

## ADDITIVE COVERS AND THE CANONICAL BASE PROPERTY

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**Abstract.** We give a new approach to the failure of the Canonical Base Property (CBP) in the so far only known counterexample, produced by Hrushovski, Palacín and Pillay. For this purpose, we will give an alternative presentation of the counterexample as an additive cover of an algebraically closed field. We isolate two fundamental weakenings of the CBP, which already appeared in work of Chatzidakis and Moosa-Pillay and show that they do not hold in the counterexample. In order to do so, a study of imaginaries in additive covers is developed. As a by-product of the presentation, we observe that a pure binding-group-theoretic account of the CBP is unlikely.

**§1. Introduction.** Internality is a fundamental notion in geometric model theory in order to understand a complete stable theory of finite Lascar rank in terms of its building blocks, its minimal types of rank one. A type  $p$  is internal, resp. almost internal to the family  $\mathbb{P}$  of all non-locally modular minimal types, if there exists a set of parameters  $C$  such that every realization  $a$  of  $p$  is definable, resp. algebraic over  $C, e$  where  $e$  is a tuple of realizations of types (each one based over  $C$ ) in  $\mathbb{P}$ .

Motivated by results of Campana [3] on algebraic coreductions, Pillay and Ziegler [19] showed that in the finite rank part of the theory of differentially closed fields in characteristic zero, the type of the canonical base of a stationary type over a realization is almost internal to the constants. With this result Pillay and Ziegler reproved the function field case of the Mordell-Lang conjecture in characteristic zero following Hrushovski's original proof but with considerable simplifications.

The above phenomena is captured in the notion of the Canonical Base Property (CBP), which was introduced and studied by Moosa and Pillay [14]: Over a realization of a stationary type, its canonical base is almost  $\mathbb{P}$ -internal. Chatzidakis [4] showed that the CBP already implies a seemingly stronger statement, the so-called uniform canonical base property (UCBP): Whenever the type of a realization of the stationary type  $p$  over some set  $C$  of parameters is almost  $\mathbb{P}$ -internal, then so is  $\text{stp}(\text{Cb}(p)/C)$ . For the proof, she isolated two remarkable properties which hold in every theory of finite rank with the CBP: Almost internality to  $\mathbb{P}$  transfers to intersections and more generally to quotients. Motivated by her work, we introduce the following two notions. A stationary type is good, resp. special, if the condition for the CBP, resp. UCBP, holds for this type. (See Definitions 2.1 and 2.3 for a precise formulation.) The following result relates these two notions to the aforementioned properties.

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**THEOREM A** (Propositions 2.5 and 2.8). *The theory  $T$  transfers internality to intersections, resp. to quotients, if and only if every stationary almost  $\mathbb{P}$ -internal type in  $T^{eq}$  is good, resp. special.*

It already follows from the proofs of [13, Theorem 3.7 and Proposition 4.2] that the theory transfers internality to quotients whenever every stationary almost  $\mathbb{P}$ -internal type is special. However, we will provide a more direct argument.

Though most relevant examples of theories satisfy the CBP, Hrushovski et al. [8] produced the so far only known example of an uncountably categorical theory without the CBP. We will give an alternative description of their counterexample in terms of additive covers of an algebraically closed field of characteristic zero. Covers are already present in early work of Hrushovski [7], Ahlbrandt and Ziegler [1] as well as of Hodges and Pillay [6]. For an additive cover  $\mathcal{M}$  of an algebraically closed field, the sort  $S$  is the home-sort and  $P$  is the field-sort. The automorphism group  $\text{Aut}(\mathcal{M}/P)$  embeds canonically in the group of all additive maps on  $P$ . Notice that if the sort  $S$  is almost  $P$ -internal, the CBP holds for trivial reasons. The counterexample to the CBP has a ring structure on the sort  $S$  and the ring multiplication  $\otimes$  is a lifting of the field multiplication. The automorphism group over  $P$  corresponds to the group of derivations, which ensures that the sort  $S$  is not almost  $P$ -internal. We prove the following result.

**THEOREM B** (Propositions 5.2 and 5.4). *The CBP holds whenever every additive map on  $P$  induces an automorphism in  $\text{Aut}(\mathcal{M}/P)$ . However, if every element of  $\text{Aut}(\mathcal{M}/P)$  corresponds to a derivation, then the CBP does not hold in  $\mathcal{M}$ .*

A standard argument shows that the CBP holds whenever it holds for all real stationary types. We show that no additive cover can eliminate imaginaries, whenever the sort  $S$  is not almost  $P$ -internal. On the other side, the counterexample to the CBP does eliminate finite imaginaries and furthermore the corresponding real versions of goodness and specialness hold, namely, every real stationary almost  $P$ -internal type is special. However the version for real types does not imply the full condition and gives a new proof of the failure of the CBP.

**THEOREM C** (Propositions 6.1 and 6.3). *The counterexample to the CBP does not transfer internality to intersections.*

Palacín and Pillay [15] considered a strengthening of the CBP, called the strong canonical base property, which we show cannot hold in any additive cover, where  $S$  is not almost  $P$ -internal. Proposition 3.11 and Lemma 5.6 support the thesis that a pure binding-group-theoretic account of the CBP is unlikely, regarding a question stated in [15].

**§2. The canonical base property and related properties.** In this section we introduce two properties related to the canonical base property. We assume throughout this article a solid knowledge in geometric stability theory [18, 20]. Most of the results in this section can be found in [4].

Let us fix a complete stable theory of finite Lascar rank. As usual, we work inside a sufficiently saturated ambient model. We denote by  $\mathbb{P}$  the  $\emptyset$ -invariant family of all non-locally modular minimal types.

The following notions provide an equivalent formulation of the CBP and the UCBP. They will play a crucial role in our attempt to weaken the CBP to other contexts.

DEFINITION 2.1. A stationary type  $p$  is:

- *good* if  $\text{stp}(\text{Cb}(p)/a)$  is almost  $\mathbb{P}$ -internal for some (any) realization  $a$  of  $p$  and
- *special* if, for every parameter set  $C$  and every realization  $a$  of  $p$ , whenever  $\text{stp}(a/C)$  almost  $\mathbb{P}$ -internal, so is  $\text{stp}(\text{Cb}(p)/C)$  almost  $\mathbb{P}$ -internal.

Let us recall the notion of *preservation of  $\mathbb{P}$ -internality* [5], which already appeared in [13] as being  $\mathbb{P}$ -Moishezon. A strong type  $\text{stp}(b/a)$  preserves  $\mathbb{P}$ -internality if the type  $\text{stp}(b/C)$  is almost  $\mathbb{P}$ -internal whenever  $\text{stp}(a/C)$  is. In the above terminology, a stationary type  $p$  is special if and only if for some realization  $a$  of  $p$  the type  $\text{stp}(\text{Cb}(p)/a)$  preserves  $\mathbb{P}$ -internality.

REMARK 2.2.

- (a) Note that every special type is good, by setting  $C = \{a\}$ .
- (b) It is immediate from the definitions that the theory  $T$  has the CBP, resp. the UCBP, if and only if every stationary type in  $T^{\text{eq}}$  is good, resp. special.
- (c) Analog to [17, Remark 2.6], it can be easily shown that whether or not every stationary type is good, resp. special, is preserved under naming parameters.

Chatzidakis showed in [4, Theorem 2.5] that the CBP already implies the UCBP for (simple) theories of finite rank. Her proof consists of two main steps:

- First, she shows in [4, Proposition 2.2] that, under the CBP, the type  $\text{tp}(b/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  is almost  $\mathbb{P}$ -internal, whenever  $\text{stp}(b/a)$  is almost  $\mathbb{P}$ -internal. Note that this was first shown by Moosa and Pillay [14, Theorem 1.3 (b)] in the stable case.
- Secondly, she proves in [4, Lemma 2.3] that  $\text{tp}(b/\text{acl}^{\text{eq}}(a_1) \cap \text{acl}^{\text{eq}}(a_2))$  is almost  $\mathbb{P}$ -internal, if both  $\text{stp}(b/a_1)$  and  $\text{tp}(b/a_2)$  are.

Motivated by her work, we now introduce two notions capturing these intermediate steps and study their relation to the CBP.

DEFINITION 2.3. The theory  $T$  *transfers internality to intersections* if the type

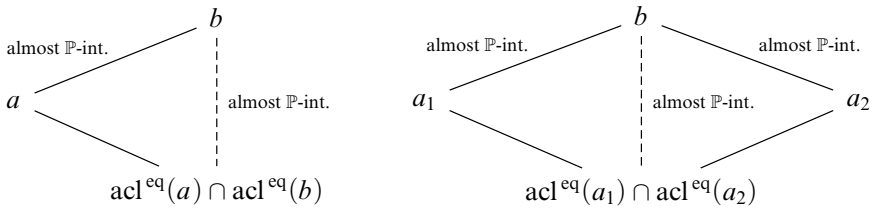
$$\text{tp}(b/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$$

is almost  $\mathbb{P}$ -internal, whenever  $\text{stp}(b/a)$  is almost  $\mathbb{P}$ -internal.

The theory *transfers internality to quotients* if the type

$$\text{tp}(b/\text{acl}^{\text{eq}}(a_1) \cap \text{acl}^{\text{eq}}(a_2))$$

is almost  $\mathbb{P}$ -internal, whenever both  $\text{stp}(b/a_1)$  and  $\text{stp}(b/a_2)$  are.



Note that transfer of internality to quotients implies transfer of internality to intersections, by setting  $a_1 = a$  and  $a_2 = b$ . It is not difficult to see that a weakening of transfer of internality to quotients holds in every complete stable theory, when the quotients are independent: If the types  $\text{stp}(b/a_1)$  and  $\text{stp}(b/a_2)$  are almost  $\mathbb{P}$ -internal and  $a_1 \perp a_2$ , then the type  $\text{stp}(b)$  is almost  $\mathbb{P}$ -internal.

The above notions have been already considered in the literature. In [13, 14] generating families and pairs of fibrations were introduced and studied. Transfer of internality to intersections is equivalent to saying that whenever a stationary type admits a generating family with almost  $\mathbb{P}$ -internal fibers, then it is itself almost  $\mathbb{P}$ -internal. Similarly the theory transfers internality to quotients if and only if every stationary type generated by a pair of fibrations whose fibres are almost  $\mathbb{P}$ -internal is itself almost  $\mathbb{P}$ -internal.

The CBP implies almost  $\mathbb{P}$ -internality of  $\text{stp}(\text{Cb}(a/b)/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  for every stationary type  $\text{stp}(a/b)$  [4, Theorem 2.1]. A close inspection of the proof of [14, Theorem 1.3 (b)] yields that, if  $\text{stp}(\text{Cb}(a/b)/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  is almost  $\mathbb{P}$ -internal whenever the stationary type  $\text{stp}(a/b)$  is almost  $\mathbb{P}$ -internal, then the theory transfers internality to intersections. Without assuming the CBP, it could be the case that the type  $\text{stp}(\text{Cb}(a/b)/a)$  is almost  $\mathbb{P}$ -internal, yet the restriction  $\text{stp}(\text{Cb}(a/b)/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  is not. We do not see how to deduce directly from the aforementioned proofs that transfer of internality to intersections already follows if every almost  $\mathbb{P}$ -internal stationary type is good (in the proof of [4, Theorem 2.1] an induction argument is applied to a type which need not be almost  $\mathbb{P}$ -internal). For the sake of completeness, we will now include a reformulation of transfer of internality in terms of good types in Proposition 2.5. In order to do so, we first need the following observation.

FACT 2.4 ([4, Proposition 1.18] and [16, Theorem 3.6]). *Let  $\text{stp}(b/A)$  and  $\text{stp}(b/C)$  be two  $\mathbb{P}$ -analysable types.*

- (a) *The type  $\text{stp}(b/\text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(C))$  is again  $\mathbb{P}$ -analysable. In particular, so is  $\text{stp}(b/\text{acl}^{\text{eq}}(A) \cap \text{acl}^{\text{eq}}(b))$  also  $\mathbb{P}$ -analysable.*
- (b) *Let  $b_A$  be the maximal subset of  $\text{acl}^{\text{eq}}(A, b)$  such that  $\text{stp}(b_A/A)$  is almost  $\mathbb{P}$ -internal. The tuple  $b_A$  (in some fixed enumeration) dominates  $b$  over  $A$ , that is, for every set of parameters  $D \supset A$ ,*

$$b \underset{A}{\perp} D \text{ whenever } b_A \underset{A}{\perp} D.$$

Furthermore, whenever  $\text{acl}^{\text{eq}}(D) \cap \text{acl}^{\text{eq}}(A, b_A) = \text{acl}^{\text{eq}}(A)$ , so is

$$\text{acl}^{\text{eq}}(D) \cap \text{acl}^{\text{eq}}(A, b) = \text{acl}^{\text{eq}}(A).$$

PROPOSITION 2.5. *The theory  $T$  transfers internality to intersections if and only if every stationary almost  $\mathbb{P}$ -internal type in  $T^{\text{eq}}$  is good.*

PROOF. We assume first that every stationary almost  $\mathbb{P}$ -internal type is good, but the conclusion fails, witnessed by two tuples  $a$  and  $b$ . By Remark 2.2, we may assume

$$\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\emptyset).$$

Thus, the type  $\text{stp}(b/a)$  is almost  $\mathbb{P}$ -internal, but the type  $\text{stp}(b)$  is not. Note that  $\text{stp}(b)$  is  $\mathbb{P}$ -analysable, by Fact 2.4.

Among all possible (imaginary) tuples in the ambient model take now  $a'$  such that  $\text{stp}(b/a')$  is almost  $\mathbb{P}$ -internal and

$$\text{acl}^{\text{eq}}(a') \cap \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\emptyset)$$

with  $U(b_\emptyset/a')$  maximal. Since  $\text{stp}(b/a')$  is almost  $\mathbb{P}$ -internal, there is a set of parameters  $A$  containing  $a'$  with  $A \downarrow_{a'} b$  such that  $b$  is algebraic over  $Ae$ , where  $e$  is a tuple of realizations of types (each one based over  $A$ ) in  $\mathbb{P}$ . Since each type in the family  $\mathbb{P}$  is minimal, we may assume, after possibly enlarging  $A$ , that  $e$  and  $b$  are interalgebraic over  $A$ .

Let now  $e'$  be a maximal subtuple of  $e$  independent from  $b_\emptyset$  over  $A$ , so

$$e' \downarrow_A b_\emptyset \quad \text{and} \quad e \in \text{acl}^{\text{eq}}(A, e', b_\emptyset).$$

Hence, the tuple  $b$  is algebraic over  $Ae'b_\emptyset$  and

$$\text{acl}^{\text{eq}}(A, e') \cap \text{acl}^{\text{eq}}(b_\emptyset) \subset \text{acl}^{\text{eq}}(a') \cap \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\emptyset).$$

Therefore  $\text{acl}^{\text{eq}}(A, e') \cap \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\emptyset)$ , by Fact 2.4.

Notice that  $\text{stp}(b/A, e')$  is almost  $\mathbb{P}$ -internal, yet this does not yield any contradiction since  $U(b_\emptyset/A, e') = U(b_\emptyset/a')$ . Choose now  $b'$  realizing  $\text{stp}(b/A, e')$  independent from  $b$  over  $A, e'$ . An easy forking computation yields

$$\text{acl}^{\text{eq}}(b') \cap \text{acl}^{\text{eq}}(b) = \text{acl}^{\text{eq}}(\emptyset).$$

By the hypothesis we have that the almost  $\mathbb{P}$ -internal type

$$\text{stp}(b'/\text{acl}^{\text{eq}}(A, e')) = \text{stp}(b/\text{acl}^{\text{eq}}(A, e'))$$

is good, so we deduce that  $\text{stp}(\text{Cb}(b/A, e')/b')$  is almost  $\mathbb{P}$ -internal. Remark that  $b$  is algebraic over  $\text{Cb}(b/A, e', b_\emptyset)$  and thus also algebraic over  $b_\emptyset \text{Cb}(b/A, e')$ .

Putting all of the above together, we conclude that the type  $\text{stp}(b/b')$  is almost  $\mathbb{P}$ -internal. Since

$$U(b_\emptyset/b') \geq U(b_\emptyset/A, e', b') = U(b_\emptyset/A, e') = U(b_\emptyset/a'),$$

we deduce by the maximality of  $U(b_\emptyset/a')$  that  $U(b_\emptyset/b') = U(b_\emptyset/A, e', b')$ , that is,

$$b_\emptyset \downarrow_{\text{acl}^{\text{eq}}(A, e') \cap \text{acl}^{\text{eq}}(b')} A, e', b'.$$

Hence  $b_\emptyset \downarrow b'$ , so  $b \downarrow b'$ , by Fact 2.4, contradicting that  $\text{stp}(b)$  is not almost  $\mathbb{P}$ -internal.

For the other direction, we need to show that the almost  $\mathbb{P}$ -internal type  $\text{stp}(a/b)$  is good, that is, that  $\text{stp}(\text{Cb}(a/b)/a)$  is almost  $\mathbb{P}$ -internal. We may assume that  $b$  equals the canonical base  $\text{Cb}(a/b)$ . Superstability yields that  $b$  is contained in the algebraic closure of finitely many  $b$ -conjugates of  $a$ . By transfer of internality to intersections, the type  $\text{tp}(a/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  is almost  $\mathbb{P}$ -internal, so it follows that

$$\text{tp}(b/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$$

is almost  $\mathbb{P}$ -internal. Hence, the type  $\text{stp}(b/a)$  is almost  $\mathbb{P}$ -internal, as desired.  $\dashv$

It follows now from Remark 2.2 that transfer of internality to intersections does not depend on constants being named.

**COROLLARY 2.6.** *Transfer of internality to intersections is invariant under naming and forgetting parameters.*

**REMARK 2.7.** By Remark 2.2 and Proposition 2.5, the CBP is equivalent to the property that whenever  $b = \text{Cb}(a/b)$ , then  $\text{tp}(b/\text{acl}^{\text{eq}}(a) \cap \text{acl}^{\text{eq}}(b))$  is almost  $\mathbb{P}$ -internal, which was already shown in [4, Theorem 2.1].

We would like to express our gratitude to the anonymous referee for pointing out that a close inspection of the proofs of [13, Theorem 3.7 and Proposition 4.2] yields that every stationary type generated by a pair of fibrations whose fibres are almost  $\mathbb{P}$ -internal is itself almost  $\mathbb{P}$ -internal (that is, the theory transfers internality to quotients), whenever every stationary almost  $\mathbb{P}$ -internal type is special. We will however provide a direct proof for the sake of completeness.

**PROPOSITION 2.8.** *The theory  $T$  transfers internality to quotients if and only if every stationary almost  $\mathbb{P}$ -internal type in  $T^{\text{eq}}$  is special.*

**PROOF.** Assume that every stationary almost  $\mathbb{P}$ -internal type is special. We want to show that

$$\text{tp}(b/\text{acl}^{\text{eq}}(a_1) \cap \text{acl}^{\text{eq}}(a_2))$$

is almost  $\mathbb{P}$ -internal, whenever both  $\text{stp}(b/a_1)$  and  $\text{stp}(b/a_2)$  are. By Remark 2.2, we may assume that

$$\text{acl}^{\text{eq}}(a_1) \cap \text{acl}^{\text{eq}}(a_2) = \text{acl}^{\text{eq}}(\emptyset).$$

Note that the type  $\text{stp}(b)$  is  $\mathbb{P}$ -analysable, by Fact 2.4, so recall that  $b_\emptyset$  is the maximal almost  $\mathbb{P}$ -internal subset of  $\text{acl}^{\text{eq}}(b)$ . As in the proof of Proposition 2.5 there is a set of parameters  $A_1$  containing  $a_1$  such that  $A_1 \downarrow_{a_1} b$  and  $b$  is interalgebraic over  $A_1$  with some tuple  $e$  of realizations of types (each one based over  $A_1$ ) in  $\mathbb{P}$ . Choosing a maximal subtuple  $e'$  of  $e$  with  $e' \downarrow_{A_1} b_\emptyset$ , it follows that  $b$  is algebraic over  $b_\emptyset A_1 e'$  and that

$$\text{acl}^{\text{eq}}(b_\emptyset) \cap \text{acl}^{\text{eq}}(A_1, e') \subset \text{acl}^{\text{eq}}(a_1).$$

Hence

$$\text{acl}^{\text{eq}}(b) \cap \text{acl}^{\text{eq}}(A_1, e') \cap \text{acl}^{\text{eq}}(a_2) = \text{acl}^{\text{eq}}(\emptyset), \tag{*}$$

by Fact 2.4. Since the almost  $\mathbb{P}$ -internal type  $\text{stp}(b/A_1, e')$  is special, we have that

$$\text{stp}(\text{Cb}(b/A_1, e')/a_2)$$

is almost  $\mathbb{P}$ -internal. Therefore

$$\text{stp}(\text{Cb}(b/A_1, e')/\text{acl}^{\text{eq}}(A_1, e') \cap \text{acl}^{\text{eq}}(a_2))$$

is almost  $\mathbb{P}$ -internal by Remark 2.2 and Proposition 2.5. Since

$$b \quad \downarrow \quad A_1, e' \\ \text{Cb}(b/A_1, e'), b_\emptyset$$

and  $b$  is algebraic over  $b_\emptyset A_1 e'$ , the tuple  $b$  is algebraic over  $\text{Cb}(b/A_1, e') b_\emptyset$ . In particular, the type

$$\text{stp}(b/\text{acl}^{\text{eq}}(A_1, e') \cap \text{acl}^{\text{eq}}(a_2))$$

is almost  $\mathbb{P}$ -internal and hence so is  $\text{stp}(b)$  because of  $(\star)$ .

In order to prove the other direction, we want to show that the almost  $\mathbb{P}$ -internal type  $\text{stp}(a/b)$  is special. Fix  $C$  some a set of parameters such that  $\text{stp}(a/C)$  is almost  $\mathbb{P}$ -internal. By transfer of internality to quotients, the type

$$\text{stp}(a/\text{acl}^{\text{eq}}(b) \cap \text{acl}^{\text{eq}}(C))$$

is almost  $\mathbb{P}$ -internal and so is

$$\text{stp}(\text{Cb}(a/b)/\text{acl}^{\text{eq}}(b) \cap \text{acl}^{\text{eq}}(C)),$$

since the canonical base  $\text{Cb}(a/b)$  is algebraic over finitely many  $b$ -conjugates of  $a$ . ⊣

We deduce now the analog of Corollary 2.6 for transfer of internality to quotients.

**COROLLARY 2.9.** *Transfer of internality to quotients is invariant under naming and forgetting parameters.*

The equivalence of the CBP and the UCBP motivates the following question, after localizing to almost  $\mathbb{P}$ -internal types.

**QUESTION 1.** *Are transfer of internality to intersections and to quotients equivalent properties for theories of finite rank?*

At the moment of writing, we do not know whether the previous question has a positive answer. Note that providing a structure which answers negatively the above question means in particular a new theory of finite rank without the CBP, since we will see in Section 4 that the so far only known counterexample to the CBP given in [8] does not transfer internality to intersections.

It was remarked in [2, Lemma 2.11] that the CBP holds whenever it holds for stationary real types, or equivalently, for real types over models. A natural question is whether the same holds for the above properties of transfer of internality.

**QUESTION 2.** *Does a theory of finite rank transfer internality to intersections, resp. to quotients, if every stationary real almost  $\mathbb{P}$ -internal type is good, resp. special?*

Additive covers of the algebraically closed field  $\mathbb{C}$ , which will be introduced in the following section, will provide a negative answer (see Corollary 6.5) to Question 2.

**§3. Additive covers.** The only known example so far of a stable theory of finite rank without the CBP appeared in [8]. We will consider this example from the perspective of additive covers of the algebraically closed field  $\mathbb{C}$ , which are based on the general notion of covers appearing in [1, 6, 7].

From now on, given the canonical projection of the sort  $S = \mathbb{C} \times \mathbb{C}$  onto the first coordinate  $P = \mathbb{C}$ , we will denote the elements of  $P$  with the greek letters  $\alpha, \beta$ , etc., while the elements of  $S$  will be seen accordingly as pairs  $(\alpha, a')$  and so on.

**DEFINITION 3.1.** An *additive cover* of the algebraically closed field  $\mathbb{C}$  is a structure  $\mathcal{M} = (P, S, \pi, \star, \oplus, \dots)$  with the distinguished sorts  $P = \mathbb{C}$  and  $S = \mathbb{C} \times \mathbb{C}$  such that the following conditions hold:

- The sort  $P$  carries the full field structure.
- The map  $\pi : S \rightarrow P$  is the projection onto the first coordinate.
- The map  $\star : P \times S \rightarrow S$  defines an action of  $P$  on  $S$  given by

$$\alpha \star (\beta, b') = (\beta, b' + \alpha).$$

- The map  $\oplus : S \times S \rightarrow S$  is a group homomorphism given by

$$((\alpha, a'), (\beta, b')) \mapsto (\alpha + \beta, a' + b').$$

- The structure  $\mathcal{M}$  is a (definitional) reduct of the *full structure*  $(\mathbb{C}, \mathbb{C} \times \mathbb{C})$ , which is equipped with both projections, so every  $\emptyset$ -definable set in  $\mathcal{M}$  is  $\emptyset$ -definable in  $(\mathbb{C}, \mathbb{C} \times \mathbb{C})$ .

**EXAMPLE 3.2.**

- The full structure  $\widetilde{\mathcal{M}} = (\mathbb{C}, \mathbb{C} \times \mathbb{C})$  is an additive cover.
- The additive cover  $\mathcal{M}_0 = (P, S, \pi, \star, \oplus)$  with no additional structure.
- The additive cover  $\mathcal{M}_1 = (P, S, \pi, \star, \oplus, \otimes)$  with the product

$$\begin{aligned} \otimes : \quad S \times S &\quad \rightarrow \quad S \\ ((\alpha, a'), (\beta, b')) &\quad \mapsto \quad (\alpha\beta, \alpha b' + \beta a'). \end{aligned}$$

Note that  $(S, \oplus, \otimes)$  is a commutative ring with multiplicative neutral element  $(1, 0)$ . The zero-divisors are exactly the elements  $a$  in  $S$  with  $\pi(a) = 0$ , that is, the pairs  $a = (0, a')$ .

Given an additive cover  $\mathcal{M}$ , there is a canonical embedding

$$\begin{aligned} \text{Aut}(\mathcal{M}/P) &\quad \leftrightarrow \quad \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ additive}\} \\ \sigma &\quad \mapsto \quad F_\sigma \end{aligned}$$

uniquely determined by the identity  $\sigma(x) = F_\sigma(\pi(x)) \star x$ . Note that  $F_\sigma$  is well-defined and the identity

$$\sigma((\alpha, a')) = (\alpha, a' + F_\sigma(\alpha)) \tag{\odot}$$

holds for all  $(\alpha, a')$  in  $S$ .

For the additive cover  $\mathcal{M}_0$  of Example 3.2, the above embedding defines a bijection

$$\text{Aut}(\mathcal{M}_0/P) \leftrightarrow \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ additive}\}$$



and a straight-forward calculation yields that

$$\text{Aut}(\mathcal{M}_1/P) \leftrightarrow \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ derivation}\}.$$

Indeed, for elements  $a = (\alpha, a')$  and  $b = (\beta, b')$  in  $S$ , we have

$$\begin{aligned} \sigma(a \otimes b) &= F_\sigma(\alpha\beta) \star (a \otimes b) \text{ and} \\ \sigma(a) \otimes \sigma(b) &= (F_\sigma(\alpha) \star a) \otimes (F_\sigma(\beta) \star b) = (\alpha\beta, \alpha(b' + F_\sigma(\beta)) + \beta(a' + F_\sigma(\alpha))) \\ &= (\alpha F_\sigma(\beta) + \beta F_\sigma(\alpha)) \star (a \otimes b). \end{aligned}$$

REMARK 3.3. Every additive cover  $\mathcal{M}$  is a saturated uncountably categorical structure, where  $P$  is the unique strongly minimal set up to non-orthogonality. The sort  $S$  has Morley rank two and degree one, and is  $P$ -analysable in two steps. Moreover, each fiber  $\pi^{-1}(\alpha)$  is strongly minimal.

Therefore, for additive covers, almost  $\mathbb{P}$ -internality in the CBP is equivalent to almost internality to  $P$ . If  $S$  is almost  $P$ -internal, then the CBP trivially holds.

REMARK 3.4. The counterexample to the CBP given in [8] is an additive cover, including for every irreducible variety  $V$  defined over  $\mathbb{Q}^{\text{alg}}$  a predicate in the sort  $S$  for the tangent bundle of  $V$ . It is easy to see that this structure has the same definable sets as the additive cover  $\mathcal{M}_1$  given in Example 3.2, since every polynomial expression over  $\mathbb{Q}^{\text{alg}}$  in  $P$  lifts to a polynomial equation in  $S$ , using the ring operations  $\oplus$  and  $\otimes$ .

A key ingredient in the proof that the sort  $S$  in the above counterexample is not almost  $P$ -internal [8, Corollary 3.3] is that every derivation on the algebraically closed field  $\mathbb{C}$  induces an automorphism in  $\text{Aut}(\mathcal{M}_1/P)$ .

For the following sections, we will need some auxiliary lemmas on the structure of additive covers, and particularly those where the sort  $S$  is not almost  $P$ -internal. However, note that there are additive covers whose sort  $S$  is  $P$ -internal, besides the full structure  $\widetilde{\mathcal{M}}$ : Consider the additive cover

$$\mathcal{M} = (P, S, \pi, \star, \oplus, R),$$

where the relation  $R$  on  $S^2 \times P$  is given by

$$R((\alpha, a'), (\beta, b'), \gamma) \iff (\beta, b') = (\gamma \cdot \alpha, \gamma \cdot a').$$

Note that the sort  $S$  is  $P$ -algebraic (actually  $P$ -definable), after naming any element in  $S$  with non-zero projection onto the first coordinate (e.g., the element  $(1, 0)$ ). Furthermore, the identity  $(\odot)$  yields that

$$\text{Aut}(\mathcal{M}/P) \cong \{F : \mathbb{C} \rightarrow \mathbb{C} \mid F(\alpha) = \mu \cdot \alpha, \mu \in \mathbb{C}\} \cong (\mathbb{C}, +),$$

so the additive cover  $\mathcal{M}$  is not equal to  $\widetilde{\mathcal{M}}$ , which has a trivial automorphism group over  $P$ .

The following notion will be helpful in the following chapter.

DEFINITION 3.5. Given elements  $a_1 = (\alpha, a'_1), \dots, a_n = (\alpha, a'_n)$  of  $S$  all in the same fiber  $\pi^{-1}(\alpha)$ , their *average* is the element

$$\left(\alpha, \frac{a'_1 + \dots + a'_n}{n}\right).$$

LEMMA 3.6. *Given a non-empty finite set  $A$  of elements of  $S$ , all lying in the same fiber, every automorphism  $\sigma$  of the additive cover maps the average of  $A$  to the average of  $\sigma[A]$ . In particular, the average of  $A$  is definable over the canonical parameter of  $A$ .*

PROOF. We proceed by induction on the size  $n$  of the non-empty set  $A$ . For  $n = 1$ , there is nothing to prove. Assume  $A$  contains at least two elements, and choose  $a$  some element of  $A$ . Set  $b = \sigma(a)$ . Inductively, we have that  $\sigma$  maps the average  $d_1$  of  $A \setminus \{a\}$  to the average  $d_2$  of  $\sigma[A] \setminus \{b\}$ . Let  $\varepsilon_1$  and  $\varepsilon_2$  be the unique elements in  $P$  such that  $\varepsilon_1 \star d_1 = a$  and  $\varepsilon_2 \star d_2 = b$ . A straight-forward computation yields that  $\frac{\varepsilon_1}{n} \star d_1$ , resp.  $\frac{\varepsilon_2}{n} \star d_2$ , is the average of  $A$ , resp. of  $\sigma[A]$ . Now the claim follows since  $\sigma$  maps  $\frac{\varepsilon_1}{n}$  to  $\frac{\varepsilon_2}{n}$ .  $\dashv$

LEMMA 3.7. *Let  $a_1 = (\alpha_1, 0), \dots, a_n = (\alpha_n, 0)$  be elements in  $S$ . The type  $tp(a_1, \dots, a_n/\alpha_1, \dots, \alpha_n)$  is stationary.*

PROOF. Since  $RM(a_i/\bar{\alpha}) = 1$ , where  $\bar{\alpha} = (\alpha_1, \dots, \alpha_n)$ , we may assume after possibly reordering that  $RM(a_1, \dots, a_m/\bar{\alpha}) = m = RM(a_1, \dots, a_n/\bar{\alpha})$ . In particular, the realization  $a_i$  of the strongly minimal type  $tp(a_i/\alpha_i)$  is algebraic over  $\bar{\alpha}, \hat{a}$ , where  $\hat{a} = (a_1, \dots, a_m)$ . Set  $b_i = (\alpha_i, b'_i)$  the average of the finite set of  $\{\bar{\alpha}, \hat{a}\}$ -conjugates of  $a_i$ . The element  $b_i$  is definable over  $\bar{\alpha}, \hat{a}$ , by Lemma 3.6.

CLAIM. *The second coordinate  $b'_i$  of the average  $b_i$  is definable (as an element of  $P$ ) over  $\bar{\alpha}$ .*

PROOF OF THE CLAIM. We need only show that  $b'_i$  is fixed by every automorphism  $\tau$  of the sort  $P$  fixing  $\bar{\alpha}$ . The map  $\sigma = (\tau, \tau \times \tau)$  is an automorphism of the full structure  $\hat{\mathcal{M}}$ , and hence of the reduct  $\mathcal{M}$ . Since  $\tau(0) = 0$ , the automorphism  $\sigma$  fixes  $\bar{\alpha}, a_1, \dots, a_n$ . Hence  $\sigma(b_i) = b_i$ , so in particular  $\tau(b'_i) = b'_i$ .  $\dashv$ Claim.

Therefore  $a_i = (-b'_i) \star b_i$  is definable over  $\bar{\alpha}, \hat{a}$ . Since the fibers of the projection  $\pi$  are strongly minimal (see Remark 3.3), the type  $tp(\hat{a}/\bar{\alpha})$  is stationary, so we obtain the desired conclusion.  $\dashv$

The above Lemma (and its proof) yields in particular the following result.

LEMMA 3.8.

- (a) *For every automorphism  $\tau$  of  $P$  fixing  $A$  pointwise the map  $\sigma = (\tau, \tau \times \tau)$  is an automorphism of the additive cover  $\mathcal{M}$  which fixes all the elements of  $S$  of the form  $(\alpha, 0)$ , with  $\alpha$  in  $A$ .*
- (b) *The definable and algebraic closure of  $P$  in the sort  $S$  coincide. Moreover, for every set  $B$  of parameters*

$$S \cap acl(B, P) = S \cap dcl(B, P).$$

- (c) *Given a set of parameters  $A$  in  $S$  and an element  $\beta$  in the sort  $P$ , all elements of the strongly minimal fiber  $\pi^{-1}(\beta)$  have the same type over  $A, P$  whenever the element  $b = (\beta, 0)$  of  $S$  is not algebraic over  $A, P$ .*

PROOF. Note that (a) is immediate since  $\sigma$  is an automorphism of the full structure  $(\mathbb{C}, \mathbb{C} \times \mathbb{C})$  and every  $\emptyset$ -definable set in  $\mathcal{M}$  is  $\emptyset$ -definable in  $(\mathbb{C}, \mathbb{C} \times \mathbb{C})$ . For (b), let  $a$  be an element in  $S$  algebraic over  $B, P$ . The average  $b$  of the finite set of  $B, P$ -conjugates of  $a$  is definable over  $B, P$ , by Lemma 3.6. Since  $\pi(b) = \pi(a)$  and

the definable action  $\star$  of  $P$  is transitive on each fiber, it follows that  $a$  is definable over  $B, P$ , as desired. For (c), let  $b_1$  and  $b_2$  be two elements in the fiber of  $\beta$  with distinct types over  $A, P$ . Strong minimality of the fiber implies that either  $b_1$  or  $b_2$  is algebraic over  $A, P$ . Now  $b = \gamma_1 \star b_1 = \gamma_2 \star b_2$  for some  $\gamma_1$  and  $\gamma_2$  in  $P$ , so  $b$  is algebraic over  $A, P$ .  $\dashv$

**LEMMA 3.9.** *Let  $a_1 = (\alpha_1, 0), \dots, a_n = (\alpha_n, 0)$  be elements in the sort  $S$  with generic independent elements  $\alpha_i$  in  $P$  (over the empty set). If the sort  $S$  is not almost  $P$ -internal, then the  $a_i$ 's are generic independent, that is, the rank  $RM(a_1, \dots, a_n)$  equals  $2n$ .*

**PROOF.** Choose some generic element  $\beta$  in  $P$  independent from  $\alpha_1$  and set  $a = \beta \star a_1 = (\alpha_1, \beta)$ . Note that the Morley rank of  $a$  is two. If  $a_1$  were not generic, then  $a_1$  must be algebraic over the generic element  $\alpha_1$  of  $P$ . Since  $a = \beta \star a_1$ , it would follow that the generic element  $a$  of  $S$  is algebraic over  $P$ , which contradicts our assumption that the sort  $S$  is not almost  $P$ -internal. Hence  $a_1$  is generic in  $S$ .

Now, we inductively assume that the tuple  $\bar{a}_{<n}(a_1, \dots, a_{n-1})$  consists of generic independent elements and want to show that  $a_n \downarrow \bar{a}_{<n}$ . Assume otherwise that  $a_n \not\downarrow \bar{a}_{<n}$ . Note that  $\alpha_n$  is not algebraic over  $\bar{a}_{<n}$ , by Lemma 3.8(a), since  $\alpha_n$  is not algebraic over  $\bar{\alpha}_{<n}$ . Thus  $a_n \not\downarrow_{\alpha_n} \bar{a}_{<n}$ , so  $a_n$  is algebraic over  $\alpha_n \bar{a}_{<n}$ . Choose now some element  $\gamma$  in  $P$  generic over  $(\alpha_1, \dots, \alpha_n)$  and set  $c = (\alpha_n, \gamma) = \gamma \star a_n$  in  $S$ . Note that  $c$  is algebraic over  $\bar{a}_{<n}P$ . Observe that  $RM(c/\bar{a}_{<n}) = 2$ , by the choice of  $\gamma$ , so we conclude that  $S$  is almost  $P$ -internal, which gives the desired contradiction.  $\dashv$

We conclude this section with a full description of binding groups of stationary  $P$ -internal types in additive covers, whenever the sort  $S$  is not almost  $P$ -internal. We consider first the case of real stationary  $P$ -internal types. Let  $(a, \gamma)$  be a realization of such a type  $q$  over  $B$ , where  $a = (a_1, \dots, a_n)$  is a tuple of elements in  $S$  and  $\gamma$  is a tuple in  $P$ . The canonical embedding

$$\text{Aut}(\mathcal{M}/P) \hookrightarrow \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ additive}\}$$

yields that the binding group  $\text{Aut}(q/P, B)$  with its action on the set  $\mathcal{Q}$  of realizations of  $q$  is naturally isomorphic to the group action of

$$G = \left\{ (g_1, \dots, g_n) \in P^n \mid \bigwedge_{i=1}^n F_{\sigma}(a_i) = g_i \text{ for some } \sigma \in \text{Aut}(\mathcal{M}/P, B) \right\}$$

on  $\mathcal{Q}$  given by  $\star$ . Note that  $G$  is definable, for it equals

$$\{(g_1, \dots, g_n) \in P^n \mid g_1 \star a_1, \dots, g_n \star a_n \equiv_{B,P} a_1, \dots, a_n\}.$$

Moreover, the type  $q$  is *fundamental*, meaning that every realization of  $q$  is algebraic over  $c, P$  for some (every) realization  $c$  of  $q$ , by the following lemma.

**LEMMA 3.10.** *Suppose the sort  $S$  of the additive cover  $\mathcal{M}$  is not almost  $P$ -internal. If the type  $tp(a/B)$  for a tuple  $a = (a_1, \dots, a_n)$  of elements in  $S$  is almost  $P$ -internal, then each projection  $\pi(a_i)$  is algebraic over  $B$ .*

**PROOF.** Clearly, it suffices to show that, whenever the real type  $\text{stp}(a/B)$  is almost  $P$ -internal, with  $a$  a single element in  $S$ , then  $\alpha = \pi(a)$  is algebraic over  $B$ . Choose a set of parameters  $B_1$  with  $B_1 \downarrow_B a$  such that  $a$  is algebraic over  $B_1, P$ . We need only show that  $\alpha$  is algebraic over  $B_1$ . Otherwise, choose an element  $a_1$  of  $S$  in the

fiber of  $\alpha$  generic over  $B_1$ . The elements  $a$  and  $a_1$  are interdefinable over  $P$ , so  $a_1$  is algebraic over  $B_1, P$ , contradicting that  $S$  is not almost  $P$ -internal.  $\dashv$

More generally, given now an imaginary  $e = f(a, \gamma)$ , where again  $a = (a_1, \dots, a_n)$  is a tuple of elements in  $S$ , the tuple  $\gamma$  is in  $P$  and  $f$  is an  $\emptyset$ -interpretable function, the binding group of the stationary  $P$ -internal type  $\text{tp}(e/B)$  equals

$$\{(g_1, \dots, g_n) \in P^n \mid f(g_1 \star a_1, \dots, g_n \star a_n, \gamma) \equiv_{B,P} e\}.$$

In order to give a complete description of binding groups appearing in additive covers, we introduce the following notation: Given a tuple  $a = (a_1, \dots, a_n)$  in  $S^n$  and  $G$  a definable subgroup of  $(P^n, +)$ , we will consider the definable set

$$G \star a = \left\{ (x_1, \dots, x_n) \in S^n \mid \bigwedge_{i=1}^n g_i \star a_i = x_i \text{ for some } (g_1, \dots, g_n) \in G \right\}$$

with canonical parameter  $\ulcorner G \star a \urcorner$ . Note that  $G \star a$  is a coset of the additive subgroup  $G \star 0_{S^n}$  of  $(S^n, \oplus)$ .

PROPOSITION 3.11. *With the above notation,*

$$\sigma(G \star a) = G \star a \iff \sigma(a) \in G \star a$$

for every automorphism  $\sigma$  in  $\text{Aut}(\mathcal{M}/P)$ . The stationary type  $q = \text{stp}(a/\ulcorner G \star a \urcorner)$  is  $P$ -internal and fundamental. Its binding group is a subgroup (with the above identifications in mind) of  $G$ . Moreover, it is equal to  $G$  whenever  $S$  is not almost  $P$ -internal and  $a$  is a tuple of generic independent elements in  $S$ .

Thus, whenever  $S$  is not  $P$ -internal, every definable subgroup of  $(\mathbb{C}^n, +)$  appears (in the sense discussed above) as a binding group and conversely every binding group is (naturally isomorphic to) such a subgroup.

PROOF. The equivalence in the statement holds trivially, since  $G \star a$  is a coset of the  $P$ -definable group  $G \star 0_{S^n}$ .

The type  $q$  is clearly  $P$ -internal and fundamental, because  $\pi(a) = (\alpha_1, \dots, \alpha_n)$  is definable over  $\ulcorner G \star a \urcorner$ . Given a realization  $b$  of  $q$ , it lies in  $G \star a$ , so  $b = g \star a$ , for some  $g$  in  $G$ . With the above identification, we deduce that the binding group of  $q$  is a subgroup of  $G$ . In case  $a_1, \dots, a_n$  are generic independent, we have equality of these two groups: the element  $a_k$ , for  $1 \leq k \leq n$ , is not algebraic over  $\bar{a}_{<k}, P$ , since  $S$  is not almost  $P$ -internal, so Lemma 3.8(c) yields that all elements in the fiber  $\pi^{-1}(\alpha_k)$  have the same type over  $\bar{a}_{<k}, P$ . For any  $g = (g_1, \dots, g_n)$  in  $G$  we can recursively construct an automorphism  $\sigma$  in  $\text{Aut}(\mathcal{M}/P)$  with  $\sigma(a_k) = g_k \star a_k$  for  $1 \leq k \leq n$ . By the previous equivalence, the automorphism  $\sigma$  permutes  $G \star a$  and therefore induces an automorphism in the binding group of  $q$ .

That every binding group is of the claimed form follows from the discussion before the proposition. Let us now conclude by showing that every definable subgroup  $G$  of  $(\mathbb{C}^n, +)$  appears as a binding group, if  $S$  is not almost  $P$ -internal. Choose a generic independent tuple  $a$  in  $S^n$ . By the above, the binding group of the  $P$ -internal type  $\text{stp}(a/\ulcorner G \star a \urcorner)$  equals  $G$ .  $\dashv$

**§4. Imaginaries in additive covers.** In order to answer Question 2, we are led to study imaginaries in additive covers, with a particular focus to the additive covers

in the Example 3.2. We will first show that neither the counterexample  $\mathcal{M}_1$  to the CBP of [8] nor the additive cover  $\mathcal{M}_0$  eliminate imaginaries.

LEMMA 4.1. *Let  $\mathcal{M}$  be an additive cover such that for every derivation  $D$  on  $\mathbb{C}$  the induced map  $\sigma_D : \mathcal{M} \rightarrow \mathcal{M}$  which is the identity on  $P$  and maps the element  $a$  of  $S$  to  $D(\pi(a)) \star a$  is an automorphism of  $\mathcal{M}$ . Then  $\mathcal{M}$  does not eliminate imaginaries.*

PROOF. Set  $\alpha = \pi(a)$  and  $\beta = \pi(b)$ , for two generic independent elements  $a$  and  $b$  in the sort  $S$ . Fix a derivation  $D$  with kernel  $\mathbb{Q}^{\text{alg}}$  and let  $G$  be the definable subgroup of  $(P^2, +)$  given by  $D(\beta)x - D(\alpha)y = 0$ . Let us assume for a contradiction that the definable set  $G \star (a, b)$  has a real canonical parameter  $e$ . By hypothesis, the induced map  $\sigma_D$  is in  $\text{Aut}(\mathcal{M}/P)$ . By Proposition 3.11, the automorphism  $\sigma_D$  permutes the set  $G \star (a, b)$ , since  $\sigma_D(a, b) = (D(\alpha) \star a, D(\beta) \star b)$  lies in  $G \star (a, b)$  because the tuple  $(D(\alpha), D(\beta))$  is an element of  $G$ . By the choice of the derivation  $D$ , every real element fixed by  $\sigma_D$  belongs to the set  $P \cup \pi^{-1}(\mathbb{Q}^{\text{alg}})$ . In particular, we deduce that the tuple  $e$  lies in this set. Thus, the definable set  $G \star (a, b)$  is permuted by every automorphism induced by a derivation, because every derivation vanishes on  $\mathbb{Q}^{\text{alg}}$  (note that we no longer need that the kernel of the derivation is exactly  $\mathbb{Q}^{\text{alg}}$ ). Choose now a derivation  $D_1$  with  $D_1(\alpha) = 1$  and  $D_1(\beta) = 0$ , and note that  $\sigma_{D_1}$  does not permute  $G \star (a, b)$ , since  $(1, 0)$  is not an element of  $G$ , which gives the desired contradiction.  $\dashv$

The proof of [8, Corollary 3.3] shows that the sort  $S$  in an additive cover  $\mathcal{M}$  is not almost  $P$ -internal, whenever every derivation on  $\mathbb{C}$  induces (in the sense of Lemma 4.1) an automorphism in  $\text{Aut}(\mathcal{M}/P)$ . We will now give a strengthening of the Lemma 4.1.

PROPOSITION 4.2. *If the additive cover  $\mathcal{M}$  eliminates imaginaries, then the sort  $S$  is  $P$ -internal.*

PROOF. We will mimic the proof of Lemma 4.1. Assume for a contradiction that the sort  $S$  is not  $P$ -internal and choose two generic independent elements  $a$  and  $b$  in  $S$ . Consider now the definable set  $G \star (a, b)$  where  $G$  is given by the equation  $F_\sigma(\beta)x - F_\sigma(\alpha)y = 0$ , for some suitable automorphism  $\sigma$  to be determined later on. If we can construct the automorphism  $\sigma$  (which plays the role of  $\sigma_D$  in the proof of Lemma 4.1) in  $\text{Aut}(\mathcal{M}/P)$  such that the only real elements fixed by  $\sigma$  are those definable over  $P$ , we conclude as before that the real canonical parameter  $e$  of the definable set  $G \star (a, b)$  is definable over  $P$ . This yields immediately the desired contradiction, if  $S$  were not  $P$ -internal, for there is an automorphism  $\tau$  in  $\text{Aut}(\mathcal{M}/P)$  which fixes  $b$  and moves  $a$ , but  $\tau(a, b) = (F_\tau(\alpha) \star a, b)$  does not belong to  $G \star (a, b)$ , since  $(0, 0) \neq (F_\tau(\alpha), 0)$  is not in  $G$ , since  $F_\sigma(\beta) \cdot F_\tau(\alpha) \neq 0$ .

Hence, we need only show in the rest of the proof that there exists such an automorphism  $\sigma$ .

Choose a transcendence basis  $\bar{\alpha} = (\alpha_i)_{i < 2^{\aleph_0}}$  of the algebraically closed field  $\mathbb{C}$  and set

$$\bar{a} = (a_i)_{i < 2^{\aleph_0}} = ((\alpha_i, 0))_{i < 2^{\aleph_0}}$$

for  $i < 2^{\aleph_0}$ . We construct a (possibly empty) subtuple

$$\bar{b} = (b_i)_{i < \kappa} = ((\beta_i, 0))_{i < \kappa}$$

of  $P \times \{0\} = ((\varepsilon, 0))_{\varepsilon \in \mathbb{C}}$  in the following way: Assume  $(b_i)_{i < \lambda}$  has already been constructed. If some element  $(\varepsilon, 0)$  is not algebraic over  $\bar{a}, (b_i)_{i < \lambda}$ , then choose it least possible according to some fixed enumeration of  $\mathbb{C}$  and set  $b_\lambda = (\varepsilon, 0)$ . Otherwise set  $\kappa = \lambda$ .

Set  $\bar{\beta} = (\beta_i)_{i < \kappa}$ . For  $i < \kappa$ , we denote by  $\langle \alpha \rangle_i$  the unique subtuple of  $\bar{\alpha}$  of smallest length such that  $\beta_i$  is algebraic over  $\langle \alpha \rangle_i$ . Write  $\mathcal{X}$  for the set of all finite subtuples of  $\bar{\alpha}$  and consider the maps

$$\begin{aligned} \Phi : \quad \mathcal{X} \quad &\rightarrow \quad 2^{\aleph_0} \\ (\alpha_{i_1}, \dots, \alpha_{i_n}) &\mapsto \max(i_1, \dots, i_n) \end{aligned}$$

and

$$F : \{\alpha_i, \beta_j \mid i < 2^{\aleph_0}, j < \kappa\} \rightarrow \{\alpha_i \mid i < 2^{\aleph_0}\}$$

defined by

$$F(\alpha_i) = \alpha_{\omega^{i+1}} \quad \text{and} \quad F(\beta_j) = \alpha_{\omega^{\max(j, \Phi(\langle \alpha \rangle_j)) + 1} + \omega^j}.$$

It follows inductively from Lemma 3.8(c) that

$$\bar{a}, \bar{b} \equiv_P F(\bar{\alpha}) \star \bar{a}, F(\bar{\beta}) \star \bar{b},$$

so there is an automorphism  $\sigma$  in  $\text{Aut}(\mathcal{M}/P)$  with

$$\sigma(a_i) = F(\alpha_i) \star a_i \quad \text{and} \quad \sigma(b_j) = F(\beta_j) \star b_j.$$

CLAIM. *Every element in  $S$  which is fixed by  $\sigma$  is definable over  $P$ .*

PROOF OF THE CLAIM. Assume that the element  $(\gamma, c')$  of  $S$  is fixed by  $\sigma$ . Note that the automorphism  $\sigma$  also fixes  $c = (\gamma, 0)$ , for  $(\gamma, c') = c' \star c$ . Thus, it suffices to show that  $c$  is definable over  $P$ . By construction of the tuples  $\bar{a}$  and  $\bar{b}$ , the element  $c$  must be algebraic over  $\bar{a}, \bar{b}$  and therefore definable over  $\bar{a}, \bar{b}, P$ , by Lemma 3.8(b). Choose subtuples of least possible length

$$\hat{a} = (a_{i_1}, \dots, a_{i_m}) \text{ of } \bar{a} \quad \text{and} \quad \hat{b} = (b_{j_1}, \dots, b_{j_n}) \text{ of } \bar{b}$$

such that  $c$  is definable over  $\hat{a}, \hat{b}, P$ . We need to show that  $m = n = 0$ . In order to reach a contradiction, assume that  $\max(n, m) > 0$ . Note that every element in the fiber of  $\gamma$  is definable over  $\hat{a}, \hat{b}, P$ . The type

$$\text{tp}(\hat{a}, \hat{b}, c/\hat{\alpha}, \hat{\beta}, \gamma)$$

is fundamental and stationary by Lemma 3.7. Its binding group  $G$  is a definable additive subgroup of  $\mathbb{C}^{m+n+1}$ , by Proposition 3.11. Note that  $\gamma$  must be algebraic over  $\hat{\alpha}, \hat{\beta}$ : otherwise, Lemma 3.9 yields that  $c \perp \hat{a}, \hat{b}$ , so  $\text{stp}(c)$  is  $P$ -internal (for  $c$  is definable over  $\hat{a}, \hat{b}, P$ ) and hence so is  $S$ , contradicting our assumption.

Hence, the binding group  $G$  of  $p = \text{tp}(\hat{a}, \hat{b}, c/\hat{\alpha}, \hat{\beta}, \gamma)$  is definable over

$$A = \text{acl}(\hat{\alpha}, \langle \alpha \rangle_{j_1}, \dots, \langle \alpha \rangle_{j_n}) \supset \{\hat{\alpha}, \hat{\beta}\}.$$

By the discussion before Proposition 3.11, the group  $G$  is given by a system  $\mathcal{G}$  of linear equations of the form

$$\lambda_1 \cdot x_1 + \dots + \lambda_{m+n+1} \cdot x_{m+n+1} = 0,$$

with coefficients  $\lambda_i$  in  $A$  and the tuple

$$(F(\alpha_{i_1}), \dots, F(\alpha_{i_m}), F(\beta_{j_1}), \dots, F(\beta_{j_n}), 0)$$

is a solution, since

$$\sigma(a_i) = F(\alpha_i) \star a_i, \quad \sigma(b_j) = F(\beta_j) \star b_j \quad \text{and} \quad \sigma(c) = 0 \star c.$$

Set now  $\varepsilon = \Phi(\hat{\alpha} \wedge \langle \alpha \rangle_{j_1} \wedge \dots \wedge \langle \alpha \rangle_{j_n})$ . If  $\alpha_\varepsilon = \alpha_{i_k}$  for some  $1 \leq k \leq m$ , denote  $i(\varepsilon) = i_k = \varepsilon$ . Otherwise set  $i(\varepsilon) = j_\ell$  if  $1 \leq \ell \leq n$  is the least index such that  $\alpha_\varepsilon$  is an element in the tuple  $\langle \alpha \rangle_{j_\ell}$ .

Observe that there is a linear equation in the system  $\mathcal{G}$  such that the coefficient  $\lambda_{i(\varepsilon)}$  is non-trivial, for otherwise every automorphism in  $\text{Aut}(\mathcal{M}/P)$  fixing all coordinates except (possibly) the element  $d_{i(\varepsilon)}$ , which is the  $i(\varepsilon)$ <sup>th</sup>-coordinate of the tuple  $(\hat{a}, \hat{b})$ , must also fix  $c$ , contradicting the minimality of  $m$  and  $n$ . The set

$$B = \{F(\alpha_{i_1}), \dots, F(\alpha_{i_m}), F(\beta_{j_1}), \dots, F(\beta_{j_n})\}$$

consists of distinct elements, since  $F$  clearly is injective. Therefore, it suffices to show that the element  $F(d_{i(\varepsilon)})$  is not algebraic over

$$A \cup (B \setminus \{F(d_{i(\varepsilon)})\})$$

to reach the desired contradiction. For this we need only show that the element  $F(d_{i(\varepsilon)})$  is not contained in the set

$$\{\hat{\alpha}, \langle \alpha \rangle_{j_1}, \dots, \langle \alpha \rangle_{j_n}\}.$$

If  $d_{i(\varepsilon)} = \alpha_{i(\varepsilon)}$ , we obtain the result since

$$\begin{aligned} \Phi(F(d_{i(\varepsilon)})) &= \Phi(F(\alpha_\varepsilon)) \\ &= \Phi(\alpha_{\omega^{\varepsilon+1}}) \\ &= \omega^{\varepsilon+1} \geq \varepsilon + 1 \\ &> \varepsilon = \Phi(\hat{\alpha} \wedge \langle \alpha \rangle_{j_1} \wedge \dots \wedge \langle \alpha \rangle_{j_n}). \end{aligned}$$

Otherwise  $d_{i(\varepsilon)} = \beta_{i(\varepsilon)}$ , so

$$\begin{aligned} \Phi(F(d_{i(\varepsilon)})) &= \Phi(F(\beta_{i(\varepsilon)})) \\ &= \omega^{\max(i(\varepsilon), \Phi(\langle \alpha \rangle_{i(\varepsilon)})) + 1} + \omega^{i(\varepsilon)} \\ &> \omega^{\Phi(\langle \alpha \rangle_{i(\varepsilon)}) + 1} = \omega^{\varepsilon+1}, \end{aligned}$$

and we conclude the result analogous to the first case.

¬Claim.  
⊥

Whilst the additive cover does not eliminate imaginaries whenever the sort  $S$  is not  $P$ -internal, the situation is different for finite imaginaries. We will see below that the additive cover  $\mathcal{M}_0$  does not eliminate finite imaginaries, but the additive cover  $\mathcal{M}_1$  does.

REMARK 4.3. Choose two generic independent elements  $\alpha$  and  $\beta$  in the sort  $P$ . The finite subset  $\{(\alpha, 0), (\beta, 0)\}$  of  $S$  has no real canonical parameter in  $\mathcal{M}_0$ .

PROOF. Assume that the tuple  $e$  is a real canonical parameter of the set  $\{a, b\}$ , with  $a = (\alpha, 0)$  and  $b = (\beta, 0)$ . We will first show that every element  $c$  in  $S$  appearing in the real tuple  $e$  is definable over  $a \oplus b, P$ . Since the canonical embedding

$$\text{Aut}(\mathcal{M}_0/P) \leftrightarrow \{F : \mathbb{C} \rightarrow \mathbb{C} \text{ additive}\}$$

is an isomorphism (see Example 3.2), every additive map vanishing on  $\alpha$  and  $\beta$  must vanish on  $\pi(c)$ . We deduce that the projection  $\pi(c)$  is a linear combination of  $\alpha$  and  $\beta$  with rational coefficients, so  $\pi(c) = \lambda \cdot \alpha + \mu \cdot \beta$  for some rational numbers  $\lambda$  and  $\mu$ . Note that there is a field automorphism  $\tau$  sending  $\alpha$  to  $\beta$  and  $\beta$  to  $\alpha$ , so the automorphism  $\sigma = (\tau, \tau \times \tau)$  of Lemma 3.8(a) permutes  $a$  and  $b$ , hence it fixes  $c$ . Therefore,

$$\lambda \cdot \alpha + \mu \cdot \beta = \pi(c) = \sigma(\pi(c)) = \sigma(\lambda \cdot \alpha + \mu \cdot \beta) = \lambda \cdot \beta + \mu \cdot \alpha,$$

so  $\lambda = \mu$ , since  $\alpha$  and  $\beta$  are generic independent. We deduce that  $c$  is definable over  $a \oplus b, P$ , because every additive map vanishing on the element  $\alpha + \beta$  vanishes on  $\lambda \cdot (\alpha + \beta) = \pi(c)$ .

Choose now an additive map  $F$  with  $F(\alpha) = 1$  and  $F(\beta) = -1$ . The induced automorphism  $\sigma_F$  in  $\text{Aut}(\mathcal{M}/P)$  with  $\sigma_F(x) = F(\pi(x)) \star x$  for elements  $x$  in  $S$  fixes  $a \oplus b$ , because

$$\sigma_F(a \oplus b) = \sigma_F(a) \oplus \sigma_F(b) = (1 \star a) \oplus (-1 \star b) = (\alpha, 1) \oplus (\beta, -1) = a \oplus b.$$

Therefore  $\sigma_F$  fixes the real tuple  $e$ , which gives the desired contradiction, since  $\sigma_F$  does not permute the set  $\{(\alpha, 0), (\beta, 0)\}$ . ⊣

In order to show that the additive cover  $\mathcal{M}_1$  eliminates finite imaginaries, we first provide a sufficient condition.

PROPOSITION 4.4. *An additive cover  $\mathcal{M}$  eliminates finite imaginaries, whenever every finite subset of  $S$  on which  $\pi$  is injective has a real canonical parameter.*

PROOF. Let  $A$  be the finite set  $\{\bar{a}_1, \dots, \bar{a}_n\}$  of real  $m$ -tuples. Every function  $\Phi : \{1, \dots, m\} \rightarrow \{P, S\}$  determines a subset  $A_\Phi$  of  $A$ , according to whether the  $j^{\text{th}}$ -coordinate lies in  $P$  or  $S$ . Every automorphism permuting  $A$  permutes each  $A_\Phi$ , so we may assume that for every tuple in  $A$ , the coordinates have the same configuration (given by the function  $\Phi$ ).

Since the canonical parameter is only determined up to interdefinability, we may further assume (after possibly permuting the coordinates) that there is a natural number  $0 \leq k \leq m$  such that for each tuple  $\bar{a}_i$  in  $A$ :

- The  $j^{\text{th}}$ -coordinate  $a_i^j$  lies in  $S$  for  $1 \leq j \leq k$ .
- The  $\ell^{\text{th}}$ -coordinate  $a_i^\ell$  lies in  $P$  for  $k < \ell \leq m$ .

For every coordinate  $1 \leq j \leq k$  set  $A^j = \{a_i^j \mid 1 \leq i \leq n\}$  and  $d_i^j$  the average of the subset  $A^j \cap \pi^{-1}(\pi(a_i^j))$ . For  $1 \leq i \leq n$  let now  $\varepsilon_i^j$  be the unique element in  $P$  with  $a_i^j = \varepsilon_i^j \star d_i^j$ . Consider the tuples  $\varepsilon_i = (\varepsilon_i^1, \dots, \varepsilon_i^k)$  and

$$\alpha_i = (\pi(a_i^1), \dots, \pi(a_i^k), a_i^{k+1}, \dots, a_i^m)$$



in  $P$ . We need only show that the tuple

$$e = (\ulcorner \{(\varepsilon_1, \alpha_1), \dots, (\varepsilon_n, \alpha_n)\} \urcorner, \ulcorner \{d_1^1, \dots, d_n^1\} \urcorner, \dots, \ulcorner \{d_1^k, \dots, d_n^k\} \urcorner)$$

is a canonical parameter of  $A$ . Note that  $e$  is a real tuple since the sort  $P$  with the full field structure eliminates imaginaries and all the sets  $\{d_1^j, \dots, d_n^j\}$  have real canonical parameters, by our assumption.

Let  $\sigma$  be an automorphism. If  $\sigma$  permutes the set  $A$ , Lemma 3.6 yields that  $\sigma$  permutes each set  $\{d_1^j, \dots, d_n^j\}$  since the image of  $A^j \cap \pi^{-1}(\pi(a_i^j))$  under  $\sigma$  is  $A^j \cap \pi^{-1}(\pi(a_{i(\sigma)}^j))$  for some index  $i$  with  $\sigma(a_i^j) = a_{i(\sigma)}^j$  and  $\sigma(\alpha_i) = \alpha_{i(\sigma)}$ . Therefore  $\sigma(\varepsilon_i) = \varepsilon_{i(\sigma)}$ , since  $\sigma(d_i^j) = d_{i(\sigma)}^j$ . Hence  $\sigma$  fixes  $e$ .

Assume now that  $\sigma$  fixes the tuple  $e$ . The tuple  $\alpha_i$  is mapped to  $\alpha_{i(\sigma)}$  and

$$\sigma(a_i^j) = \sigma(\varepsilon_i^j) \star \sigma(d_i^j) = \varepsilon_{i(\sigma)}^j \star \sigma(d_i^j).$$

It suffices to show that  $\sigma(d_i^j) = d_{i(\sigma)}^j$  to conclude that  $\sigma$  permutes  $A$ . This follows immediately from

$$\pi(\sigma(d_i^j)) = \sigma(\pi(d_i^j)) = \sigma(\alpha_i^j) = \alpha_{i(\sigma)}^j,$$

since  $\sigma$  permutes the set  $\{d_1^j, \dots, d_n^j\}$ . ◻

Thus, we will deduce that the additive cover  $\mathcal{M}_1$  eliminates finite imaginaries, by applying Proposition 4.4, lifting the corresponding canonical parameters of finite subsets of  $P$  to  $S$  using the ring operations.

**COROLLARY 4.5.** *The additive cover  $\mathcal{M}_1$  eliminates finite imaginaries.*

**PROOF.** By Proposition 4.4, we need only show that every subset  $\{a_1, \dots, a_n\}$  of  $S$ , with pairwise distinct projections  $\pi(a_i) = \alpha_i$ , has a real canonical parameter.

For  $1 \leq i \leq n$  lift the  $i^{\text{th}}$ -symmetric function to  $S$ :

$$b_i = \sum_{1 \leq j_1 < \dots < j_i \leq n} a_{j_1} \otimes \dots \otimes a_{j_i}. \tag{♠}$$

We claim that the tuple  $b = (b_1, \dots, b_n)$  is a canonical parameter of the set  $A = \{a_1, \dots, a_n\}$ . If the automorphism  $\sigma$  permutes  $A$ , then it clearly fixes  $b$ . Assume now that  $\sigma$  fixes the tuple  $b$ . Since

$$\beta_i = \sum_{1 \leq j_1 < \dots < j_i \leq n} \alpha_{j_1} \dots \alpha_{j_i},$$

the tuple  $(\beta_1, \dots, \beta_n)$  encodes the finite set  $\{\alpha_1, \dots, \alpha_n\}$ . In particular the automorphism  $\sigma$  induces a permutation of the set  $\{1, \dots, n\}$  (which we will also denote by  $\sigma$ ) such that

$$\sigma(a_i) = \lambda_{\sigma(i)} \star a_{\sigma(i)}$$

for some  $\lambda_{\sigma(i)}$  in  $P$ . We need only show that each  $\lambda_j$  equals 0.

Write each element  $a_i$  of  $A$  as  $a_i = (\alpha_i, a'_i)$ , and similarly  $b_i = (\beta_i, b'_i)$ . In the full structure  $\widehat{\mathcal{M}}$  the definable condition  $(\spadesuit)$  is equivalent to

$$\beta_i = \sum_{1 \leq j_1 < \dots < j_i \leq n} \alpha_{j_1} \cdots \alpha_{j_i}$$

together with the system of linear equations:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \sum_{j \neq 1} \alpha_j & \sum_{j \neq 2} \alpha_j & \cdots & \sum_{j \neq n} \alpha_j \\ \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \neq 1}} \alpha_{j_1} \alpha_{j_2} & \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \neq 2}} \alpha_{j_1} \alpha_{j_2} & \cdots & \sum_{\substack{j_1 < j_2 \\ j_1, j_2 \neq n}} \alpha_{j_1} \alpha_{j_2} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{j \neq 1} \alpha_j & \prod_{j \neq 2} \alpha_j & \cdots & \prod_{j \neq n} \alpha_j \end{pmatrix} \begin{pmatrix} a'_1 \\ a'_2 \\ \vdots \\ a'_n \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$$

Since

$$b_i = \sigma(b_i) = \sum_{1 \leq j_1 < \dots < j_i \leq n} (\lambda_{\sigma^{-1}(j_1)} \star a_{j_1}) \otimes \cdots \otimes (\lambda_{\sigma^{-1}(j_i)} \star a_{j_i}),$$

we conclude that  $\lambda_1 = \dots = \lambda_n = 0$ , because the above matrix has determinant  $\prod_{i < j} (\alpha_i - \alpha_j) \neq 0$ . ⊣

**§5. The CBP in additive covers.** As already stated in Remark 3.4, the CBP does not hold in the additive cover  $\mathcal{M}_1$  (see Example 3.2). For the sake of completeness, we will now sketch a proof, using the terminology introduced so far. For generic independent elements  $a, b$  and  $c$  in  $S$ , set  $d = (a \otimes c) \oplus b$ . The canonical base  $\text{Cb}(c, d/a, b)$  is interdefinable with  $(a, b)$ , since the intersection of the two lines  $y = (a_1 \otimes x) \oplus b_1$  and  $y = (a_2 \otimes x) \oplus b_2$  with  $(a_1, b_1) \neq (a_2, b_2)$  has Morley rank at most one. Assuming the CBP, the type  $\text{stp}(a/c, d)$  is therefore almost  $P$ -internal. As the elements  $a, c$  and  $d$  are again (generic) independent, we conclude that the type  $\text{stp}(a)$  is almost  $P$ -internal, contradicting the fact that  $S$  is not almost  $P$ -internal.

The above is a lifting to the sort  $S$  of a configuration witnessing that the field  $P$  is not one-based. We will now present another proof that the additive cover  $\mathcal{M}_1$  does not have the CBP, using the so called group version of the CBP [8, Fact 1.3], which was already present in [10, Theorem 4.1].

**FACT 5.1.** *Let  $G$  be a definable group in a theory with the CBP. If the type  $p = \text{stp}(g/B)$  of an element  $g$  of  $G$  has finite stabilizer, then  $p$  is almost internal to the family of all non-locally modular minimal types.*

The failure of the group version of the CBP is another example of such a lifting approach: Set  $c = a \otimes b$  for two generic independent elements  $a$  and  $b$  of  $S$  and consider  $(a, b, c)$  as an element of the definable group  $(S^3, \oplus)$ . Note that elements  $(g_1, g_2, g_3)$  of the stabilizer of  $\text{stp}(a, b, c)$  must satisfy  $\alpha \cdot \pi(g_2) + \beta \cdot \pi(g_1) = \pi(g_3)$ . Assuming the tuple  $(g_1, g_2, g_3)$  to be independent from  $a, b, c$ , we deduce that  $0 = \pi(g_1) = \pi(g_2) = \pi(g_3)$ . Hence, there are  $\gamma_i$  in  $P$  with  $\gamma_i \star 0_S = g_i$  and we deduce further that  $\alpha \cdot \gamma_2 + \beta \cdot \gamma_1 = \gamma_3$ . The independence  $\gamma_1, \gamma_2, \gamma_3 \perp \alpha, \beta$  now

yields  $0 = \gamma_1 = \gamma_2 = \gamma_3$  and we conclude that the stabilizer of  $\text{stp}(a, b, c)$  is trivial. Therefore the above Fact 5.1 implies, assuming the CBP, that the sort  $S$  is almost  $P$ -internal, which is a contradiction.

Now we will see that the failure of the CBP in the additive cover  $\mathcal{M}_1$  is already determined by its automorphism group over  $P$ .

**PROPOSITION 5.2.** *If  $\mathcal{M}$  is an additive cover such that  $\text{Aut}(\mathcal{M}/P)$  corresponds to the group of derivations on  $\mathbb{C}$ , then the CBP does not hold in  $\mathcal{M}$ .*

**PROOF.** Choose two generic independent elements  $\alpha$  and  $\beta$  in  $P$  and consider the elements  $a = (\alpha, 0)$ ,  $b = (\beta, 0)$  and  $c = (\alpha \cdot \beta, 0)$  in  $S$ . We first show that  $c$  is definable over  $a, b$ . So, let  $\sigma$  be an automorphism of  $\mathcal{M}$  fixing  $a$  and  $b$ . The restriction  $\sigma|_P$  is a field automorphism and induces an automorphism  $\tilde{\sigma} = (\sigma|_P, \sigma|_P \times \sigma|_P)$  of  $\mathcal{M}$ , by Lemma 3.8(a). Note that the automorphism  $\tilde{\sigma}^{-1} \circ \sigma$  fixes  $P, a, b$ . By our assumption, it corresponds to a derivation  $D$  with  $D(\alpha) = 0 = D(\beta)$ . In particular

$$\tilde{\sigma}^{-1} \circ \sigma(c) = D(\alpha \cdot \beta) \star c = c,$$

so

$$\sigma(c) = \tilde{\sigma}((\tilde{\sigma}^{-1} \circ \sigma)(c)) = \tilde{\sigma}(c) = c,$$

as desired.

By Fact 5.1, we need only show that  $\text{stp}(a, b, c)$  has trivial stabilizer (since  $S$  is not almost  $P$ -internal). As in the discussion above, it follows that every tuple  $(g_1, g_2, g_3)$  in the stabilizer must have trivial projection onto the first coordinates, so write  $g_i = \gamma_i \star 0_S$ . Note that  $g_i$  and  $\gamma_i$  are interdefinable, so we may assume that

$$\gamma_1, \gamma_2, \gamma_3 \downarrow a, b, c.$$

We have that  $(\gamma_1 \star a, \gamma_2 \star b) \equiv_P (a, b)$  by Lemma 3.8(c). Hence, there is an automorphism  $\sigma$  fixing  $P$  pointwise (and hence a derivation  $D_\sigma$ , by assumption) such that  $D_\sigma(\alpha) = \gamma_1$  and  $D_\sigma(\beta) = \gamma_2$ . As

$$(a \oplus g_1, b \oplus g_2, c \oplus g_3) = (\gamma_1 \star a, \gamma_2 \star b, \gamma_3 \star c) \equiv (a, b, c),$$

and both  $\sigma(c)$  and  $\gamma_3 \star c$  are definable over  $\gamma_1 \star a, \gamma_2 \star b$  by the same formula, we deduce that

$$\gamma_3 = D_\sigma(\alpha \cdot \beta) = \gamma_1 \cdot \beta + \alpha \cdot \gamma_2.$$

The independence

$$\gamma_1, \gamma_2, \gamma_3 \downarrow \alpha, \beta$$

yields that all  $\gamma_i$ 's are trivial, as desired. ⊣

**REMARK 5.3.** Choosing a formula  $\varphi(a, b, z)$  which defines  $c$  over the elements  $a$  and  $b$ , it is easy to conclude, following Marker and Pillay's work [12, Fact 1.5], that the multiplication  $\otimes$  is globally definable (with parameters) in  $\mathcal{M}$  as the composition of germs of elements in

$$X = \{a \in S \mid \varphi(\varepsilon \star a, b, (\varepsilon \star a) \otimes b) \text{ for every generic } b \text{ independent from } a \text{ and every element } \varepsilon \text{ in } P\}.$$

We will now show that the CBP holds in the additive cover  $\mathcal{M}_0$  and, more generally, whenever there is essentially no additional structure on the sort  $S$ .

**PROPOSITION 5.4.** *The CBP holds in an additive cover  $\mathcal{M}$ , whenever every additive map on  $\mathbb{C}$  induces an automorphism in  $\text{Aut}(\mathcal{M}/P)$ .*

In particular, the additive cover  $\mathcal{M}_0$  has the CBP.

**PROOF.** Recall that we need only consider real types over models in order to deduce that the CBP holds. Let  $p(x)$  be the type of some finite real tuple  $\bar{a}$  of length  $k$  over an elementary substructure  $N$ . In order to show that the type  $\text{stp}(\text{Cb}(p)/\bar{a})$  is almost  $P$ -internal, choose a formula  $\varphi(x; \bar{b}, \gamma)$  in  $p$  of least Morley rank and Morley degree one, where  $\bar{b}$  is a tuple of elements in  $S \cap N$  and  $\gamma$  is a tuple of elements in  $P \cap N$ .

We claim that every automorphism in  $\text{Aut}(\mathcal{M}/P, \bar{a})$  fixes the canonical base  $\text{Cb}(p)$ , which is (interdefinable with) the canonical parameter  $\ulcorner d_p x \varphi(x; y) \urcorner$ . For this, it suffices to show that every such automorphism  $\sigma$  sends the tuple  $\bar{b}$  to another realization of the formula  $d_p x \varphi(x; y_1, \gamma)$ .

Write  $\bar{a} = (a_1, \dots, a_k)$  and

$$\alpha_i = \begin{cases} \pi(a_i), & \text{if } a_i \text{ is in } S, \\ a_i & \text{otherwise.} \end{cases}$$

For  $\bar{b} = (b_1, \dots, b_n)$ , set  $\beta_i = \pi(b_i)$ . We may assume (after possibly reordering) that  $(\beta_1, \dots, \beta_m)$  is a maximal subtuple of  $\bar{\beta}$  which is  $\mathbb{Q}$ -linearly independent over  $\bar{a}$ . So,

$$\beta_j = \sum_{i=1}^m q_i \cdot \beta_i + \sum_{i=1}^k r_i \cdot \alpha_i$$

for  $m + 1 \leq j \leq n$  and rational numbers  $q_i$  and  $r_i$ . In order to show that  $\bar{b}$  is mapped by the automorphism  $\sigma$  of  $\text{Aut}(\mathcal{M}/P, \bar{a})$  to another realization of the formula  $d_p x \varphi(x; y_1, \gamma)$ , it suffices to show that

$$N \models \forall \varepsilon_1, \dots, \varepsilon_m \in P \quad d_p x \varphi(x; \bar{\varepsilon} \star \bar{b}, \gamma),$$

where  $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$  with

$$\varepsilon_j = \sum_{i=1}^m q_i \cdot \varepsilon_i$$

for  $m + 1 \leq j \leq n$ . Indeed: since  $N$  is an elementary substructure of  $\mathcal{M}$ , the above implies that

$$\mathcal{M} \models \forall \varepsilon_1, \dots, \varepsilon_m \in P \quad d_p x \varphi(x, \bar{\varepsilon} \star \bar{b}, \gamma),$$

so we deduce from

$$F_\sigma(\beta_j) = \sum_{i=1}^m q_i \cdot F_\sigma(\beta_i) + \sum_{i=1}^k r_i \cdot F_\sigma(\alpha_i) = \sum_{i=1}^m q_i \cdot F_\sigma(\beta_i) + 0$$

for  $m + 1 \leq j \leq n$ , that  $\sigma(\bar{b}) = F_\sigma(\bar{b}) \star \bar{b}$  realizes  $d_p x \varphi(x; \bar{y}_1, \gamma)$ , as desired.

So, let  $\varepsilon_1, \dots, \varepsilon_m$  be in  $P \cap N$  and set  $\varepsilon_j = \sum_{i=1}^m q_i \cdot \varepsilon_i$  for  $m + 1 \leq j \leq n$ . Choose an additive map  $G$  vanishing on  $\alpha_i$  for  $1 \leq i \leq k$  and with  $G(\beta_i) = \varepsilon_i$  for  $1 \leq i \leq m$ . Hence

$$G(\beta_j) = \sum_{i=1}^m q_i \cdot \varepsilon_i,$$

so the image of  $\bar{b}$  under the automorphism  $\sigma_G$  induced by  $G$  lies in  $N$ . Hence  $\sigma_G(\bar{b}) = \bar{\varepsilon} \star \bar{b}$  realizes  $d_{p,x}\varphi(x, y_1, \gamma)$  since  $\sigma_G(\bar{a}) = \bar{a}$ , as desired.  $\dashv$

REMARK 5.5. The above proof shows that the canonical base of a real stationary type  $\text{stp}(a/N)$  is definable over  $a, P$  which is stronger than  $P$ -internality. As we will see below this does not hold for all imaginary types.

Palacín and Pillay [15] considered a strengthening of the CBP, called the *strong canonical base property*, which we reformulate in the setting of additive covers: Given a (possibly imaginary) type  $p = \text{stp}(a/B)$ , its canonical base  $\text{Cb}(p)$  is algebraic over  $a, \bar{d}$ , where  $\text{stp}(\bar{d})$  is  $P$ -internal. If we denote by  $\mathcal{Q}$  the family types over  $\text{acl}^{\text{eq}}(\emptyset)$  which are  $P$ -internal, then the strong CBP holds if and only if every binding group  $G$  relative to  $\mathcal{Q}$  is *rigid* [15, Theorem 3.4], that is, the connected component of every definable subgroup of  $G$  is definable over  $\text{acl}^{\text{eq}}(\ulcorner G \urcorner)$ .

Notice that no additive cover where the sort  $S$  is not almost  $P$ -internal can have the strong CBP: For the two generic independent elements  $a = (\alpha, 0)$  and  $b = (\beta, 0)$  in  $S$ , the stationary  $P$ -internal type  $p = \text{tp}(a, b/\alpha, \beta)$  is fundamental and has (relative to  $P$ ) binding group  $G = (\mathbb{C}^2, +)$ , see the discussion before Proposition 3.11. This is clearly a  $\mathcal{Q}$ -internal type whose binding group  $H$  (relative to  $\mathcal{Q}$ ) is a definable subgroup of the connected group  $G$ . Since vector groups are never rigid, it suffices to show that  $H = G$  (compare to [9, Proposition 4.9]). Otherwise, the Morley rank of  $H$  is at most 1, so

$$\text{RM}(a, b/\alpha, \beta, \mathcal{Q}) \leq 1,$$

by [15, Fact 1.2]. In particular, there is a tuple  $\bar{d}$  such that  $\text{stp}(\bar{d})$  belongs to  $\mathcal{Q}$  and (up to relabeling) the element  $b$  is algebraic over  $a, \bar{d}$  (note that  $\text{stp}(\beta)$  is in  $\mathcal{Q}$ ). Hence, the type  $\text{stp}(b/a)$ , and thus  $S$ , is almost  $P$ -internal.

The question whether a binding-theoretic interpretation of the CBP exists arose in [15]. We conclude this section with an observation that a pure binding-group-theoretic account of the CBP is heuristically unlikely. We already noticed in Proposition 3.11 that, whenever the sort  $S$  in an additive cover is not almost  $P$ -internal, then the binding groups relative to  $P$  are precisely all definable subgroups of  $(\mathbb{C}^n, +)$ , as  $n$  varies. In particular, all such additive covers share the same binding groups (relative to  $P$ ). We will now see that the same holds for the binding groups relative to  $\mathcal{Q}$ .

LEMMA 5.6. *Every definable subgroup of  $(\mathbb{C}^n, +)$ , for  $n$  in  $\mathbb{N}$ , occurs as a binding group relative to  $\mathcal{Q}$  in each additive cover where the sort  $S$  is not almost  $P$ -internal. In particular all such additive covers share the same binding groups relative to  $\mathcal{Q}$ .*

PROOF. Note that  $\mathcal{Q}$ -internality coincides with  $P$ -internality. Moreover, the binding group relative to  $\mathcal{Q}$  is a subgroup of the binding group relative to  $P$ , which

by Proposition 3.11 is a definable subgroup of some  $(\mathbb{C}^n, +)$ . So it suffices to show that every definable subgroup  $G$  of  $(\mathbb{C}^n, +)$  appears as a binding group relative to  $\mathcal{Q}$ .

Choose a tuple  $\bar{a}$  of elements  $a_1 = (\alpha_1, 0), \dots, a_n = (\alpha_n, 0)$  in the sort  $S$  with generic independent elements  $\alpha_i$  in  $P$  and consider the set  $G \star \bar{a}$ . By Proposition 3.11, the stationary type  $\text{stp}(\bar{a}/\ulcorner G \star \bar{a} \urcorner)$  is  $P$ -internal and fundamental with binding group  $G$ .

We now show that the binding group  $H$  of  $\text{stp}(\bar{a}/\ulcorner G \star \bar{a} \urcorner)$  relative to  $\mathcal{Q}$  equals  $G$ . Assume for a contradiction that  $H$  is a proper subgroup of  $G$ . Since  $G$  is connected, the Morley rank  $\text{RM}(H) < \text{RM}(G)$ , so

$$\text{RM}(\bar{a}/\ulcorner G \star \bar{a} \urcorner, \mathcal{Q}) < \text{RM}(\bar{a}/\ulcorner G \star \bar{a} \urcorner, P),$$

by [15, Fact 1.2]. Note that  $\pi(\bar{a})$  is definable over  $\ulcorner G \star \bar{a} \urcorner$ . It follows that for some  $1 \leq k \leq n$  there is a tuple  $\bar{d}$  with  $\text{stp}(\bar{d})$  in  $\mathcal{Q}$  such that the coordinate  $a_k$  of  $\bar{a}$  is not algebraic over  $\bar{a}_{<k}, \ulcorner G \star \bar{a} \urcorner$ , yet it is algebraic over  $\bar{a}_{<k}, \ulcorner G \star \bar{a} \urcorner, \bar{d}$ .

By  $P$ -internality of  $\text{stp}(\bar{d})$ , there is a set of parameters  $C$  with  $C \perp \bar{d}, \bar{a}, \ulcorner G \star \bar{a} \urcorner$  such that  $\bar{d}$  is definable over  $C, P$ . Therefore  $a_k$  is algebraic over

$$\bar{a}_{<k}, \ulcorner G \star \bar{a} \urcorner, C, P.$$

We deduce now from Lemma 3.8(c) that

$$\text{stp}(\bar{a}/\ulcorner G \star \bar{a} \urcorner, C) \vdash \text{tp}(\bar{a}/\ulcorner G \star \bar{a} \urcorner, C, P),$$

which contradicts that  $a_k$  is not algebraic over  $\bar{a}_{<k}, \ulcorner G \star \bar{a} \urcorner$ , by the independence  $C \perp \bar{a}, \ulcorner G \star \bar{a} \urcorner$ . ⊥

**§6. Transfer of internality in additive covers.** In this section we will show that the additive cover  $\mathcal{M}_1$  does not transfer internality to intersections nor internality to quotients. We will start with the latter, whose proof is considerably simpler.

PROPOSITION 6.1. *The additive cover  $\mathcal{M}_1$  does not transfer internality to quotients.*

PROOF. Choose generic independent elements  $a, b$  and  $c$  in the sort  $S$  and set  $d = (a \otimes c) \oplus b$ . Consider now the following definable set:

$$E = \{(x, y) \in S^2 \mid \pi(x) = \pi(a) \ \& \ \pi(y) = \pi(b) \ \& \ d = (x \otimes c) \oplus y\}.$$

Since the canonical parameter  $\ulcorner E \urcorner$  is clearly definable over  $c, d, \pi(a), \pi(b)$  and the type  $\text{stp}(c, d, \pi(a), \pi(b)/\pi(c), \pi(d))$  is  $P$ -internal, we deduce that the type

$$\text{stp}(\ulcorner E \urcorner/\pi(c), \pi(d))$$

is  $P$ -internal.

CLAIM. *The type  $\text{stp}(\ulcorner E \urcorner/\pi(a), \pi(b))$  is  $P$ -internal.*

PROOF OF THE CLAIM. Choose elements  $a_1$  and  $b_1$  in the fiber of  $\pi(a)$ , resp.  $\pi(b)$ , such that

$$a_1, b_1 \perp_{\pi(a), \pi(b)} \ulcorner E \urcorner.$$

Note that every automorphism  $\sigma$  in  $\text{Aut}(\mathcal{M}_1/P)$  fixing the elements  $a_1$  and  $b_1$  must fix  $\pi^{-1}(\pi(a)) \times \pi^{-1}(\pi(b)) \supseteq E$ , so  $\sigma$  fixes  $E$  pointwise. In particular, the canonical parameter  $\ulcorner E \urcorner$  is definable over  $a_1, b_1, P$ , as desired.  $\dashv$ Claim.

We assume now that  $\mathcal{M}_1$  transfers internality to quotients in order to reach a contradiction. Since

$$\text{acl}^{\text{eq}}(\pi(a), \pi(b)) \cap \text{acl}^{\text{eq}}(\pi(c), \pi(d)) = \text{acl}^{\text{eq}}(\emptyset),$$

we deduce that the type  $\text{stp}(\ulcorner E \urcorner)$  is almost  $P$ -internal. It follows that the type  $\text{stp}(b/a)$  is almost  $P$ -internal, since  $b$  is definable over  $\ulcorner E \urcorner, a$ . Now, the independence  $b \perp a$  yields the desired contradiction, because the sort  $S$  is not almost  $P$ -internal.  $\dashv$

REMARK 6.2. The previous set is definable in every additive cover, since  $E$  equals  $G \star (a, b)$ , where  $G$  is the definable subgroup of  $(P^2, +)$  given by the equation  $\pi(c) \cdot x + y = 0$ . The main cause for the failure of transfer of internality to quotients in  $\mathcal{M}_1$  is that  $E$  is definable over  $c, d, P$ .

PROPOSITION 6.3. *The additive cover  $\mathcal{M}_1$  does not transfer internality to intersections.*

PROOF. Choose generic independent elements  $a_1$  and  $a_2$  in  $S$  and  $\varepsilon$  in  $P$  generic over  $a_1, a_2$ . Set  $\bar{\alpha} = (\alpha_1, \alpha_2) = (\pi(a_1), \pi(a_2))$  and let  $G$  be the definable subgroup of  $(P^2, +)$  given by  $\varepsilon \cdot x + y = 0$ . Consider now the definable set  $G \star (a_1, a_2)$  and choose  $\beta_1$  in  $P$  generic over  $\ulcorner G \star (a_1, a_2) \urcorner, \bar{\alpha}, \varepsilon$  as well as elements  $\beta_2$  and  $\beta_3$  in  $P$  with

$$0 = \beta_1 \alpha_1 + \frac{1}{2} \beta_2 \alpha_1^2 + \frac{1}{3} \beta_3 \alpha_1^3 + \alpha_2 \tag{1}$$

$$0 = \beta_1 + \beta_2 \alpha_1 + \beta_3 \alpha_1^2 - \varepsilon \tag{2}$$

This is possible because the matrix

$$\begin{pmatrix} \frac{\alpha_1^2}{2} & \frac{\alpha_1^3}{3} \\ \alpha_1 & \alpha_1^2 \end{pmatrix}$$

has determinant  $\frac{\alpha_1^4}{2} - \frac{\alpha_1^4}{3} \neq 0$ . Since  $\beta_2$  and  $\beta_3$  are definable over  $\beta_1, \bar{\alpha}, \varepsilon$ , we get the independence

$$\bar{\beta} \underset{\bar{\alpha}, \varepsilon}{\perp} \ulcorner G \star (a_1, a_2) \urcorner, \tag{\diamond}$$

where  $\bar{\beta} = (\beta_1, \beta_2, \beta_3)$ .

CLAIM 1. *The type  $\text{stp}(\ulcorner G \star (a_1, a_2) \urcorner / \bar{\beta})$  is  $P$ -internal.*

PROOF OF THE CLAIM 1. Let  $b_1, b_2$  and  $b_3$  be elements in  $S$  such that  $b_i$  is in the fiber of  $\beta_i$  with

$$b_1, b_2, b_3 \underset{\bar{\beta}}{\perp} \ulcorner G \star (a_1, a_2) \urcorner, \bar{\alpha}, \varepsilon.$$

We show that every automorphism  $\sigma$  in  $\text{Aut}(\mathcal{M}_1/P)$  fixing  $b_1, b_2$  and  $b_3$  must permute  $G \star (a_1, a_2)$ . Recall that  $F_\sigma$  is the derivation on  $\mathbb{C}$  induced by the automorphism  $\sigma$ . Since  $F_\sigma(\beta_i) = 0$ , we deduce from Equations (1) and (2) that  $\varepsilon \cdot F_\sigma(\alpha_1) + F_\sigma(\alpha_2) = 0$ . Since the tuple  $(F_\sigma(\alpha_1), F_\sigma(\alpha_2))$  lies in  $G$ , we have that

$$\sigma(a_1, a_2) = (F_\sigma(\alpha_1) \star a_1, F_\sigma(\alpha_2) \star a_2) \text{ belongs to } G \star (a_1, a_2),$$

so the automorphism  $\sigma$  permutes the set  $G \star (a_1, a_2)$ , by Proposition 3.11.  $\dashv$ Claim 1

CLAIM 2. *The intersection  $\text{acl}^{\text{eq}}(\ulcorner G \star (a_1, a_2) \urcorner) \cap \text{acl}^{\text{eq}}(\bar{\beta}) = \text{acl}^{\text{eq}}(\emptyset)$ .*

PROOF OF THE CLAIM 2. Because of the independence ( $\blacklozenge$ ), we need only show that

$$\text{acl}^{\text{eq}}(\bar{\beta}) \cap \text{acl}^{\text{eq}}(\bar{\alpha}, \varepsilon) = \text{acl}^{\text{eq}}(\emptyset).$$

Choose tuples  $\bar{\beta}', \bar{\alpha}', \varepsilon', \bar{\beta}'', \bar{\alpha}'', \varepsilon'', \bar{\beta}''''$  such that

$$\bar{\beta}, \bar{\alpha}, \varepsilon \stackrel{\text{st}}{\equiv} \bar{\beta}', \bar{\alpha}, \varepsilon \stackrel{\text{st}}{\equiv} \bar{\beta}', \bar{\alpha}', \varepsilon' \stackrel{\text{st}}{\equiv} \bar{\beta}'', \bar{\alpha}', \varepsilon' \stackrel{\text{st}}{\equiv} \bar{\beta}'', \bar{\alpha}'', \varepsilon'' \stackrel{\text{st}}{\equiv} \bar{\beta}''', \bar{\alpha}'', \varepsilon''$$

with

$$\bar{\beta}' \downarrow_{\bar{\alpha}, \varepsilon} \bar{\beta} \quad \bar{\alpha}', \varepsilon' \downarrow_{\bar{\beta}'} \bar{\beta}, \bar{\alpha}, \varepsilon \quad \bar{\beta}'' \downarrow_{\bar{\alpha}', \varepsilon'} \bar{\beta}, \bar{\alpha}, \varepsilon, \bar{\beta}' \quad \bar{\alpha}'', \varepsilon'' \downarrow_{\bar{\beta}''} \bar{\beta}, \bar{\alpha}, \varepsilon, \bar{\beta}', \bar{\alpha}', \varepsilon'$$

and

$$\bar{\beta}'''' \downarrow_{\bar{\alpha}'', \varepsilon''} \bar{\beta}, \bar{\alpha}, \varepsilon, \bar{\beta}', \bar{\alpha}', \varepsilon', \bar{\beta}''.$$

Since

$$\text{acl}^{\text{eq}}(\bar{\beta}) \cap \text{acl}^{\text{eq}}(\bar{\alpha}, \varepsilon) \subset \text{acl}^{\text{eq}}(\bar{\beta}) \cap \text{acl}^{\text{eq}}(\bar{\beta}''''),$$

we need only show the independence  $\bar{\beta} \perp \bar{\beta}''''$ . Note first that the whole configuration has Morley rank 9:

$$\text{RM}(\bar{\beta}, \bar{\alpha}, \varepsilon, \bar{\beta}', \bar{\alpha}', \varepsilon', \bar{\beta}'', \bar{\alpha}'', \varepsilon'', \bar{\beta}''') = \text{RM}(\beta_1, \alpha_1, \alpha_2, \varepsilon, \beta'_1, \alpha'_1, \beta''_1, \alpha''_1, \beta'''_1) = 9.$$

Since

$$\begin{aligned} \text{RM}(\bar{\beta}''', \bar{\beta}, \alpha_1, \alpha'_1, \alpha''_1) &= \text{RM}(\bar{\beta}''''/\bar{\beta}, \alpha_1, \alpha'_1, \alpha''_1) + \text{RM}(\alpha''_1/\bar{\beta}, \alpha_1, \alpha'_1) \\ &\quad + \text{RM}(\alpha'_1/\bar{\beta}, \alpha_1) + \text{RM}(\alpha_1/\bar{\beta}) + \text{RM}(\bar{\beta}) \\ &= \text{RM}(\bar{\beta}''''/\bar{\beta}, \alpha_1, \alpha'_1, \alpha''_1) + 6, \end{aligned}$$

it suffices to show that  $\alpha_2, \varepsilon, \bar{\beta}', \alpha'_2, \varepsilon', \bar{\beta}'', \alpha''_2$  and  $\varepsilon''$  are all algebraic over the tuple  $(\bar{\beta}''', \bar{\beta}, \alpha_1, \alpha'_1, \alpha''_1)$ . Clearly  $\alpha_2, \varepsilon, \alpha''_2$  and  $\varepsilon''$  are algebraic over  $\bar{\beta}''', \bar{\beta}, \alpha_1, \alpha''_1$ .



Furthermore we have the following system of linear equations:

$$\begin{pmatrix} 6\alpha_1 & 3\alpha_1^2 & 2\alpha_1^3 & 0 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1 & \alpha_1^2 & 0 & 0 & 0 & 0 & 0 \\ 6\alpha_1' & 3\alpha_1'^2 & 2\alpha_1'^3 & 6 & 0 & 0 & 0 & 0 \\ 1 & \alpha_1' & \alpha_1'^2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 6\alpha_1' & 3\alpha_1'^2 & 2\alpha_1'^3 \\ 0 & 0 & 0 & 0 & 1 & 1 & \alpha_1' & \alpha_1'^2 \\ 0 & 0 & 0 & 0 & 0 & 6\alpha_1'' & 3\alpha_1''^2 & 2\alpha_1''^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_1'' & \alpha_1''^2 \end{pmatrix} \begin{pmatrix} \beta_1' \\ \beta_2' \\ \beta_3' \\ \alpha_2' \\ \varepsilon' \\ \beta_1'' \\ \beta_2'' \\ \beta_3'' \end{pmatrix} = \begin{pmatrix} -6\alpha_2 \\ \varepsilon \\ 0 \\ 0 \\ 0 \\ 0 \\ -6\alpha_2'' \\ \varepsilon'' \end{pmatrix}$$

Thus, we need only show that the above matrix has non-zero determinant

$$6 \begin{vmatrix} 6\alpha_1 & 3\alpha_1^2 & 2\alpha_1^3 \\ 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_1' & \alpha_1'^2 \end{vmatrix} \begin{vmatrix} 6\alpha_1' & 3\alpha_1'^2 & 2\alpha_1'^3 \\ 6\alpha_1'' & 3\alpha_1''^2 & 2\alpha_1''^3 \\ 1 & \alpha_1'' & \alpha_1''^2 \end{vmatrix} - 6 \begin{vmatrix} 6\alpha_1 & 3\alpha_1^2 & 2\alpha_1^3 \\ 1 & \alpha_1 & \alpha_1^2 \\ 6\alpha_1' & 3\alpha_1'^2 & 2\alpha_1'^3 \end{vmatrix} \begin{vmatrix} 1 & \alpha_1' & \alpha_1'^2 \\ 6\alpha_1'' & 3\alpha_1''^2 & 2\alpha_1''^3 \\ 1 & \alpha_1'' & \alpha_1''^2 \end{vmatrix} \\ = 72\alpha_1^2\alpha_1'^2\alpha_1''^2(\alpha_1 - \alpha_1')(\alpha_1 - \alpha_1'')(\alpha_1'' - \alpha_1') \neq 0. \tag{Claim 2}$$

If  $\mathcal{M}_1$  had transfer of internality to intersections, then the type

$$\text{stp}(\ulcorner G \star (a_1, a_2) \urcorner / \text{acl}^{\text{eq}}(\ulcorner G \star (a_1, a_2) \urcorner) \cap \text{acl}^{\text{eq}}(\beta_1, \beta_2, \beta_3))$$

would be almost  $P$ -internal, by Claim 1, and so would be  $\text{stp}(\ulcorner G \star (a_1, a_2) \urcorner)$ , by the previous claim, which yields a contradiction, exactly as in the proof of Proposition 6.1.  $\dashv$

Recall that an additive cover transfers internality to intersections, resp. to quotients, if and only if every almost  $P$ -internal type is good, resp. special, by Propositions 2.5 and 2.8. For real types, the property of being special follows directly from almost internality.

REMARK 6.4. Almost  $P$ -internal real types are special in every additive cover.

PROOF. We may assume that the sort  $S$  is not almost  $P$ -internal. We want to show that the almost  $P$ -internal type  $\text{stp}(a, \gamma/B)$  is special, where  $a$  is a tuple  $(a_1, \dots, a_n)$  of elements in  $S$  and  $\gamma$  is a tuple in  $P$ . So, assume that for some set  $C$  of parameters the type  $\text{stp}(a, \gamma/C)$  is almost  $P$ -internal. By Lemma 3.10, the projections  $\pi(a_i)$  are algebraic over  $B$  and over  $C$ . Hence, the type

$$\text{stp}(a, \gamma / \text{acl}^{\text{eq}}(B) \cap \text{acl}^{\text{eq}}(C))$$

is almost  $P$ -internal and so is

$$\text{stp}(\text{Cb}(a, \gamma/B) / \text{acl}^{\text{eq}}(B) \cap \text{acl}^{\text{eq}}(C)),$$

since  $\text{Cb}(a, \gamma/B)$  is algebraic over finitely many  $B$ -conjugates of  $a$ .  $\dashv$

Propositions 6.3 and 6.1 and the above remark give a negative answer to Question 2.

COROLLARY 6.5. *There is a stable theory of finite Morley rank, where every stationary real almost  $\mathbb{P}$ -internal type is special, yet  $\mathbb{P}$ -internality is not transferred to intersections.*

In [11] we give necessary conditions for an additive cover to transfer internality to quotients. This approach yields as a by-product new counterexamples to the CBP. Furthermore, we will deduce that no additive cover which transfers internality to quotients can eliminate finite imaginaries, whenever the sort  $S$  is not almost  $P$ -internal, which generalizes Proposition 6.1.

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