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RADIAL GROWTH AND BOUNDEDNESS FOR BLOCH FUNCTIONS

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Let B be the Bloch space of all those functions f holomorphic in the open unit disc $\mathbb{D}$ of the complex plane satisfying $\sup_{|z|<1} \left(1 - |z|^2\right) |f'(z)| < \infty$. We establish sufficient conditions for the boundedness of functions f belonging to B satisfying a certain uniform radial boundedness condition, and, by introducing a wide class of subsets E of $\partial\mathbb{D}$, which we call negligible sets for boundedness, we show that if $f \in B$ and there is a constant $K > 0$ such that $\limsup_{z \to e^{i\theta}} |f(z)| \leqslant K$ for $e^{i\theta} \in \partial\mathbb{D} \setminus E$, then f is bounded in $\mathbb{D}$. Hence a significant extension of a theorem of Goolsby is obtained.

1. INTRODUCTION

As usual, B denotes the Bloch space of all those holomorphic functions f in the open unit disc $\mathbb{D}$ of the complex plane $\mathbb{C}$ which satisfy

$$\|f\|_B = |f(0)| + \sup_{|z|<1} \left(1 - |z|^2\right) |f'(z)| < \infty.$$

Endowed with the Bloch norm $\|.\|_B$, B is a Banach space. The space $H^\infty(\mathbb{D})$ of all bounded analytic functions in $\mathbb{D}$ is strictly contained in B, since the function $\log(1+z)/(1-z) \in B \setminus H^\infty(\mathbb{D})$. $\mathbb{T}$ will denote the unit circle.

The following result was shown in [5]:

THEOREM A. (Goolsby). *Let E be a finite subset of $\mathbb{T}$, and let $f \in B$. If there exists a constant $K > 0$ such that*

$$\limsup_{z \to a} |f(z)| \leqslant K$$

for any $a \in \mathbb{T} \setminus E$, then f is bounded in $\mathbb{D}$.

The proof of this result depends on Theorem 4.2 in [1] which we reproduce below.

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Theorem B. (Anderson, Clunie, Pommerenke). *Let $f \in B$ and let Γ be an arc ending at $e^{i\theta}$. Let $A \subseteq \mathbb{C}$. If*

$$\lim_{\substack{z \to e^{i\theta} \\ z \in \Gamma}} \operatorname{dist}[f(z), A] = 0,$$

then

$$\limsup_{r \to 1-0} \operatorname{dist}[f(re^{i\theta}), A] \leqslant K_1 \|f\|_B,$$

where K_1 is an absolute constant.

The constant K_1 comes from the Lehto–Virtanen maximum principle (see [1], p.30 or [7]), and depends neither on the point $e^{i\theta}$ nor on the function f. In fact, if for a fixed $\beta > 0$, we consider the expression (of Lehto–Virtanen):

$$\delta_0(\alpha,\beta) = \frac{\sin\beta}{\beta}\left[1 + \sqrt{1 + \left(\frac{\alpha\beta}{\sin\beta}\right)^2}\right].\exp\left[-\sqrt{1 + \left(\frac{\alpha\beta}{\sin\beta}\right)^2}\right] \tag{1.1}$$

and choose some number $\alpha > 0$ such that $\delta_0(\alpha,\beta) \geqslant 1$, we can assume $K_1 = 3/\alpha$. In particular, for $\beta = 3\pi/4$, we have $\delta_0(\alpha,\beta) \geqslant 1$ whenever $\alpha \leqslant 0.0001989$, and hence, if $\alpha = 1/10$, then $K_1 = 30$. From now on K_1 will always represent an absolute constant.

Our major goal in this research has been to obtain a significant generalisation of Theorem A for subsets E of $\mathbb{T}$ bigger than a finite set. In Section 3 of this paper we show that this is the case (Theorem 3). In fact, Theorem 3 is derived from the results in Section 2, which is devoted to the establishing of sufficient conditions for the boundedness of functions $f \in B$ satisfying a certain radial uniform boundedness condition. To the best of our knowledge, although Theorem 2 is a corollary of [3], Theorem 4, the theorems in this section are new and cannot be deduced from any recent result in this area. (See [4, 6,8]). Although Theorem 1 is a particular case of Theorem 2, we have included an independent proof based upon the ideas and geometric constructions in [1] and [5]. This allows us to extend Theorem A by using the same methods invoked in its proof.

2. Radial growth and boundedness

We shall need two lemmas.

Lemma 1. ([5], p.720). *Let (r_n) be a sequence of real numbers in $\mathbb{D}$ such that $\lim_{n\to\infty} r_n = 1$. Then there exists a sequence of discs, Δ_n, satisfying:*

(a) $r_n \in \Delta_n$;
(b) $1 \notin \overline{\Delta}_n$;
(c) *the angle between* $\mathbb{T}$ *and* $\partial\Delta_n$ *is* $3\pi/4$;
(d) *diameter*$(\Delta_n) \longrightarrow 0$, *as* $n \longrightarrow \infty$; *and,*
(e) $\Delta_n \cap (\mathbb{C} \setminus \mathbb{D}) \neq \emptyset$.

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REMARK. The construction of the Δ_n's can be made in such a way that, for each n, the two points in $\partial\Delta_n \cap \mathrm{T}$ remain in the semiplane $\{z\colon \mathrm{Im} z > 0\}$.

LEMMA 2. *Let $f \in B$ and assume that there is a point $e^{i\theta_0}$ such that $\limsup_{r\to 1-0} |f(re^{i\theta_0})| < \infty$. Suppose also that U is a domain in C such that*

(i) $e^{i\theta_0} \in \overline{U}$, and

(ii) *$\overline{U}$ does not meet the segment $\left[r_0^{i\theta_0}, e^{i\theta_0}\right)$ for some r_0, $0 \leqslant r_0 < 1$.*

If $A = f(\overline{U} \setminus \mathrm{T})$, then

$$\limsup_{r\to 1-0} \operatorname{dist}[f(re^{i\theta_0}, A)] \leqslant K_1 \|f\|_B. \tag{2.1}$$

PROOF: We shall exploit ideas from [1] and [5]. Without loss of generality, we suppose $\theta_0 = 0$, and $\|f\|_B \leqslant 1$. Further we assume that, at least, U peaks at 1 across the domain $R = \mathrm{D} \cap \{z\colon \mathrm{Im} z > 0\}$. Let $\beta = 3\pi/4$ and choose α so small that $\delta_0(\alpha,\beta) \geqslant 1$ is as in (1.1), and put $K_1 = 3/\alpha$.

If (2.1) fails, then by the hypothesis on radial boundedness, a complex number w_0 and a sequence (r_n) of real numbers with $r_n \to 1-0$ as $n \to \infty$, can be found such that

$$\begin{gathered} f(r_n) \longrightarrow w_0, \ \text{as } n \longrightarrow \infty \\ \operatorname{dist}[w_0, A] > K_1. \end{gathered} \tag{2.2}$$

Next, let (Δ_n) be a sequence of discs having the properties stated in Lemma 1 and the Remark above. Let A_n and C_n be the two arcs of $\partial\Delta_n$ in R. Since U is connected and $\overline{U} \cap [r_0, 1)$ is empty, there is a number N such that, for any $n \geqslant N$, we have

$$\begin{gathered} r_n \in [r_0, 1), \\ A_n \cap U \neq \emptyset, \\ C_n \cap U \neq \emptyset. \end{gathered} \tag{2.3}$$

If G_n denotes the (open) connected component of $(\mathrm{C} \setminus \overline{U}) \cap \Delta_n$ containing r_n, then $\overline{G}_n \subset \overline{\Delta}_n$ and $1 \notin \partial G_n \subset \partial\Delta_n \cup (\overline{U} \setminus 1)$. We assume for the moment the crucial inclusion

$$\overline{G}_n \subset \mathrm{D}. \tag{2.4}$$

Then $\partial G_n \subseteq \partial\Delta_n \cup (\overline{U} \setminus \mathrm{T})$, and if $g(z)$ is defined to be

$$g(z) = \frac{1}{\alpha(f(z) - w_0)}, \quad z \in \mathrm{D},$$

then g is a meromorphic function which is normal:

$$\left(1-|z|^2\right)\frac{|g'(z)|}{1+|g(z)|^2}=\frac{\left(1-|z|^2\right)\alpha|f'(z)|}{\alpha^2|f(z)-w_0|^2+1}\leqslant\alpha\left(1-|z|^2\right)|f'(z)|\leqslant\alpha\|f\|_B\leqslant\alpha.$$

Now, if $z\in\partial G_n\setminus\partial\Delta_n$, then $z\in\overline{U}\setminus\mathrm{T}$, so that $f(z)\in A$, and the inequality in (2.2) leads to $|g(z)|\leqslant 1/(\alpha K_1)=1/3$. We deduce from the Lehto–Virtanen maximum principle that g is analytic and bounded in every G_n, with a bound independent of n. However $|g(r_n)|\to\infty$, as $n\to\infty$, and we reach a contradiction.

It only remains to show the inclusion in (2.4). The construction of the sequence Δ_n makes it clear that $\Delta_n\cap(\mathrm{C}\setminus R)\subseteq\mathrm{D}$, so we need only check that $\overline{G}_n\cap R\subset\mathrm{D}$. If this were not the case, there would exist a point q in $\overline{G}_n\cap R$ with $|q|\geqslant 1$. We may assume q is on T, for if q were off D the arcwise connectedness of G_n would yield an arc in G_n joining q with r_n, which would intersect T. Next, let E_1 and E_2 be the components of $R\cap(\mathrm{C}\setminus\Delta_n)$ whose closures contain A_n and C_n respectively. By (2.3), there are two points $t\neq 1$ and $s\neq 1$ such that $t\in E_1\cap U$ and $s\in E_2\cap U$. Since U is connected, we can take a curve Γ_1 defined in $[0,1]$ such that $\Gamma_1\subseteq U$, $\Gamma_1(0)=t$ and $\Gamma_1(1)=s$. Let $x_1=\sup\{x\in[0,1]/\Gamma_1(x)\in A_n\}$ and let $x_2=\inf\{x\in[x_1,1]:\Gamma_1(x)\in C_n\}$. Since $\Gamma_1(1)\notin A_n$, $x_1\neq 1$ and $\Gamma_1(x_1)\in A_n$. Similarly $\Gamma_1(x_2)\in C_n$, and $x_1\neq x_2$. Let Γ be the closed (Jordan) curve formed by Γ_1 on $[x_1,x_2]$ and $\partial\Delta_n$ so that r_n is in the interior of Γ. Since $\Gamma_1\subseteq U\subseteq\mathrm{D}$, q does not meet $\Gamma_{1|[x_1,x_2]}$. This means that if $q'\in D(q,r)\cap G_n$, for some $r>0$, then q' can be joined to r_n by a curve γ in G_n. But since r_n is in the interior of Γ and q' is outside Γ, we would have that $\Gamma_1\cap\gamma\neq\emptyset$, which is impossible because $\gamma\subseteq G_n\subseteq[\Delta_n\cap(\mathrm{C}\setminus\overline{U})]$ and $\Gamma_1\subseteq U$. ∎

We now establish our first theorem.

Theorem 1. *Let $f\in B$. If there is a constant $K>0$ such that*

$$\limsup_{r\to 1-0}\left|f\left(re^{i\theta}\right)\right|\leqslant K$$

for any $e^{i\theta}$ in T, then f is bounded in D.

Proof: Suppose that f is not bounded, and let M be an arbitrary constant, $M>K_1\|f\|_B+K$. Since $\limsup_{r\to 1-0}\left|f\left(re^{i\theta}\right)\right|\leqslant K$ for any $e^{i\theta}$, it follows that

$$\limsup_{r\to 1-0}\operatorname{dist}[f(re^{i\theta}),\mathrm{C}\setminus\overline{D}(O,M)]>K_1\|f\|_B \tag{2.5}$$

for each point $e^{i\theta}$. Let $U=f^{-1}(\mathrm{C}\setminus\overline{D}(0,M))$. We shall assume that $U\neq\emptyset$, for otherwise f would be bounded.

Let V be any component of U. If $\partial V \cap \mathbb{T} = \emptyset$, then $\overline{V} \subseteq \mathbb{D}$ and f attains its maximum in $\overline{V}$ in a point $z_0 \in (\partial V \setminus \mathbb{T}) \cap \mathbb{D}$, with $|f(z_0)| > M$; by continuity, $|f| > M$ in a disc $D(z_0, \varepsilon) \subseteq \mathbb{D}$. This contradicts the maximality of V and shows that $\partial V \cap \mathbb{T} \neq \emptyset$.

Next, let $e^{i\alpha}$ be any fixed point in $\partial V \cap \mathbb{T}$. The condition on radial boundednes guarantees that there exists an r_0, $0 \leqslant r_0 < 1$, such that $\overline{V} \cap [r_0 e^{i\theta}, e^{i\alpha}) = \emptyset$. Put $A = f(\overline{V} \setminus \mathbb{T})$. By Lemma 2,

$$\limsup_{r \to 1-0} \operatorname{dist}[f(re^{i\alpha}, A)] \leqslant K_1 \|f\|_B .$$

Since $f(\overline{V} \setminus \mathbb{T}) \subseteq \mathbb{C} \setminus D(O, M)$, we find that $\limsup_{r \to 1-0} \operatorname{dist}[f(re^{i\alpha}), \mathbb{C} \setminus \overline{D}(O, M)] \leqslant K_1 \|f\|_B$, contradicting (2.5). $\square$

We do not know whether Theorem 1 holds when the condition $f \in B$ is replaced by $f \in L^1_a$, the Bergman space of all those analytic functions in $\mathbb{D}$ that are L^1 with respect to Lebesgue area measure. In this direction the following example shows that it is not true for an arbitrary analytic function in $\mathbb{D}$.

Let f be the function defined by

$$f(z) = (1 - z) \exp\left[-\exp\left(\frac{1+z}{1-z}\right)\right], \qquad z \in \mathbb{D}.$$

It is not hard to check that $f^*(e^{i\theta}) = \lim_{r \to 1-0} f(re^{i\theta})$ exists and $|f^*(e^{i\theta})| \leqslant 2e$, for any $e^{i\theta} \in \mathbb{T}$. Now let us consider the sequence $a_k = ((k-1) + i\pi)/((k+1) + i\pi)$, $k = 1, 2, \ldots$. Then $a_k \in \mathbb{D}$ and $|f(a_k)|^2 = 4/\left((1+k)^2 + \pi^2\right) e^{2e^k} \to \infty$, as $k \to \infty$. This implies that $f \notin H^\infty(\mathbb{D})$. Further, $f \notin L^1_a$ because the area integral of $|f|$ on discs centred at a_k and with radii $(1 - |a_k|)/2$ can become very large as $k \to \infty$.

On the other hand, we have the next result which is a consequence of [**3**], Theorem 4. As usual, "Dim" means Hausdorff dimension.

Theorem 2. *Let $f \in B$ and let E be a subset of $\mathbb{T}$ with $\operatorname{Dim}(E) < 1$. If there exists a constant $K > 0$ such that $\limsup_{r \to 1-0} |f(re^{i\theta})| \leqslant K$ for any $e^{i\theta} \in \mathbb{T} \setminus E$, then f is bounded.*

Proof: Since $f \in B$, $|f(z)| \leqslant C \log 1/(1 - |z|)$. Hence the function $u(z) = \log|f(z)|$ satisfies $u(z) \leqslant \log\log 1/(1 - |z|) + C$. In particular, $u^*(e^{i\theta}) = \limsup_{r \to 1-0} u(re^{i\theta}) \leqslant$ constant for any $e^{i\theta} \in \mathbb{T} \setminus E$, and $M(r, u) = \operatorname{Max}_\theta u(re^{i\theta}) = o(1/((1-r)^\alpha))$ for each $\alpha > 0$. Choosing α_0 such that $0 < \alpha_0 < 1 - \operatorname{Dim}(E)$, Theorem 4 in [**3**] shows that $u \leqslant$ constant in $\mathbb{D}$. $\square$

3. Negligible sets for boundedness

In order to extend Theorem A it will be convenient to introduce a class of subsets of $\mathbb{T}$.

Definition: Let E be a subset of $\mathbb{T}$. We shall say that E is a *negligible set for boundedness* (in short, negligible) if for every $e^{i\theta} \in E$, there is a subarc $I \subseteq \mathbb{T}$ ending at $e^{i\theta}$ and $I \subseteq \mathbb{T} \setminus E$.

It is clear that any negligible set is a countable set, and that any subset of a negligible set is also negligible. We list some examples.

(i) Every finite set in $\mathbb{T}$ is plainly negligible.

(ii) Let $E = \left(e^{i\theta_n}\right)$ be a sequence of points on $\mathbb{T}$ whose arguments converge strictly to $\theta_0 \in [0, 2\pi]$. Then E is negligible.

(iii) Let $E = \left(e^{i\theta_n}\right) \cup \{1\}$, where $2\pi > \theta_n \to 0$ and $\theta_{n+1} - \theta_n \simeq 1/\left(n(\log n)^2\right)$ for each n. Then E is negligible but not a Carleson set.

We are in a position to prove the following result in two independent ways.

Theorem 3. *Let $f \in B$ and let E be a negligible set. If there is a constant $K > 0$ such that*

$$\limsup_{z \to e^{i\theta}} |f(z)| \leqslant K$$

for any $e^{i\theta} \in \mathbb{T} \setminus E$, then f is bounded in $\mathbb{D}$.

Proof: First of all since $\operatorname{Dim}(E) = 0$, the conclusion follows immediately from Theorem 2. On the other hand, we note that if J is a subarc of $\mathbb{T} \setminus E$ and $e^{i\theta_0}$ is an endpoint of J, then

$$\limsup_{f \to 1-0} \left|f\left(re^{i\theta_0}\right)\right| \leqslant K_1 \|f\|_B + K + 2;$$

this can be proved just as in [5], Theorem 2.60. But the same holds for any $e^{i\theta}$ belonging to E, since there exists an arc I ending at $e^{i\theta}$. If M is a constant satisfying $M > K_1 \|f\|_B + K + 2$, then $\limsup_{r \to 1-0} \left|f\left(re^{i\alpha}\right)\right| \leqslant M$ for any α. Now Theorem 1 implies that f is bounded. ∎

References

[1] J.M. Anderson, J. Clunie and C. Pommerenke, 'On Bloch functions and normal functions', *J. Reine. Angew. Math.* **270** (1974), 12–37.

[2] J.A. Cima, 'The basic properties of Bloch functions', *Internat. J. Math. Math. Sci.* **2-3** (1979), 369–413.

[3] B.E.I. Dahlberg, 'On the radial boundary values of subharmonic functions', *Math. Scand.* **40** (1977), 301–317.

[4] D. Girela, 'Integral means and radial growth of Bloch functions', *Math. Z.* **195** (1987), 37–50.

[5] R.C. Goolsby, 'Boundedness for Bloch functions', *Rocky Mountain J. Math.* **16** (1986), 717–726.

[6] B. Korenblum, 'Estimates and radial growth of Bloch functions', *Bull. Amer. Math. Soc. (N.S.)* **12** (1985), 99–102.

[7] O. Lehto and K.I. Virtanen, 'Boundary behaviour and normal meromorphic functions', *Acta Math.* **97** (1957), 47–65.

[8] N.G. Makarov, 'On the distortion of boundary sets under conformal mappings', *Proc. London Math. Soc.* **51** (1985), 369–384.

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