

## A MONOTONE PROCESS MAINTENANCE MODEL FOR A MULTISTATE SYSTEM

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### Abstract

In this paper, we study a monotone process maintenance model for a multistate system with  $k$  working states and  $\ell$  failure states. By making different assumptions, we can apply the model to a multistate deteriorating system as well as to a multistate improving system. We show that the monotone process model for a multistate system is equivalent to a geometric process model for a two-state system. Then, for both the deteriorating and the improving system, we analytically determine an optimal replacement policy for minimizing the long-run average cost per unit time.

*Keywords:* Geometric process; replacement policy  $N$ ; harmonic mean; renewal reward process

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### 1. Introduction

In practice, because of ageing effects and accumulated wearing, many systems deteriorate in such a way that the successive operating times become shorter and shorter, while the repair times become longer and longer. Therefore, the use of a monotone process model is a direct approach to the maintenance problem of a deteriorating system. To introduce such a model, we first introduce the concept of stochastic order.

**Definition 1.** A random variable  $X$  is said to be stochastically greater or less than a random variable  $Y$  if, respectively,  $P(X > \alpha) \geq P(Y > \alpha)$  or  $P(X > \alpha) \leq P(Y > \alpha)$  for all real  $\alpha$ . This ordering is denoted by  $X \geq_{st} Y$  or  $X \leq_{st} Y$ , as appropriate (see Ross (1996)). Furthermore, a stochastic process  $\{X_n, n = 1, 2, \dots\}$  is said to be stochastically increasing or decreasing if  $X_{n+1} \geq_{st} X_n$  or  $X_{n+1} \leq_{st} X_n$ , respectively, for  $n = 1, 2, \dots$

Note that the stochastic order defined by Definition 1 is the usual one. However, the stochastically monotone concept for a stochastic process  $\{X_n, n = 1, 2, \dots\}$  defined here is simply based on the distributions of  $X_n$  and  $X_{n+1}$  for  $n = 1, 2, \dots$ . This is different from the stochastically monotone concept for a Markov process. Recall that a Markov process  $\{X_n, n = 1, 2, \dots\}$  with state space  $\{0, 1, \dots\}$  is said to be stochastically monotone if

$$(X_{n+1} | X_n = i_1) \leq_{st} (X_{n+1} | X_n = i_2)$$

for any  $0 \leq i_1 \leq i_2$  (see, e.g. Shaked and Shanthikumar (1994)). Clearly, the stochastically monotone concept for a Markov process is defined for a Markov process and is based on the probabilities of transition from one state to another state, conditioning on the former state.

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However, the stochastically monotone concept for a stochastic process defined here is for a general process, and is based on the unconditional distributions of two successive random variables in the process.

Lam (1988a), (1988b) introduced the geometric process as a simple monotone process.

**Definition 2.** A stochastic process  $\{X_n, n = 1, 2, \dots\}$  is a geometric process (GP), if there exists a real  $a > 0$  such that  $\{a^{n-1}X_n, n = 1, 2, \dots\}$  forms a renewal process. The number  $a$  is called the ratio of the GP.

Obviously, if  $0 < a \leq 1$ , the GP is stochastically increasing; if  $a \geq 1$ , it is stochastically decreasing; and if  $a = 1$ , the GP will reduce to a renewal process.

The GP was applied to the maintenance problem of a one-component system by Lam (1988a), (1988b), (2003). Thereafter, the GP was also used in the study of the maintenance problem for a two-component system; see Lam (1995), Lam and Zhang (1996), and Zhang (1999). Furthermore, Lam (1992a), Lam and Chan (1998), and Lam *et al.* (2004) applied the GP model to the analysis of data from a series of events. For further reference, see Leung and Lee (1998), Sheu (1999), and Pérez-Ocón and Torres-Castro (2002). Moreover, see Lam (2005) for a brief review and more references.

However, most of the research work has so far considered a two-state system only, with up and down states, say. Little attention has been paid to the study of an improving system. In many practical situations, a system may have more than two states. For example, the system constituted by an electronic instrument may break down due to a short circuit or an open circuit, in which case the system will have two failure states and one working state. Moreover, although many practical systems are deteriorating, there do exist some improving systems. For example, the successive operating times of a system after repair might be prolonged due to the system operator's accumulation of working experience and/or the replacement, during repair, of some failed parts of the system by more advanced parts. Consecutive repair times after failure might, furthermore, be shortened because the repair facility becomes more and more familiar with the system. Therefore, it is necessary to study a maintenance model for both a multistate deteriorating system and a multistate improving system. Lam *et al.* (2002) considered a monotone process model for a multistate deteriorating system with  $k$  failure states and one working state. Zhang *et al.* (2002) also studied a monotone process model for the maintenance problem of a multistate system.

In this paper, we will study a monotone process model for a one-component multistate system with  $k + \ell$  states, namely  $k$  working states and  $\ell$  failure states. By making different assumptions, the model can be applied to a multistate deteriorating system as well as to a multistate improving system. The replacement policy  $N$  is applied, by which the system will be replaced following the  $N$ th failure. Then, we shall show that the monotone process model for the multistate system is equivalent to a GP model for a two-state system in the sense that two systems will have the same long-run average cost per unit time and the same optimal replacement policy. Finally, the optimal replacement policy is determined analytically for both the deteriorating system and the improving system.

## 2. The monotone process model

Suppose that we are given a one-component multistate system. We will first describe the probability structure of the model. To do this, we note that the system state at time  $t$  can be

denoted by

$$S(t) = \begin{cases} i & \text{if the system is in the } i\text{th working state at time } t \ (i = 1, \dots, k), \\ k + i & \text{if the system is in the } i\text{th failure state at time } t \ (i = 1, \dots, \ell). \end{cases}$$

The set of working states is  $W = \{1, \dots, k\}$ , the set of failure states is  $F = \{k + 1, \dots, k + \ell\}$ , and the state space is  $\Omega = W \cup F$ . Initially, assume that a new system in working state 1 is installed. Whenever the system fails, it will be repaired. Let  $t_n$  be the completion time of the  $n$ th repair,  $n = 0, 1, \dots$ , with  $t_0 = 0$ , and let  $s_n$  be the time of the  $n$ th failure,  $n = 1, 2, \dots$ . Then, clearly we have

$$t_0 < s_1 < t_1 < \dots < s_n < t_n < \dots$$

Furthermore, assume that the transition probability from working state  $i, i = 1, \dots, k$ , to failure state  $k + j, j = 1, \dots, \ell$ , is given by

$$P(S(s_{n+1}) = k + j \mid S(t_n) = i) = q_j,$$

with  $\sum_{j=1}^{\ell} q_j = 1$ . Moreover, the transition probability from failure state  $k + i, i = 1, \dots, \ell$ , to working state  $j, j = 1, \dots, k$ , is given by

$$P(S(t_n) = j \mid S(s_n) = k + i) = p_j,$$

with  $\sum_{j=1}^k p_j = 1$ .

Let  $X_1$  be the operating time of a system after installation. In general, let  $X_n, n = 2, 3, \dots$ , be the operating time of the system after the  $(n - 1)$ th repair and let  $Y_n, n = 1, 2, \dots$ , be the repair time after the  $n$ th failure. Assume that there exist a lifetime distribution  $U(t)$  and an  $a_i > 0, i = 1, \dots, k$ , such that

$$P(X_1 \leq t) = U(t)$$

and

$$P(X_2 \leq t \mid S(t_1) = i) = U(a_i t), \quad i = 1, \dots, k.$$

In general,

$$\begin{aligned} P(X_n \leq t \mid S(t_1) = i_1, \dots, S(t_{n-1}) = i_{n-1}) \\ = U(a_{i_1} \dots a_{i_{n-1}} t), \quad i_j = 1, \dots, k, \ j = 1, \dots, n - 1. \end{aligned} \quad (1)$$

Similarly, there exist a lifetime distribution  $V(t)$  and a  $b_i > 0, i = 1, \dots, \ell$ , such that

$$P(Y_1 \leq t \mid S(s_1) = k + i) = V(b_i t)$$

and, in general,

$$\begin{aligned} P(Y_n \leq t \mid S(s_1) = k + i_1, \dots, S(s_n) = k + i_n) \\ = V(b_{i_1} \dots b_{i_n} t), \quad i_j = 1, \dots, \ell, \ j = 1, \dots, n. \end{aligned} \quad (2)$$

We now make additional assumptions on the policy used, the cost structure, and the monotonicity conditions of the model.

**Assumption 1.** A replacement policy  $N$  is applied, by which the system is replaced by a new (identical) one following the  $N$ th failure.

**Assumption 2.** *If a system in working state  $i$  is operating, the reward rate is  $r_i$ ,  $i = 1, \dots, k$ . If the system in failure state  $k + i$  is under repair, the repair-cost rate is  $c_i$ ,  $i = 1, \dots, \ell$ . The replacement cost comprises two parts: one part is the basic replacement cost  $R$  and the other is proportional to the replacement time  $Z$  at rate  $c_p$ . In other words, the replacement cost is given by  $R + c_p Z$ .*

**Assumption 3.** *We assume that*

$$1 = a_1 \leq a_2 \leq \dots \leq a_k, \quad (3)$$

$$1 = b_1 \geq b_2 \geq \dots \geq b_\ell > 0. \quad (4)$$

Under the additional Assumptions 1–3, we shall argue that the model is a maintenance model for a multistate deteriorating system. For a multistate improving system, Assumption 3 will be replaced by the following assumption.

**Assumption 3'.** *We assume that*

$$1 = a_1 \geq a_2 \geq \dots \geq a_k > 0, \quad (5)$$

$$1 = b_1 \leq b_2 \leq \dots \leq b_\ell. \quad (6)$$

*Note that the equalities in (5) and (6) do not all hold simultaneously.*

**Remarks.** The use of policy  $N$  has a long history (see, e.g. Morimura (1970) and Park (1979)). However, in maintenance problems, replacement policy  $T$  – by which the system is replaced at a stopping time  $T$  – is also used, where  $T$  is the total operating time. It is interesting to compare policies  $N$  and  $T$ . In studying the long-run average cost per unit time, Stadjc and Zuckerman (1990) and Lam (1991b) showed that, under some mild conditions, an optimal policy  $N^*$  is at least as good as an optimal policy  $T^*$ . In the case of the total expected discounted cost, the same result was proven by Lam (1991a), (1992b). The implementation of a policy  $N$  is also much simpler than that of a policy  $T$ . These are the reasons we are using policy  $N$  here.

Since the system has  $k$  different working states, it follows that in each one it should have a different reward rate. Similarly, in different failure states, the system should have different repair-cost rates.

Under Assumption 3, for two working states  $0 \leq i_1 < i_2 \leq k$ , we have

$$(X_2 \mid S(t_1) = i_1) \geq_{\text{st}} (X_2 \mid S(t_1) = i_2).$$

Therefore, working state  $i_1$  is better than working state  $i_2$  in the sense that the system in state  $i_1$  has a stochastically larger operating time than it does in state  $i_2$ . Consequently, the  $k$  working states are arranged in decreasing order, such that state 1 is the best working state and state  $k$  is the worst working state. Similarly, for two failure states  $k + i_1$  and  $k + i_2$  such that  $k + 1 \leq k + i_1 < k + i_2 \leq k + \ell$ ,

$$(Y_1 \mid S(s_1) = k + i_1) \leq_{\text{st}} (Y_1 \mid S(s_1) = k + i_2).$$

Therefore, failure state  $k + i_1$  is better than failure state  $k + i_2$  in the sense that the system in state  $k + i_1$  has a stochastically smaller repair time than it does in state  $k + i_2$ . Thus, the  $\ell$  failure states are also arranged in decreasing order, such that state  $k + 1$  is the best failure state and state  $k + \ell$  is the worst failure state.

In particular, if  $p_1 = q_1 = 0$ ,  $a_2 = \dots = a_k = a$ , and  $b_2 = \dots = b_\ell = b$ , then the system reduces to a two-state system. In fact, (1) and (2) now become  $P(X_n \leq t) = U(a^{n-1}t)$  and  $P(Y_n \leq t) = V(b^n t)$ , respectively. Thus,  $\{X_n, n = 1, 2, \dots\}$  will form a GP with ratio  $a \geq 1$  and  $X_1 \sim U$ , while  $\{Y_n, n = 1, 2, \dots\}$  will form a GP with ratio  $0 < b \leq 1$  and  $Y_1 \sim G$  where  $G(t) = V(bt)$ . As a result, our model reduces to the GP model for the one-component two-state system introduced by Lam (1988a), (1988b).

Under Assumption 3', for two working states  $0 \leq i_1 < i_2 \leq k$  a similar argument shows that working state  $i_1$  is worse than working state  $i_2$  since the system in state  $i_1$  has a stochastically smaller operating time than it does in state  $i_2$ . This means that the  $k$  working states are arranged in increasing order, such that state 1 is the worst working state and state  $k$  is the best working state. On the other hand, for two failure states  $k + 1 \leq k + i_1 < k + i_2 \leq k + \ell$ , failure state  $k + i_1$  is worse than failure state  $k + i_2$  since the system in state  $k + i_1$  has a stochastically larger repair time than it does in state  $k + i_2$ . Therefore, the  $\ell$  failure states are also arranged in increasing order, such that state  $k + 1$  is the worst failure state and state  $k + \ell$  is the best failure state. Note that if all the equalities in (5) and (6) hold, this will be a special case of (3) and (4). In this case, Assumption 3 will hold and it should be excluded from Assumption 3'.

Now we shall demonstrate that our model is a monotone process model for the multistate system. To start with, we first observe that

$$P(X_2 \leq t) = \sum_{i=0}^{k-1} P(X_2 \leq t \mid S(t_1) = i) P(S(t_1) = i) = \sum_{i=1}^k p_i U(a_i t).$$

In general,

$$\begin{aligned} P(X_n \leq t) &= \sum_{i_1=1}^k \dots \sum_{i_{n-1}=1}^k P(X_n \leq t \mid S(t_1) = i_1, \dots, S(t_{n-1}) = i_{n-1}) \\ &\quad \times P(S(t_1) = i_1, \dots, S(t_{n-1}) = i_{n-1}) \\ &= \sum_{i_1=1}^k \dots \sum_{i_{n-1}=1}^k p_{i_1} \dots p_{i_{n-1}} U(a_{i_1} \dots a_{i_{n-1}} t) \end{aligned} \tag{7}$$

$$= \sum_{\{j_1, \dots, j_k \in \mathbb{Z}^+ : \sum_{i=1}^k j_i = n-1\}} \frac{(n-1)!}{j_1! \dots j_k!} p_1^{j_1} \dots p_k^{j_k} U(a_1^{j_1} \dots a_k^{j_k} t) \tag{8}$$

and, similarly,

$$\begin{aligned} P(Y_n \leq t) &= \sum_{i_1=1}^\ell \dots \sum_{i_n=1}^\ell P(Y_n \leq t \mid S(s_1) = k + i_1, \dots, S(s_n) = k + i_n) \\ &\quad \times P(S(s_1) = k + i_1, \dots, S(s_n) = k + i_n) \\ &= \sum_{i_1=1}^\ell \dots \sum_{i_n=1}^\ell q_{i_1} \dots q_{i_n} V(b_{i_1} \dots b_{i_n} t) \end{aligned} \tag{9}$$

$$= \sum_{\{j_1, \dots, j_\ell \in \mathbb{Z}^+ : \sum_{i=1}^\ell j_i = n\}} \frac{n!}{j_1! \dots j_\ell!} q_1^{j_1} \dots q_\ell^{j_\ell} V(b_1^{j_1} \dots b_\ell^{j_\ell} t). \tag{10}$$

Second, we will show that, under the additional Assumption 3,  $\{X_n, n = 1, 2, \dots\}$  is stochastically decreasing while  $\{Y_n, n = 1, 2, \dots\}$  is stochastically increasing. This is the following theorem.

**Theorem 1.** *Under Assumption 3, for  $n = 1, 2, \dots$ , we have  $X_n \geq_{st} X_{n+1}$  and  $Y_n \leq_{st} Y_{n+1}$ .*

*Proof.* For any  $t > 0$ , it follows from (7) that

$$\begin{aligned} P(X_{n+1} \leq t) &= \sum_{i_1=1}^k \cdots \sum_{i_n=1}^k p_{i_1} \cdots p_{i_n} U(a_{i_1} \cdots a_{i_n} t) \\ &= \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k p_{i_1} \cdots p_{i_{n-1}} \left[ \sum_{i_n=1}^k p_{i_n} U(a_{i_1} \cdots a_{i_{n-1}} a_{i_n} t) \right] \\ &\geq \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k p_{i_1} \cdots p_{i_{n-1}} \left[ \sum_{i_n=1}^k p_{i_n} U(a_{i_1} \cdots a_{i_{n-1}} t) \right] \\ &= \sum_{i_1=1}^k \cdots \sum_{i_{n-1}=1}^k p_{i_1} \cdots p_{i_{n-1}} U(a_{i_1} \cdots a_{i_{n-1}} t) = P(X_n \leq t). \end{aligned}$$

Thus,  $X_n \geq_{st} X_{n+1}$ . By a similar argument, from (9) we can prove that  $Y_n \leq_{st} Y_{n+1}$ . This completes the proof of Theorem 1.

Theorem 1 shows that, under the additional Assumptions 1–3, the model is a monotone process model for a multistate deteriorating system. However, under Assumption 3', instead of Theorem 1, we have the following result.

**Theorem 2.** *Under Assumption 3', for  $n = 1, 2, \dots$ , we have  $X_n \leq_{st} X_{n+1}$  and  $Y_n \geq_{st} Y_{n+1}$ .*

The proof is similar to that of Theorem 1.

Therefore, according to Theorem 2, under the additional Assumptions 1, 2, and 3', our model is a monotone process model for a multistate improving system.

### 3. Long-run average cost per unit time

In this section, we shall derive results based on the probability structure of the model and the additional Assumptions 1 and 2 only. To derive the long-run average cost per unit time, we should first evaluate the expected values of  $X_n$  and  $Y_n$ . To do this, let  $\int_0^\infty t dU(t) = \lambda$  and  $\int_0^\infty t dV(t) = \mu$ . Then, from (8), we have

$$\begin{aligned} E(X_n) &= \sum_{\{j_1, \dots, j_k \in \mathbb{Z}^+ : \sum_{i=1}^k j_i = n-1\}} \frac{(n-1)!}{j_1! \cdots j_k!} p_1^{j_1} \cdots p_k^{j_k} \int_0^\infty t dU(a_1^{j_1} \cdots a_k^{j_k} t) \\ &= \sum_{\{j_1, \dots, j_k \in \mathbb{Z}^+ : \sum_{i=1}^k j_i = n-1\}} \frac{(n-1)!}{j_1! \cdots j_k!} \left(\frac{p_1}{a_1}\right)^{j_1} \cdots \left(\frac{p_k}{a_k}\right)^{j_k} \lambda \\ &= \lambda \left(\frac{p_1}{a_1} + \cdots + \frac{p_k}{a_k}\right)^{n-1} = \frac{\lambda}{a^{n-1}}, \end{aligned}$$

where  $a^{-1} = \sum_{i=1}^k p_i/a_i$ . Similarly, from (10), we have

$$E(Y_n) = \mu \left( \frac{q_1}{b_1} + \dots + \frac{q_\ell}{b_\ell} \right)^n = \frac{\mu}{b^n},$$

where  $b^{-1} = \sum_{i=1}^\ell q_i/b_i$ .

Second, we should calculate the expected reward earned after the  $(n - 1)$ th repair and the expected repair cost incurred after the  $n$ th failure. For this purpose, define  $R_n$ , the reward rate after the  $(n - 1)$ th repair, as

$$R_n = r_i \quad \text{if } S(t_{n-1}) = i, \quad i = 1, \dots, k.$$

Because  $S(t_0) = 1$ , we have  $R_1 = r_1$  and, so, the expected reward after installation or one replacement is given by

$$E(R_1 X_1) = E(r_1 X_1) = r_1 \lambda.$$

In general, for  $n \geq 2$ , the expected reward after the  $(n - 1)$ th repair is

$$\begin{aligned} E(R_n X_n) &= \sum_{i_1=1}^k \dots \sum_{i_{n-1}=1}^k E(R_n X_n \mid S(t_1) = i_1, \dots, S(t_{n-1}) = i_{n-1}) \\ &\quad \times P(S(t_1) = i_1, \dots, S(t_{n-1}) = i_{n-1}) \\ &= \sum_{i_1=1}^k \dots \sum_{i_{n-1}=1}^k p_{i_1} \dots p_{i_{n-1}} \int_0^\infty r_{i_{n-1}} t \, dU(a_{i_1} \dots a_{i_{n-1}} t) \\ &= \left( \sum_{i_1=1}^k \dots \sum_{i_{n-2}=1}^k \frac{p_{i_1} \dots p_{i_{n-2}}}{a_{i_1} \dots a_{i_{n-2}}} \right) \left( \sum_{i_{n-1}=1}^k \frac{r_{i_{n-1}} p_{i_{n-1}}}{a_{i_{n-1}}} \right) \lambda \\ &= r \lambda \left( \frac{p_1}{a_1} + \dots + \frac{p_k}{a_k} \right)^{n-2} = \frac{r \lambda}{a^{n-2}}, \end{aligned}$$

where

$$r = \sum_{i=1}^k \frac{r_i p_i}{a_i}. \tag{11}$$

To evaluate the expected repair cost incurred after the  $n$ th failure, define  $C_n$ , the repair-cost rate after the  $n$ th failure, as

$$C_n = c_i \quad \text{if } S(s_n) = k + i, \quad i = 1, \dots, \ell.$$

By a calculation similar to that of  $E(R_n X_n)$ , the expected repair cost after the  $n$ th failure is found to be

$$E(C_n Y_n) = c \mu \left( \frac{q_1}{b_1} + \dots + \frac{q_\ell}{b_\ell} \right)^{n-1} = \frac{c \mu}{b^{n-1}},$$

where

$$c = \sum_{i=1}^\ell \frac{c_i q_i}{b_i}. \tag{12}$$

Now suppose that a replacement policy  $N$  is adopted. We say that a cycle is completed if a replacement is completed; in other words, a cycle is the time interval between the installation and the first replacement or between two successive replacements. Thus, the successive cycles, and the costs incurred in each one, form a renewal reward process (see, e.g. Ross (1996)). Let  $\tau = E(Z)$  be the expected replacement time. Then, by applying a standard result on renewal reward processes, the long-run average cost per unit time (or, simply, the average cost) is given by

$$\begin{aligned}
 C(N) &= \frac{\text{expected cost incurred in a cycle}}{\text{expected length of a cycle}} \\
 &= \frac{E(\sum_{n=1}^{N-1} C_n Y_n + R + c_p Z - \sum_{n=1}^N R_n X_n)}{E(\sum_{n=1}^N X_n + \sum_{n=1}^{N-1} Y_n + Z)} \\
 &= \frac{c\mu \sum_{n=1}^{N-1} (1/b^{n-1}) + R + c_p \tau - [r_1 \lambda + r \lambda \sum_{n=2}^N (1/a^{n-2})]}{\lambda \sum_{n=1}^N (1/a^{n-1}) + \mu \sum_{n=1}^{N-1} (1/b^n) + \tau} \\
 &= \frac{c\mu \sum_{n=1}^{N-1} (1/b^{n-1}) - r \lambda \sum_{n=1}^N (1/a^{n-2}) + R'}{\lambda \sum_{n=1}^N (1/a^{n-1}) + \mu \sum_{n=1}^{N-1} (1/b^n) + \tau} \tag{13}
 \end{aligned}$$

or

$$C(N) = \frac{bc(\mu/b) \sum_{n=1}^{N-1} (1/b^{n-1}) - ar \lambda \sum_{n=1}^N (1/a^{n-1}) + R'}{\lambda \sum_{n=1}^N (1/a^{n-1}) + (\mu/b) \sum_{n=1}^{N-1} (1/b^{n-1}) + \tau}, \tag{14}$$

where

$$R' = R + c_p \tau - (r_1 - ar)\lambda. \tag{15}$$

Clearly,  $a$  and  $b$  are two important constants in (13) and (14). To understand the meaning of  $a$  and  $b$  better, we now introduce the harmonic mean of a random variable.

**Definition 3.** Given a random variable  $X$  with  $E(1/X) \neq 0$ ,  $m_H = [E(1/X)]^{-1}$  is the harmonic mean of  $X$ .

The harmonic mean has the following properties.

1. If  $X$  is a discrete, uniformly distributed random variable such that  $X = x_i$  with probability  $1/n, i = 1, \dots, n$ , then the harmonic mean  $m_H = n/\sum_{i=1}^n (1/x_i)$  is the harmonic mean of the numbers  $x_1, \dots, x_n$ .
2. If  $0 < \alpha \leq X \leq \beta$  then  $\alpha \leq m_H \leq \beta$ .
3. If  $X$  is nonnegative with  $E(X) > 0$  then  $m_H \leq E(X)$ . This is because  $h(x) = 1/x$  is a convex function, from which the result follows by the Jensen inequality.

Thus, in our model,  $a$  is the harmonic mean of random variable  $A$  with  $P(A = a_i) = p_i, i = 1, \dots, k$ , and  $b$  is the harmonic mean of random variable  $B$  with  $P(B = b_i) = q_i, i = 1, \dots, \ell$ . For this reason, we may call  $a$  and  $b$  the harmonic means of  $a_1, \dots, a_k$  and  $b_1, \dots, b_\ell$ , respectively.

Our problem now is to determine an optimal replacement policy for minimizing the average cost  $C(N)$ . To this end, we first observe from (14) that

$$C(N) = D(N) - ar,$$



where

$$D(N) = \frac{(bc + ar)(\mu/b) \sum_{n=1}^{N-1} (1/b^{n-1}) + R' + ar\tau}{\lambda \sum_{n=1}^N (1/a^{n-1}) + (\mu/b) \sum_{n=1}^{N-1} (1/b^{n-1}) + \tau}.$$

Obviously, the problem of minimizing  $C(N)$  is equivalent to the problem of minimizing  $D(N)$ . Therefore, consider

$$\begin{aligned} &D(N+1) - D(N) \\ &= \left\{ (bc+ar)\mu \left[ \lambda \left( \sum_{n=1}^N a^n - \sum_{n=1}^{N-1} b^n \right) + \tau a^N \right] - (R'+ar\tau)[\lambda b^N + \mu a^N] \right\} \\ &\quad \times \left\{ a^N b^N \left[ \lambda \sum_{n=1}^N \frac{1}{a^{n-1}} + \frac{\mu}{b} \sum_{n=1}^{N-1} \frac{1}{b^{n-1}} + \tau \right] \times \left[ \lambda \sum_{n=1}^{N+1} \frac{1}{a^{n-1}} + \frac{\mu}{b} \sum_{n=1}^N \frac{1}{b^{n-1}} + \tau \right] \right\}^{-1}. \end{aligned} \tag{16}$$

For  $R' + ar\tau \neq 0$ , we define an auxiliary function

$$g(N) = \frac{(bc + ar)\mu[\lambda(\sum_{n=1}^N a^n - \sum_{n=1}^{N-1} b^n) + \tau a^N]}{(R' + ar\tau)[\lambda b^N + \mu a^N]} \tag{17}$$

while, for  $R' + ar\tau = 0$ , we define

$$g_0(N) = (bc + ar)\mu \left[ \lambda \left( \sum_{n=1}^N a^n - \sum_{n=1}^{N-1} b^n \right) + \tau a^N \right]. \tag{18}$$

As the denominator of  $D(N + 1) - D(N)$  (i.e. the quantity in braces in (16)) is always positive, it is clear that the sign of  $D(N + 1) - D(N)$  is the same as that of its numerator. Consequently, we have the following lemma.

**Lemma 1.** 1. If  $R' + ar\tau > 0$  then

$$\begin{aligned} D(N + 1) > D(N) &\Leftrightarrow g(N) > 1, \\ D(N + 1) = D(N) &\Leftrightarrow g(N) = 1, \\ D(N + 1) < D(N) &\Leftrightarrow g(N) < 1. \end{aligned}$$

2. If  $R' + ar\tau < 0$  then

$$\begin{aligned} D(N + 1) > D(N) &\Leftrightarrow g(N) < 1, \\ D(N + 1) = D(N) &\Leftrightarrow g(N) = 1, \\ D(N + 1) < D(N) &\Leftrightarrow g(N) > 1. \end{aligned}$$

3. If  $R' + ar\tau = 0$  then

$$\begin{aligned} D(N + 1) > D(N) &\Leftrightarrow g_0(N) > 0, \\ D(N + 1) = D(N) &\Leftrightarrow g_0(N) = 0, \\ D(N + 1) < D(N) &\Leftrightarrow g_0(N) < 0. \end{aligned}$$

Lemma 1 shows that the monotonicity of  $D(N)$  can be determined by the value of either  $g(N)$  or  $g_0(N)$ .

Let  $h(N) = \lambda b^N + \mu a^N$ . Then, for  $R' + ar\tau \neq 0$ , it follows from (17) that

$$\begin{aligned}
 g(N + 1) - g(N) &= \frac{(bc + ar)\mu}{(R' + ar\tau)h(N)h(N + 1)} \\
 &\times \left\{ \lambda^2 b^N (1 - b) \sum_{n=1}^N a^n + \lambda^2 b^N (a^{N+1} - b) + \lambda \mu a^N (a - b^N) \right. \\
 &\quad \left. + \lambda \mu a^N (a - 1) \sum_{n=1}^{N-1} b^n + \lambda \tau a^N b^N (a - b) \right\}. \tag{19}
 \end{aligned}$$

For  $R' + ar\tau = 0$ , it follows from (18) that

$$g_0(N + 1) - g_0(N) = (bc + ar)\mu \{ \lambda (a^{N+1} - b^N) + \tau (a^{N+1} - a^N) \}. \tag{20}$$

Note that, in this section, the results are all developed under the probability structure of the model and the additional Assumptions 1 and 2 only. Therefore, all the results, including Lemma 1, are true for our model of deteriorating systems and improving systems.

#### 4. The optimal policy for the deteriorating system

Under Assumption 3, it follows from property 2 of the harmonic mean that

$$1 = a_0 \leq a \leq a_{k-1} \quad \text{and} \quad b_{k-1} \leq b \leq b_0 = 1. \tag{21}$$

Consider a GP model for a two-state (up and down, say) deteriorating system. Assume that the successive operating times after repair  $\{X'_n, n = 1, 2, \dots\}$  form a GP with ratio  $a \geq 1$  and  $E(X'_1) = \lambda$ . Assume further that the consecutive repair times after failure  $\{Y'_n, n = 1, 2, \dots\}$  constitute a GP with ratio  $0 < b \leq 1$  and  $E(Y'_1) = \mu/b$ . The replacement time is still  $Z$ , with  $E(Z) = \tau$ . The reward rate is  $ar$ , the repair-cost rate is  $bc$ , and the replacement cost is  $R'$ . Then, under policy  $N$ , an argument similar to that presented in the previous section will show that the average cost for the two-state system is exactly the same as that given by (14) (see Lam (1988a), (1988b), and (2003) for details). In other words, the monotone process model for a multistate system is equivalent to a GP model for the two-state system, in the sense that they will have the same average cost and, hence, the same optimal maintenance policy. In conclusion, we have proved the following theorem.

**Theorem 3.** *The monotone process model for a one-component multistate deteriorating system is equivalent to a GP model for a one-component two-state deteriorating system in the sense that they will have the same average cost and the same optimal replacement policy. The successive operating times after repair  $\{X'_n, n = 1, 2, \dots\}$  of the two-state system will form a GP with ratio  $a \geq 1$  (the harmonic mean of  $a_1, \dots, a_k$ ) and  $E(X'_1) = \lambda$ . The consecutive repair times after failure  $\{Y'_n, n = 1, 2, \dots\}$  will constitute a GP with ratio  $0 < b \leq 1$  (the harmonic mean of  $b_1, \dots, b_\ell$ ) and  $E(Y'_1) = \mu/b$ . The reward rate of the two-state system is  $ar$  with  $r$  given by (11), the repair-cost rate is  $bc$  with  $c$  given by (12), and the replacement cost is  $R'$ , given by (15).*

From (19), (20), and (21), the proof of the following lemma is straightforward.

- Lemma 2.** 1. If  $R' + ar\tau > 0$  then  $g(N)$  is nondecreasing in  $N$ .  
 2. If  $R' + ar\tau < 0$  then  $g(N)$  is nonincreasing in  $N$ .  
 3. The function  $g_0(N)$  is nondecreasing in  $N$ .

The combination of Lemmas 1 and 2 yields the following theorem.

**Theorem 4.** 1. For  $R' + ar\tau > 0$ , an optimal replacement policy  $N_d^*$  for the multistate deteriorating system is given by

$$N_d^* = \min\{N \mid g(N) \geq 1\}. \tag{22}$$

The optimal policy  $N_d^*$  is unique if and only if  $g(N_d^*) > 1$ .

2. For  $R' + ar\tau < 0$ , an optimal replacement policy  $N_d^*$  for the multistate deteriorating system is given by

$$N_d^* = \min\{N \mid g(N) \leq 1\}.$$

The optimal policy  $N_d^*$  is unique if and only if  $g(N_d^*) < 1$ .

3. For  $R' + ar\tau = 0$ , an optimal replacement policy  $N_d^*$  for the multistate deteriorating system is given by

$$N_d^* = \min\{N \mid g_0(N) \geq 0\}.$$

The optimal policy  $N_d^*$  is unique if and only if  $g_0(N_d^*) > 0$ .

In applications of Theorem 4, we should determine the value of  $R' + ar\tau$  first. As an example, suppose that  $R' + ar\tau > 0$ ; we then determine  $N_d^*$  from (22), and we have

$$g(N) < 1 \Leftrightarrow N < N_d^*$$

and

$$g(N) \geq 1 \Leftrightarrow N \geq N_d^*.$$

Thus,  $N_d^*$  is the minimum integer satisfying (22). By Lemma 1, we have

$$D(N) > D(N_d^*) \quad \text{if } N < N_d^*$$

and

$$D(N) \geq D(N_d^*) \quad \text{if } N \geq N_d^*.$$

In other words, policy  $N_d^*$  is indeed an optimal replacement policy. Clearly, it is unique if and only if  $g(N_d^*) > 1$ .

Of course, Theorem 4 can be applied not only to a multistate deteriorating system but also to a two-state deteriorating system.

### 5. The monotone process model for an improving system

Now, under Assumption 3', we have

$$a_k \leq a \leq a_1 = 1 \quad \text{and} \quad 1 = b_1 \leq b \leq b_\ell. \tag{23}$$

Also, the equalities in (23) do not all hold simultaneously.

As in the case of the multistate deteriorating system, we can consider a GP model for a two-state improving system. Consequently, we have the following theorem, which is analogous to Theorem 3.

**Theorem 5.** *The monotone process model for a one-component multistate improving system is equivalent to a GP model for a one-component two-state improving system in the sense that they have the same average cost and the same optimal replacement policy. The successive operating times after repair  $\{X'_n, n = 1, 2, \dots\}$  of the two-state system will form a GP with ratio  $0 < a \leq 1$  (the harmonic mean of  $a_1, \dots, a_k$ ) and  $E(X'_1) = \lambda$ . The consecutive repair times after failure  $\{Y'_n, n = 1, 2, \dots\}$  will constitute a GP with ratio  $b \geq 1$  (the harmonic mean of  $b_1, \dots, b_\ell$ ) and  $E(Y'_1) = \mu/b$ . The reward rate of the two-state system is  $ar$  with  $r$  given by (11), the repair-cost rate is  $bc$  with  $c$  given by (12), and the replacement cost is  $R'$ , given by (15).*

Note that, for an improving system, the average cost due to the adoption of a replacement policy  $N$  is also given by (14). Moreover, we can also determine an optimal replacement policy for the improving system on the basis of Lemma 1. To do so, recall that not all equalities in (23) will hold simultaneously. Then, from (19), (20), and (23),  $g(N)$  and  $g_0(N)$  are clearly strictly monotone. Thus in analogy with Lemma 2, we have the following result.

- Lemma 3.**
1. *If  $R' + ar\tau > 0$  then  $g(N)$  is decreasing in  $N$ .*
  2. *If  $R' + ar\tau < 0$  then  $g(N)$  is increasing in  $N$ .*
  3. *The function  $g_0(N)$  is decreasing in  $N$ .*

Now we are available to determine the optimal replacement policy for the multistate improving system.

**Theorem 6.** *For the multistate improving system, if*

$$r_1 \leq r_i, \quad i = 2, \dots, k, \tag{24}$$

*then  $N_1^* = \infty$  is the unique optimal replacement policy.*

*Proof.* We shall prove Theorem 6 by considering three cases.

*Case 1.* For  $R' + ar\tau > 0$ ,  $g(N)$  is decreasing, so let

$$N_+ = \min\{N \mid g(N) \leq 1\}.$$

Thus, we have

$$g(N) > 1 \Leftrightarrow N < N_+$$

and

$$g(N) \leq 1 \Leftrightarrow N \geq N_+.$$

By Lemma 1, it is easy to see that  $C(N)$  and  $D(N)$  are unimodal and will take their maxima at  $N_+$ . This implies that the minimum of  $C(N)$  must be given by  $\min\{C(1), C(\infty)\}$ . Note, from (24), that  $r_1 \leq ar$ . Then (14) yields

$$C(1) = \frac{R + c_p\tau - r_1\lambda}{\lambda + \tau} > C(\infty) = -ar.$$

Consequently,

$$\min C(N) = \min\{C(1), C(\infty)\} = C(\infty) = -ar.$$

Therefore,  $N_1^* = \infty$  is the optimal replacement policy. The optimal policy is also unique because, from Lemma 3,  $g(N)$  is strictly decreasing.

Case 2. For  $R' + ar\tau < 0$ , the proof is similar.

Case 3. For  $R' + ar\tau = 0$ , because  $g_0(N)$  is decreasing there exists an

$$N_0 = \min\{N \mid g_0(N) \leq 0\}.$$

Thus, we have

$$g_0(N) > 0 \Leftrightarrow N < N_0$$

and

$$g_0(N) \leq 0 \Leftrightarrow N \geq N_0.$$

An argument similar to the above shows that  $C(N)$  and  $D(N)$  will attain their maxima at  $N_0$  and, consequently, that  $N_i^* = \infty$  is the unique optimal replacement policy.

This completes the proof.

Because the  $k$  working states are arranged in increasing order (in the sense defined above) with state 1 the worst, the worse the working state is, the lower is the reward rate. Therefore, condition (24) is a natural and reasonable assumption. Consequently, for most practical multistate improving systems, Theorem 6 is applicable. As a result, we should continually repair the system as it ages, without ever replacing it.

Like Theorem 4, Theorem 6 can apply not only to a multistate improving system but also to a two-state improving system.

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