on a Hilbert space H into itself, the range of S is closed in H and the codimension of the range of S is finite, then, $(TS)^* = S^*T^*$. With a somewhat different approach and more restricted condition on S, the same assertion was obtained by Holland [5] recently, that S is a bounded everywhere-defined linear operator whose range is a closed subspace of finite codimension in H.

The purpose of the present note is to generalize this result to the case of domain-dense closed linear operators on Banach spaces over the same field of real or complex numbers. In particular, if S and T are Fredholm operators on reflexive Banach spaces, then, $(TS)^{**} = TS$. We will also prove these results for adjoint operators between normed dual systems of Banach spaces.

2. We shall denote by $D(S)$ the domain, $R(S)$ the range and $N(S)$ the null space of a linear operator S on a Banach space. For convenience we sometimes write (S, x) instead of Sx for every $x \in D(S)$.

LEMMA 1. *Let X, Y and Z be Banach spaces, S and T linear operators (not necessarily closed or bounded) on X into Y with $D(S) \subseteq X$ and on Y into Z with $D(T) \subseteq Y$, respectively. If TS is densely defined on X, then $(TS)^* \supseteq S^*T^*$. Furthermore, if T is bounded and defined everywhere on Y, then $(TS)^* = S^*T^*$.*

Proof. Let $f \in D(S^*T^*)$, then $f \in D(T^*)$ and $T^*f \in D(S^*)$. It follows that for any $x \in D(TS)$, $(f, TSx) = (T^*f, Sx) = (S^*T^*f, x)$. $(TS)^*$ is defined uniquely, because TS is densely defined by assumption. Thus, $f \in D((TS)^*)$, $(TS)^*f = S^*T^*f$ and hence $(TS)^* \supseteq S^*T^*$. To prove the second part, let $f \in D((TS)^*)$ and $x \in D(S)$. T is bounded and defined everywhere which assures that T^* takes every $f \in Z^* = D(T^*)$ into Y^*. $((TS)^*f, x) = (f, TSx) = (T^*f, Sx)$. Since TS is densely defined, so is S and hence S^* is defined uniquely. Thus, $T^*f \in D(S^*)$ and $f \in D(S^*T^*)$. Therefore, $(TS)^* \subseteq S^*T^*$, and the equality holds. Q.E.D.

THEOREM 1. *Let X, Y and Z be Banach spaces, S and T domain-dense closed linear operators on X into Y and on Y into Z, respectively. If* codim $R(S) < \infty$ *and there is a closed complementary subspace of $N(S)$, then $(TS)^* = S^*T^*$.*

Proof. Let us remark here that codim $R(S) < \infty$ implies that $R(S)$ is closed, and $N(S)$ is closed since S is closed. According to Lemma 1, $(TS)^* \supseteq S^*T^*$ holds if TS

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is densely defined. However, it is well-known that if S and T are domain-dense closed linear operators on X into Y and on Y into Z, respectively, and codim $R(S)<\infty$, then $D(TS)$ is dense in X ([4] p. 103). To show that $(TS)^* \subseteq S^*T^*$, let $f \in D((TS)^*)$ and we shall first prove that $f \in D(T^*)$ (i.e., there is a number $c>0$ such that $|(f, Ty)| \leq c \|y\|$ for every $y \in D(T)$). Since codim $R(S)=\dim Y/R(S)<\infty$, we have $Y=R(S) \oplus M'$, where M' is some finite-dimensional subspace of Y. Also $Y=R(S) \oplus M$, where M is some finite-dimensional subspace of $D(T)$ since $D(T)$ is dense in Y ([4] p. 103). Hence we have

$$D(T) = D(T) \cap R(S) \oplus M. \tag{1}$$

By assumption, there exists a closed subspace M'' of X such that $X=M'' \oplus N(S)$. Let $S_0=S \mid (M'' \cap D(S))$, then S_0 is a closed operator which is one-to-one and $R(S_0)=R(S)$. Accordingly, S_0^{-1} exists and is a closed operator on Banach space $R(S_0)$ into Banach space X, and thus S_0^{-1} is a bounded operator by the closed-graph theorem, or equivalently,

$$\|x\| \leq c_0\|S_0x\| = c_0\|Sx\| \quad \text{for every } x \in M'' \cap D(S) \text{ with } c_0 > 0. \tag{2}$$

Suppose $y \in D(T) \cap R(S)$, then, there is an $x \in M'' \cap D(S)$ with $Sx=y$ and $\|x\| \leq c_0\|y\|$. Since $f \in D((TS)^*)$, we have

$$|(f, Ty)| = |(f, TSx)| \leq c_1\|x\| \leq c_0c_1\|y\|, \text{ with } c_1 > 0. \tag{3}$$

On the other hand, suppose $y \in M$, then,

$$|(f, Ty)| \leq \|f\| \, \|Ty\| \leq \|f\| \, \|T\| \, \|y\| = c_2\|y\|, \text{ with } c_2 > 0, \tag{4}$$

since the operator T on M is bounded due to M being finite-dimensional. By (3) and (4) we see that $|(f, Ty)| \leq c \|y\|$ for every $y \in D(T)$ with $c>0$, and thus $f \in D(T^*)$. Now, let us next show that $T^*f \in D(S^*)$. $f \in D((TS)^*)$ and $f \in D(T^*)$ imply that

$$|(f, TSx)| = |(T^*f, Sx)| \leq c_3\|x\|, \quad \text{for every } x \in D(TS) \text{ with } c_3 > 0. \tag{5}$$

It suffices to prove that (5) holds for every $x \in D(S) \cap M''$. Since $D(T) \cap R(S)$ is dense in $R(S)$ ([4] p. 103) and $R(S)=R(S_0)$, $D(T) \cap R(S_0)$ is dense in $R(S_0)$ which means that for any $S_0x \in R(S_0)$ ($x \in D(S) \cap M''$), there exists a sequence $\{S_0x_n\}$ of elements in $D(T) \cap R(S_0)$ ($\{x_n\} \subseteq D(TS_0)$) such that $S_0x_n \to S_0x$, with $x_n - x \in D(S) \cap M''$ for every n. Therefore, by (2),

$$\|x_n - x\| \leq c_0\|S_0(x_n - x)\| = c_0\|S_0x_n - S_0x\| \to 0. \tag{6}$$

We see that $x_n \to x$, and (5) holds for every $x \in D(S) \cap M''$. Q.E.D.

Corollary 1. *Let X, Y and Z be reflexive Banach spaces, S and T domain-dense closed linear operators on X into Y and on Y into Z, respectively. If* codim $R(S)<\infty$, codim $R(T^*)<\infty$ *and both $N(S)$ and $N(T^*)$ have closed complementary subspaces, then, $(TS)^{**}=TS$.*

The proof follows by applying Theorem 1 and the following wellknown result:

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if X and Y are reflexive Banach spaces and S is a domain-dense closed linear operator on X into Y, then, S^* is also a domain-dense closed linear operator on Y^* into X^*. Moreover, $S^{**}=(S^*)^*=S$.

As usual, a domain-dense closed linear operator S on Banach space X into Banach space Y is said to be a Fredholm operator if both codim $R(S)$ and dim $N(S)$ are finite. Accordingly, for such operator S there is a closed complementary subspace of $N(S)$, and hence we have

THEOREM 2. *If X, Y and Z are Banach spaces, S is a Fredholm operator on X into Y and T is a domain-dense closed linear operator on Y into Z (T is not necessarily a Fredholm operator), then, $(TS)^*=S^*T^*$.*

THEOREM 3. *If X, Y and Z are reflexive Banach spaces, both S and T are Fredholm operators on X into Y and on Y into Z, respectively, then, $(TS)^{**}=TS$.*

Proof. It is wellknown that T is Fredholm if and only if T^* is Fredholm (in this case, Y and Z are not necessarily reflexive). This and Corollary 1 imply the desired result.

3. In this section we will prove some properties of the adjoint operator between normed dual systems which will be needed in the next section.

Let X_1 and X_2 be normed linear spaces and let f be a bounded bilinear form on $X_1 \times X_2$, if f is non-degenerate, the pair (X_1, X_2) is said to be the normed dual system on f ([9] Chap. 2, p. 62). Suppose that X_1' and X_2' are dense subspaces of X_1 and X_2, respectively, in virtue of f being bounded it is easily seen that the non-degeneracy of f is equivalent with the following condition

$$(7)\qquad \begin{aligned} f(x, y) &= 0 \text{ for every } x \in X_1' \text{ implies that } y = 0, \text{ and} \\ f(x, y) &= 0 \text{ for every } y \in X_2' \text{ implies that } x = 0. \end{aligned}$$

Let Y_1 and Y_2 be normed linear spaces, g a bounded bilinear form on $Y_1 \times Y_2$, the pair (Y_1, Y_2) the normed dual system on g, and S a domain dense linear operator on X_1 into Y_1. An operator S^* is said to be the adjoint of S if

$$(8)\qquad D(S^*) = \{w \in Y_2 : \exists y \in X_2 \cdot\ni\cdot\ g(Sx, w) = f(x, y), \forall x \in D(S)\},$$

and since $y \in X_2$ is uniquely determined by w due to (7), S^* is defined by $S^*w = y$. Clearly, S^* is a uniquely defined linear operator on Y_2 into X_2. In other words, a linear operator S^* is the adjoint of S if

$$(9)\qquad g(Sx, w) = f(x, S^*w) \quad \text{for every } x \in D(S) \text{ and } w \in D(S^*).$$

It may be noted that no matter whether S is a closed operator or not, S^* is always closed, although it may happen that $D(S^*)=\{0\}$. If S is not densely defined, S^* is in general not unique.

Let A and B be subsets of X_1 and X_2, respectively. If $A^{\perp}=\{y \in X_2 : f(x, y)=0, \forall x \in A\}$ and ${}^{\perp}B=\{x \in X_1 : f(x, y)=0, \forall y \in B\}$, then, $A^{\perp}$ and ${}^{\perp}B$ are closed subspaces

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of X_2 and X_1, respectively. Moreover, if A and B are subspaces, then $A^{\perp}=\bar{A}^{\perp}$, ${}^{\perp}B={}^{\perp}\bar{B}$, ${}^{\perp}(A^{\perp})=\bar{A}$ and $({}^{\perp}B)^{\perp}=\bar{B}$, where $\bar{A}$ is the closure of A, etc.

Let us denote by X_1^* the adjoint space of X_1, and A' the orthogonal complement in X_1^* of $A\subseteq X_1$, etc. As is easily seen, for the normed dual system (X_1, X_2) on f, X_2 (resp. X_1) may be regarded as a linear subspace of X_1^* (resp. X_2^*) due to the non-degeneracy of f. Consequently, $\dim X_1=\dim X_2$, since $\dim X_2\leq\dim X_1^*=\dim X_1\leq\dim X_2^*=\dim X_2$.

Throughout the remainder of this section we shall assume that (X_1, X_2) and (Y_1, Y_2) are normed dual systems on f and on g, respectively.

Lemma 2. *If S is a domain-dense linear operator on X_1 into Y_1 and S^* is the adjoint of S in the sense of* (9), *then*

(a) $N(S^*) = R(S)^{\perp}$.
(b) $R(S^*) \subseteq N(S)^{\perp}$.
(c) ${}^{\perp}N(S^*) = \overline{R(S)}$.

We shall omit the proof since it is completely standard. As a simple consequence of this lemma, we have

Corollary 2. *Notation as in Lemma 2, then*
(a′) *$R(S)$ is dense if and only if S^* is one-to-one.*
(b′) *That $R(S^*)$ is dense implies that S is one-to-one.*

Lemma 3. *If S is a domain-dense closed linear operator on X_1 into Y_1, then*
(a) *The pair $(X_1/N(S), N(S)^{\perp})$ is a normed dual system.*
(b) *The pair $\overline{(R(S)}, Y_2/R(S)^{\perp})$ is a normed dual system.*

The closedness of S implies the closedness of $N(S)$ in X_1, and the proof follows from Proposition 5 [N. Bourbaki: *Espaces vectoriels topologiques*, Chap. 4, p. 54] and a simple calculation. Indeed, Lemma 3 is true for arbitrary closed subspaces.

Lemma 4. *If S is a domain-dense closed linear operator on X_1 into Y_1 and S^* is the adjoint of S in the sense of* (9), *then $D(S^*)$ is dense in Y_2, and $S^{**}=S$.*

Proof. Let us first show that if $D(S)$ is dense and for any nonzero element $y\in Y_1$, there is a $w\in D(S^*)$ such that $g(y, w)\neq 0$. In fact, if $y\neq 0$, then $(0, y)$ is not in the graph of S which is closed subspace of $X_1\times Y_1$, since S is closed. By the Hahn–Banach theorem, there is a $z^*\in(X_1\times Y_1)^*$ such that $z^*(0, y)\neq 0$ and $z^*(x, Sx)=0$ for every $x\in D(S)$. Due to the non-degeneracy of f and g, we may define $x'\in X_2$ and $w\in Y_2$ by $f(x, x')=z^*(x, 0)$ and $g(y, w)=z^*(0, y)$, respectively. Then, $0=z^*(x, Sx)=f(x, x')+g(Sx, w)$ for every $x\in D(S)$, hence $w\in D(S^*)$ and $0\neq z^*(0, y)=g(y, w)$. Now, (Y_2^*, Y_2) is the normed dual system on g' if g' is defined by $g'(F, w)=Fw$. Suppose that $D(S^*)$ is not dense in Y_2, then there is a nonzero element $F\in Y_2^*$ such that $g'(F, w)=Fw=0$ for every $w\in D(S^*)$. This contradicts the above claim. That $S^{**}=S$ is, therefore, easy to see by the definition. Q.E.D.

Throughout the remainder of this section, we shall assume that X_1, X_2, Y_1 and Y_2 are Banach spaces.

LEMMA 5. *If A is a closed subspace of X_1, then* $\dim A^\perp = \dim A'$.

Proof. It is wellknown that $\dim A' = \dim X_1/A$ ([7] Chap. 3, p. 141). But by Lemma 3, $(X_1/A, A^\perp)$ is the normed dual system on some operator, hence,

$$\dim A^\perp = \dim X_1/A = \dim A'. \qquad \text{Q.E.D.}$$

THEOREM 4. (The closed range theorem of Banach). *If S is a domain-dense closed linear operator on X_1 into Y_1 and S^* is the adjoint of S in the sense of (9), then the following statements are equivalent:*

(a) *$R(S)$ is closed.*
(b) $R(S^*) \supseteq N(S)^\perp$.
(c) *$R(S^*)$ is closed.*
(d) $R(S) = {}^\perp N(S^*)$.

Proof. (a)$\Rightarrow$(b): Since S is a domain-dense closed linear operator, the induced operator S_0 on $X_1/N(S)$ into Y_1, which is defined by $S_0(x+N(S)) = Sx$, is one-to-one and closed with $R(S_0) = R(S)$. Thus, S_0^{-1} is bounded on $R(S)$ since $R(S)$ is closed. The operator S_0^* on Y_2 into $N(S)^\perp$, by (a) of Lemma 3, is the adjoint of S_0 in the sense of (9), which exists uniquely due to the denseness of $D(S_0)$. $R(S_0^*) \supseteq N(S)^\perp$. In fact, due to the non-degeneracy of g, if $y \in N(S)^\perp$ we may define $y' \in Y_2$ by

$$g(x', y') = f_0(S_0^{-1}x', y), \quad x' \in R(S_0),$$

where f_0 on $X_1/N(S) \times N(S)^\perp$ is defined by $f_0(x+N(S), y) = f(x, y)$.

For every $x+N(S) \in D(S_0)$, $f_0(x+N(S), y) = g(S_0(x+N(S)), y')$, hence $S_0^* y' = y$ and $R(S_0^*) \supseteq N(S)^\perp$. Now, $g(Sx, y') = g(S_0(x+N(S)), y') = f_0(x+N(S), S_0^* y') = f(x, S_0^* y')$ for every $x \in D(S)$. Thus, $y = S_0^* y' = S^* y'$. Since $y \in N(S)^\perp$ was arbitrary, $R(S^*) \supseteq N(S)^\perp$.

(b) $\Rightarrow$ (c): By (b) of Lemma 2, $R(S^*) = N(S)^\perp$ which is closed.

(c) $\Rightarrow$ (a): Let $S_0 = S$ be an operator on X_1 into Banach space $\overline{R(S)} \subseteq Y_1$, clearly S_0 is closed and $\overline{R(S_0)} = \overline{R(S)}$. The operator S_0^* on $Y_2/R(S)^\perp$ into X_2, by (b) of Lemma 3, is the adjoint of S_0 in the sense of (9), which exists uniquely due to the denseness of $D(S_0)$. S_0^* is one-to-one by (a') of Corollary 2. Since $g(Sx, y) = f(x, x')$ for every $x \in D(S)$, if and only if $g_0(S_0 x, y + R(S)^\perp) = f(x, x')$ for every $x \in D(S_0)$, where g_0 on $\overline{R(S)} \times Y_2/R(S)^\perp$ is defined by $g_0(x, y+R(S)^\perp) = g(x, y)$, it is easily seen that $R(S_0^*) = R(S^*)$ which is closed. Thus, S_0^* has a bounded inverse. $\overline{R(S)} \subseteq R(S_0)$. In fact, if $y \in \overline{R(S)}$ we may define $x \in X_1$ by

$$f(x, x') = g_0(y, S_0^{*-1}x'), \quad x' \in R(S_0^*).$$

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For every $y'+R(S)^{\perp} \in D(S_0^*)$, $g_0(y, y'+R(S)^{\perp})=f(x, S_0^*(y'+R(S)^{\perp}))$. $x \in D(S_0^{**})=D(S_0)$ and $S_0x=S_0^{**}x=y$ due to Lemma 4. Therefore, $\overline{R(S)} \subseteq R(S_0)=R(S)$, i.e., $R(S)$ is closed.

(a) and (d) are equivalent by (c) of Lemma 2. Q.E.D.

COROLLARY 3. *Notation as in Theorem 4. If $R(S)$ is closed, then $R(S^*)=N(S)^{\perp}$ and the converse to* (b′) *of Corollary 2 holds.*

A domain-dense closed linear operator S on a Banach space into another one is said to be a semi-Fredholm operator if $R(S)$ is closed and at least one of codim $R(S)$ and dim $N(S)$ is finite. The index of a Fredholm (semi-Fredholm) operator S is defined by ind $S=\dim N(S)-\operatorname{codim} R(S)$. The following theorem is wellknown if S^* is the adjoint of S in the usual sense ([4] p. 102; [7] p. 234; [8]).

THEOREM 5. *If S is a domain-dense closed linear operator on X_1 into Y_1, then, S is a Fredholm operator* (*resp. semi-Fredholm operator*) *if and only if S^*, the adjoint of S in the sense of* (9), *is a Fredholm operator* (*resp. semi-Fredholm operator*). *In this case we have* ind $S=-$ind S^*.

Proof. By Lemma 5 and Lemma 2, we have

$$\dim Y_1/R(S)=\dim R(S)'=\dim R(S)^{\perp}=\dim N(S^*).$$

On the other hand, $\dim X_1^*/A'=\dim A^*$ for any subspace $A\subseteq X_1$, since X_1^*/A' and A^* are isometrically isomorphic. By this, Corollary 3 and Lemma 5, we have

$$\begin{aligned}\dim X_2/R(S^*) &= \dim X_2/N(S)^{\perp} = \dim (N(S)^{\perp})' = \dim (N(S)')' \\ &= \dim X_1^*/N(S)' = \dim N(S)^* = \dim N(S).\end{aligned}$$

Hence, the first part of the theorem is proved. Next, we have

$$\begin{aligned}\text{ind } S &= \dim N(S)-\dim Y_1/R(S) \\ &= \dim X_2/R(S^*)-\dim N(S^*) = -\text{ind } S^*.\end{aligned} \qquad \text{Q.E.D.}$$

4. Unless mention is made, we shall assume throughout this section that X_1, X_2, Y_1, Y_2, Z_1 and Z_2 are Banach spaces, (X_1, X_2), (Y_1, Y_2) and (Z_1, Z_2) are normed dual systems on f, on g and on h, respectively. We will investigate the adjoint, in the sense of (9), of the product of operators for such systems.

LEMMA 6. *Let S and T be linear operators* (*not necessarily closed or bounded*) *on X_1 into Y_1 with $D(S)\subseteq X_1$ and on Y_1 into Z_1 with $D(T)\subseteq Y_1$, respectively. If TS is densely defined on X_1, then, $(TS)^*\supseteq S^*T^*$. Furthermore, if T is bounded and defined everywhere on Y_1, then $(TS)^*=S^*T^*$.*

The proof follows from the same argument we employed in Lemma 1.

LEMMA 7. *Let S be a domain-dense linear operator on normed linear space X_1 into normed linear space Y_1, then, $w \in D(S^*)$ if and only if there is a number $c>0$ and $y \in X_2$ such that $|g(Sx, w)| \le c\,\|x\|\,\|y\|$ for every $x \in D(S)$.*

Proof. The "only if" part follows from (8) and the boundedness of f. To show the "if" part, it is easily seen that there is an $F \in X_1^*$ such that $g(Sx, w) = \|y\| Fx$ for every $x \in D(S)$. (X_1, X_1^*) is the normed dual system on f' if $f'(x, F) = Fx$. This and the uniqueness of S^* imply that $w \in D(S^*)$. Q.E.D.

THEOREM 6. *Let S and T be domain-dense closed linear operators on X_1 into Y_1 and on Y_1 into Z_1, respectively. If* codim $R(S) < \infty$ *and there is a closed complementary subspace of $N(S)$, then $(TS)^* = S^*T^*$.*

By making use of Lemma 6 and 7, the proof may be carried out in a manner similar to that of Theorem 1. Also by Lemma 4 and Theorem 6, we have

COROLLARY 4. *Let S and T be domain-dense closed linear operators on X_1 into Y_1 and on Y_1 into Z_1, respectively. If* codim $R(S) < \infty$, codim $R(T^*) < \infty$ *and both $N(S)$ and $N(T^*)$ have closed complementary subspaces, then $(TS)^{**} = TS$.*

The next theorem is easy to see.

THEOREM 7. *If S is a Fredholm operator on X_1 into Y_1 and T is a domain-dense closed linear operator on Y_1 into Z_1 (T is not necessarily a Fredholm operator), then, $(TS)^* = S^*T^*$.*

Finally, by Theorem 5 and Corollary 4 we also have

THEOREM 8. *If both S and T are Fredholm operators on X_1 into Y_1 and on Y_1 into Z_1, respectively, then $(TS)^{**} = TS$.*

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