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DISTINGUISHED DOMAINS

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Introduction. This paper introduces a class of domains which we hope to show merits some attention.

Definition. The domain R is said to be a *distinguished* domain if for any $0 \neq z \in K$, the quotient field of R, $(1 : z)$ does not consist entirely of zero divisors modulo $(1 : z^{-1})$. (Note: Here we use the fact that a zero module has no zero divisors. Thus if $z^{-1} \in R$, so that $(1 : z^{-1}) = R$, then the condition holds trivially.)

Section 1 of this paper gives numerous examples of distinguished domains, foremost among them being Krull domains and Prüfer domains. In fact Prüfer domains are shown to be exactly those distinguished domains whose prime lattice forms a tree. Other distinguished domains can be constructed by the $D + M$ construction. It is shown that distinguished domains are integrally closed but the converse fails.

Krull domains and Prüfer domains are both defined in terms of valuation rings. In Section 2 of this paper, it is shown that valuation rings also play an important role in distinguished domains, but in a subtle (and not wholly understood) way. Specifically, it is shown that prime ideals in a distinguished domain come in two flavours, called U-primes and V-primes. If P is a V-prime, then there is a unique largest prime, Q, properly contained in P and R_P/Q_P is a valuation ring. Also $R = \cap R_P$, the intersection taken over all V-primes. On the other hand, U-primes (which seem to play a much less important role) are primes P for which $R_P = \cap R_p$, the intersection taken over all p properly contained in P.

In Section 3, an understanding of V-primes is used to show that if R is a distinguished domain, then so is $R[X]$, X an indeterminate. (We do not know if the converse holds.) Finally, Section 4 gives an example of a domain which is locally, but not globally, distinguished. This example also serves to show that having every prime be either a U-prime or a V-prime is not enough to guarantee the domain is distinguished.

Throughout this paper, R will be an integral domain with quotient field K. We will repeatedly use the fact that for $z \in K$, and P a prime ideal of R, $(R : zR)R_P = (R_P : zR_P)$. In particular, it is not hard to see that if $(1 : z)R_P$ consists of zero divisors modulo $(1 : z^{-1})R_P$ then $(1 : z)$

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consists of zero divisors modulo $(1 : z^{-1})$, and so we have that if R is a distinguished domain, then so is R_P.

1. Examples of distinguished domains.

PROPOSITION 1.1. *Krull domains are distinguished domains.*

Proof. For $0 \neq z \in K$, the associated primes of $(1 : z^{-1})$ are the finitely many height one primes containing it. As none of these contain $(1 : z)$ (since height one prime localizations are valuation rings) the prime avoidance lemma shows there is a $b \in (1 : z)$ which is not a zero divisor modulo $(1 : z^{-1})$.

THEOREM 1.2. *R is a Prüfer domain if and only if R is a distinguished domain such that for any maximal ideal M, the prime ideals of R_M are linearly ordered by inclusion.*

Proof. Let R be a Prüfer domain. It is well known that the primes of R_M are linearly ordered. Now for $0 \neq z \in K$, $(1 : z) + (1 : z^{-1}) = R$ and so $(1 : z)$ cannot consist entirely of zero divisors modulo $(1 : z^{-1})$. Thus R is a distinguished domain.

For the converse, it is enough to assume that R is a quasi-local distinguished domain whose primes are linearly ordered, and show that R is a valuation ring. Pick $0 \neq z \in K$ and suppose that z and z^{-1} are not in R. We will get a contradiction. Let P and Q be primes minimal over the (proper) ideals $(1 : z^{-1})$ and $(1 : z)$ respectively. By linear ordering and symmetry, assume that $Q \subseteq P$. Then $(1 : z) \subseteq Q \subseteq P$ and since P consists of zero divisors modulo $(1 : z^{-1})$, we have contradicted that R is a distinguished domain.

THEOREM 1.3. *Let R be a domain, and let Q be a prime ideal in R which is comparable to every ideal of R. Then R is a distinguished domain if and only if R_Q and R/Q are distinguished domains.*

Proof. Choose $c, d \in R - Q$. We first claim that $(c : d)$ does not consist of zero divisors modulo $(d : c)$ if and only if in $\bar{R} = R/Q$, $(\bar{c} : \bar{d})$ does not consist of zero divisors modulo $(\bar{d} : \bar{c})$. To see this, note that $Q \subseteq dR$ so that we easily see that $(\bar{d} : \bar{c}) = \overline{(d : c)}$. Similarly $(\bar{c} : \bar{d}) = \overline{(c : d)}$. In particular, $R/(d : c) \approx \bar{R}/(\bar{d} : \bar{c})$, and our claim becomes obvious.

Now suppose that R is a distinguished domain. It follows easily from the above claim that $\bar{R}$ is also a distinguished domain. We have previously noted that R_Q is a distinguished domain.

Conversely, suppose that both R_Q and $\bar{R}$ are distinguished domains. Pick $0 \neq z \in K$. We will show that $(1 : z)$ does not consist entirely of zero divisors modulo $(1 : z^{-1})$. If $(1 : z) = R$ this is obvious. Thus assume that $z \notin R$. As a first case, assume that $(1 : z^{-1}) \subseteq Q$. Since R_Q

is a distinguished domain, there is a $b \in (1 : z)R_Q$ which is not a zero divisor modulo $(1 : z^{-1})R_Q$. Obviously we may assume $b \in (1 : z)$. We now claim that in fact b is not a zero divisor modulo $(1 : z^{-1})$. For this, suppose that $ba \in (1 : z^{-1})$. Then $ba \in (1 : z^{-1})R_Q$ so that $a \in (1 : z^{-1})R_Q$. Now write $a = a'/s$ with $a' \in (1 : z^{-1})$ and $s \in R - Q$. Since $z \notin R$, clearly $a' \notin a'z^{-1}R$. However $a' \in (1 : z^{-1}) \subseteq Q$, showing that $Q \nsubseteq a'z^{-1}R$. By hypothesis, $a'z^{-1}R \subseteq Q \subseteq sR$. Thus $az^{-1} = (a'/s)z^{-1} \in R$, so that $a \in (1 : z^{-1})$. Therefore b is not a zero divisor modulo $(1 : z^{-1})$, as desired.

In the second case, we have $(1 : z^{-1}) \nsubseteq Q$. Write $z^{-1} = c/d$ with $d \notin Q$. If $c \in Q$ then $c \in Q \subseteq dR$ and so $z^{-1} \in R$, $(1 : z^{-1}) = R$, and our assertion holds trivially. Therefore assume that $c \notin Q$, as well. As $\bar{R}$ is a distinguished domain, $(\bar{c} : \bar{d})$ does not consist entirely of zero divisors modulo $(\bar{d} : \bar{c})$. By the opening paragraph of this proof, $(1 : z) = (c : d)$ does not consist entirely of zero divisors modulo $(d : c) = (1 : z^{-1})$.

COROLLARY 1.4. *Let (T, Q) be a quasi-local domain. Let D be a domain whose quotient field is T/Q. Let $R = \{t \in T | t + Q \in D\}$. Then R is a distinguished domain if and only if both T and D are distinguished domains.*

Proof. Obviously Q is a prime in R. If $s \in R - Q$, then s is a unit in T and so $Q = sQ \subset sR$. Thus Q is comparable to any ideal of R. We claim that $R_Q = T$ and $R/Q = D$. The second fact is obvious. For the first, clearly $R_Q \subseteq T$. For $t \in T$, if $t \in Q$ then $t \in Q \subset R \subset R_Q$. If $t \notin Q$, then $t + Q$ is nonzero in T/Q, the quotient field of D. Thus there are t_1, t_2 in $R - Q$ with

$$t + Q = \frac{t_1 + Q}{t_2 + Q}.$$

Thus $t_2t = t_1 + q$, $q \in Q$. Therefore $t_2t \in R$ and so $t \in R_Q$. We now have $R_Q = T$ and $R/Q = D$ as claimed. The corollary is now immediate from Theorem 1.3.

Example. As a simple application of Corollary 1.4, let F be a field and let X, Y, and Z be indeterminates. Let $T = F(X, Y)[[Z]]$ with $Q = ZT$. Then T is a D.V.R., hence a distinguished domain. As $T/Q = F(X, Y)$, let $D = F[X, Y]$. Forming R as in Corollary 1.4, we see that R consists of those power series in T whose constant term comes from $F[X, Y]$. As T and D are distinguished domains, R is a distinguished domain.

Example. Let $R = \mathbf{Z}[\{X/p | p \text{ is a prime integer}\}]$. This ring was constructed by Eakin and Silver [**1**] as an example of a domain which is locally, but not globally, a polynomial ring. It is also a distinguished domain (details of verification are omitted), though it is not in any of the previously mentioned classes of examples.

LEMMA 1.5. *Let* $z = a/b$, $a, b \in R$. *Then* $b \in (1 : z)$ *is not a zero divisor modulo* $(1 : z^{-1})$ *if and only if* $(a : b) = (a : b^2)$.

Proof. This is straightforward.

PROPOSITION 1.6. *Let R be an integrally closed domain and let T be an integrally closed integral extension domain of R. If* $a, b \in R$ *with* $(a : b) = (a : b^2)$, *then*

$$(aT : bT) = (aT : b^2T).$$

Proof. First, by induction we prove that $(a^n : b^n) = (a^n : b^{2n})$, $n = 1, 2, \ldots$. Let

$$r \in (a^{n+1} : b^{2n+2}) \subseteq (a^n : b^{2n+2}) \subseteq (a^n : b^{4n}) = (a^n : b^n),$$

the last step by induction. Thus $rb^n/a^n \in R$ so that

$$rb^n/a^n \in (a : b^{n+2}) = (a : b).$$

This shows that $r \in (a^{n+1} : b^{n+1})$ as desired.

Now suppose that $c \in (aT : b^2T)$. As c is integral over R, which is integrally closed, the minimal polynomial of c over K is in $R[X]$, say

$$c^m + r_1c^{m-1} + \ldots + r_m = 0.$$

Since $b^2/a \in K$, the minimal polynomial of $c(b^2/a)$ is just

$$(cb^2/a)^m + (b^2/a)r_1(cb^2/a)^{m-1} + \ldots + (b^2/a)^m r_m = 0.$$

However $cb^2/a \in T$, and so each coefficient of this last polynomial is in R, as well. Thus $r_i \in (a^i : b^{2i}) = (a^i : b^i)$, so that $r_ib^i/a^i \in R$. However

$$(cb/a)^m + (b/a)r_1(cb/a)^{m-1} + \ldots + (b/a)^m r_m = 0$$

showing $cb/a \in T$. Thus $c \in (aT : bT)$.

PROPOSITION 1.7. *Let R be a Krull domain and let L be an algebraic extension of K. Let T be the integral closure of R in L. Then T is a distinguished domain.*

Proof. Say $0 \neq z \in L$. Let $F = K(z)$ and let D be the integral closure of R in F. Then D is a Krull domain, hence a distinguished domain. Thus we may write $z = a/b$, $a, b \in D$ with $(aD : bD) = (aD : b^2D)$. By Proposition 1.6, $(aT : bT) = (aT : b^2T)$ and so we are done, using Lemma 1.5.

Question 1. Let R be a distinguished domain and let T be the integral closure of R in some algebraic extension of K. Is T a distinguished domain?

Using Proposition 1.6 (as we did in Proposition 1.7) we see that if the answer is yes for simple algebraic extensions of K, then it is always yes.

We now show that distinguished domains are integrally closed, but that the converse fails.

PROPOSITION 1.8. *If R is a distinguished domain, then R is integrally closed.*

Proof. Suppose $0 \neq z \in K$ is integral over R, with

$$z^n + r_{n-1}z^{n-1} + \ldots + r_0 = 0, \quad r_i \in R.$$

Write $z = a/b$ as in Lemma 1.5. Thus

$$a^n + br_{n-1}a^{n-1} + \ldots + b^n r_0 = 0,$$

so that

$$r_0 \in (a : b^n) = \ldots = (a : b^2) = (a : b).$$

Writing $br_0 = ar_0'$ we find that

$$z^{n-1} + r_{n-1}z^{n-2} + \ldots + (r_1 + r_0') = 0.$$

By induction on n, $z \in R$.

Remark. The existence of an integrally closed 1-dimensional quasi-local domain R which is not a valuation domain is well known. Such an R cannot be a distinguished domain, since by Theorem 1.2 it would then be a Prüfer domain and hence a valuation ring. We now exhibit another integrally closed domain which is not a distinguished domain.

Example. Let $R = \mathbf{Z} + 2X\mathbf{Z}[X]$. It is easily verified that R is integrally closed. However it is not a distinguished domain. Let $z = X^{-1} = a/b$ with $a, b \in R$. Then

$$(a : b) = (1 : X) = 2\mathbf{Z} + 2X\mathbf{Z}[X].$$

However $b = Xa$ shows that $b \in 2X\mathbf{Z}[X]$. Thus $b^2/a = bX \in R$ so that $(a : b^2) = R \neq (a : b)$. By Lemma 1.5, R is not distinguished.

2. U-primes and V-primes. In any integral domain, prime ideals naturally fall into two categories; those which are minimal over the conductor of some element in the quotient field and those which are not. If we denote the first set by F, $R = \bigcap_{q \in F} R_q$ and localizations are merely sub-intersections. In particular, if P is a prime of the second type, $P = \bigcup_{q \subsetneq P} q$ and $R_P = \bigcap_{q \subsetneq P} R_q$ [**2**, p. 118]. Since primes of the second type are basically determined by those in F, it is those in F which hold the most interest. In distinguished domains, these primes prove to have a very distinctive and useful character.

PROPOSITION 2.1. *Let P be a prime ideal in the distinguished domain R which is minimal over the conductor of some $0 \neq \alpha \in K$. Then there is a* *unique prime Q maximal with respect to the property of being properly contained in P. Further, if $0 \neq w \in R_Q$, then either w or $w^{-1} \in R_P$ and Q_P is comparable to each ideal of R_P. So ideals I_P and J_P can be incomparable only if $I, J \subsetneq Q$.*

Proof. We may assume R is quasi-local with maximal ideal P. Let Q be the union of all primes properly contained in P. As P is minimal over $(1 : \alpha)$, P consists of zero divisors modulo $(1 : \alpha)$ and so $(1 : \alpha^{-1}) \not\subset P$; thus $\alpha^{-1} \in R$. $(1 : \alpha) = \alpha^{-1}R$ and so $\alpha^{-1} \in P - Q$, yielding $Q \subsetneq P$. If $x \in P - Q$ and $y \in Q$, then $y \in p \subsetneq P$ and so $(y : x) \subset p \subset P$. Since $x \notin Q$, either $(x : y) = R$ or P is minimal over $(x : y)$. If P were minimal over $(x : y)$, P, and hence also $(y : x)$, would consist of zero divisors modulo $(x : y)$. Thus $(x : y) = R$ and $y/x \in R$. As $x^n \notin Q$, we get $y/x^n \in R$ and so $Q \subset \bigcap_{n=1}^{\infty} x^nR$. Also, since $x^n \notin x^{n+1}R$, no power of x is contained in $\bigcap_{n=1}^{\infty} x^nR$ and there exists a prime $p \supseteq \bigcap_{n=1}^{\infty} x^nR$ with $x \notin p$. Thus

$$Q \subset \bigcap_{n=1}^{\infty} x^nR \subset p \subset Q$$

and so $Q = \bigcap x^nR$ is an ideal. Since Q is a union of prime ideals, it is clearly a prime ideal. If $x \in R - Q$, then we already have $Q \subseteq xR$. Thus Q is comparable to any ideal of R. Next, if $w \in R_Q$ and $w \notin R_P$, then P is minimal over $(1 : w)$ and so consists of zero divisors modulo $(1 : w)$. Hence $(1 : w^{-1}) \not\subset P$ and $w^{-1} \in R$. The last statement is evident.

Definition. Call the nonzero prime ideal P of the domain R a *U-prime* if $R_P = \bigcap R_p$ over all primes p properly contained in P. Call P a *V-prime* if there is a prime Q contained in P with height $(P/Q) = 1$ and for any $0 \neq w \in R_Q$, either w or w^{-1} is in R_P.

Remarks. (a) If P is a U-prime, it is easily seen that P is the union of the primes it properly contains and that P is not minimal over $(1 : w)$ for any $0 \neq w \in K$.

(b) If P is a V-prime and if Q is as in the definition, then it is easily seen that Q_P is comparable to any ideal of R_P, and that R_P/Q_P is a valuation ring.

(c) By Proposition 2.1 and the comments preceding it, we see that in a distinguished domain any nonzero prime is either a U-prime or a V-prime.

(d) A domain can have every nonzero prime be either a U-prime or a V-prime and yet not be a distinguished domain. In Section 4 we exhibit a domain which is locally, but not globally, distinguished. Being locally distinguished obviously implies that every nonzero prime is either a U-prime or a V-prime.

PROPOSITION 2.2. *Let R be a domain in which every nonzero prime is either a U-prime or a V-prime. Let $0 \neq z \in K$. Then the zero divisors modulo $(1 : z^{-1})$ are the union of those V-primes which contain $(1 : z^{-1})$ but do not contain $(1 : z)$.*

Proof. First, let P be a V-prime containing $(1 : z^{-1})$ but not containing $(1 : z)$. It is enough to show that P_P consists of zero divisors modulo $(1 : z^{-1})R_P$. Thus we assume that R is quasi-local at P. Since $(1 : z) \not\subseteq P$, $z \in R$, and $(1 : z^{-1}) = zR$. Let Q be the largest prime properly contained in P. If $z \notin Q$, then P is minimal over zR and so consists of zero divisors modulo $zR = (1 : z^{-1})$. Thus suppose that $z \in Q$. For any $c \in P - Q$ we have $Q \subseteq cR$ so that $z = cd$ for some $d \in R - zR$. Therefore $P - Q$ consists of zero divisors modulo zR, and so P is obviously a maximal prime of zero divisors modulo $zR = (1 : z^{-1})$.

Conversely, suppose that $xy \in (1 : z^{-1})$ with $y \in R - (1 : z^{-1})$. Then $x \in (1 : yz^{-1})$ and this ideal is proper. Let P be a prime minimal over $(1 : yz^{-1})$. As U-primes cannot be minimal over ideals of this form, P is a V-prime. Let Q be the largest prime properly contained in P. Now $(1 : yz^{-1}) \not\subseteq Q$, so $yz^{-1} \in R_Q$. By the definition of V-prime, we must have $y^{-1}z \in R_P$, so that $(1 : y^{-1}z) \not\subseteq P$. Now

$$(1 : z^{-1}) \subseteq (1 : yz^{-1}) \subseteq P \quad \text{and} \quad (1 : y^{-1}z) \subseteq (1 : z)$$

so that $(1 : z) \not\subseteq P$. As P is a V-prime and $x \in (1 : yz^{-1}) \subseteq P$, our result is proved.

Remark. The distinction between domains in which every nonzero prime is either a U-prime or a V-prime, and distinguished domains, is a matter of prime avoidance. Specifically, let R be a domain in which every nonzero prime is either a U-prime or a V-prime. Now consider $0 \neq z \in K$, and

$$S = \{P | P \text{ is a } V\text{-prime of } R, (1 : z^{-1}) \subseteq P, \text{ and } (1 : z) \not\subseteq P\}.$$

By definition, $(1 : z)$ is not contained in any $P \in S$. However, is $(1 : z) \subseteq \cup P$, $P \in S$? R is a distinguished domain if and only if the answer is no, for all $0 \neq z \in K$.

PROPOSITION 2.3. *Let R be a distinguished domain and let $0 \neq z \in K$. There are elements $b_1, b_2 \in (1 : z)$ such that if P is a V-prime with $(1 : z) \not\subseteq P$ then $\{b_1, b_2\} \not\subseteq P$.*

Proof. Choose $b_1 \in (1 : z)$ to be not a zero divisor modulo $(1 : z^{-1})$ and let $a_1 = b_1 z$. By Lemma 1.5, $(a_1 : b_1) = (a_1 : b_1^2)$. We now pick $a_2 \in (a_1 : b_1^2)$ to be not a zero divisor modulo $(b_1^2 : a_1)$. As $a_2 \in (a_1 : b_1^2) = (a_1 : b_1)$, let

$$b_2 = a_2 b_1 / a_1 \in (b_1 : a_1) = (1 : z).$$

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Now suppose that P is a V-prime not containing $(1 : z)$, but which does contain b_1. We will show $b_2 \notin P$. Since $b_1 \in P$, P does not consist entirely of zero divisors modulo $(1 : z^{-1})$. Since $(1 : z) \not\subseteq P$, Proposition 2.2 shows that $(1 : z^{-1}) \not\subseteq P$, so that z is a unit in R_P. Since $b_1 \in P$,

$$(b_1{}^2 : a_1) = (b_1 : z) \subseteq P,$$

while

$$(a_1 : b_1{}^2) = (a_1 : b_1) = (1 : z^{-1}) \not\subseteq P.$$

By Proposition 2.2, P consists of zero divisors modulo $(b_1{}^2 : a_1)$, and in particular, $a_2 \notin P$. Now $(b_2 : a_2) = (1 : z^{-1}) \not\subseteq P$, and since $a_2 \notin P$ we must have $b_2 \notin P$.

Question 2. Let R be a domain in which every nonzero prime is either a U-prime or a V-prime. Suppose that for each $0 \neq z \in K$, there are finitely many $b_1, \ldots, b_n \in (1 : z)$ such that any V-prime which does not contain $(1 : z)$ does not contain $\{b_1, \ldots, b_n\}$. Is R a distinguished domain?

In the proof of Proposition 3.3 we will note that the hypothesis of this question does imply that $R[X]$ is a distinguished domain.

Notation. Let $T = \{\alpha_0, \ldots, \alpha_n\} \subseteq K$. We will use J_T to denote $(1 : \alpha_0) \cap \ldots \cap (1 : \alpha_0)$.

COROLLARY 2.4. *Let R be a distinguished domain and let $T = \{\alpha_0, \ldots, \alpha_n\} \subseteq K$. Then there is a finite set $S \subseteq J_T$ such that if P is a V-prime of R with $J_T \not\subseteq P$, then $S \not\subseteq P$.*

Proof. For $i = 0, \ldots, n$, find $b_{i1}, b_{i2} \in (1 : \alpha_i)$ satisfying Proposition 2.3. Then the desired S is

$$S = \left\{ \prod_{i=0}^{n} b_{i\sigma(i)} \middle| \sigma \text{ varies over all functions from } \{0, 1, \ldots, n\} \text{ to } \{1,2\} \right\}.$$

3. Adjoining an indeterminate.

LEMMA 3.1. *Let P be a V-prime of R, and let Q be the largest prime properly contained in P. Let $\{\alpha_0, \ldots, \alpha_n\} \subseteq R_Q$ with $\{\alpha_0, \ldots, \alpha_n\} \not\subseteq R_P$. Then for some $i = 0, 1, \ldots, n$, $\{\alpha_0/\alpha_i, \ldots, \alpha_0/\alpha_i\} \subseteq R_P$.*

Proof. We may assume R is quasi-local at P. Since each $\alpha_j \in R_Q$, there is a $c \notin Q$ with $t_j = c\alpha_j \in R$ for each j. As $Q \subseteq cR$ and not every α_j is in R, some $t_j \notin Q$. We also have that t_jR and t_kR are comparable whenever $t_j \notin Q$. Thus, for some i, $t_iR \supseteq t_jR$ for every j and so $t_j/t_i \in R$. Finally,

$$\alpha_j/\alpha_i = c^{-1}t_j/c^{-1}t_i = t_j/t_i.$$

Recall that if R is integrally closed and if $\alpha(X)$, $\beta(X) \in K[X]$, then $\alpha(X)\beta(X) \in R[X]$ if and only if any coefficient of $\alpha(X)$ times any coefficient of $\beta(X)$ is in R.

THEOREM 3.2. *If R is a distinguished domain, then $R[X]$ is a distinguished domain.*

Proof. For $0 \neq z \in K(X)$, write $z = f(X)/g(X)$ with $f(X)$, $g(X) \in R[X]$ and relatively prime in $K[X]$. Let $g_0, \ldots, g_n$ be the nonzero coefficients of $g(X)$, and for $i = 0, 1, \ldots, n$, let

$$T_i = \{a/g_i | a \text{ is a coefficient of either } f(X) \text{ or } g(X)\}.$$

By Corollary 2.4 there is a finite set $S_i \subseteq J_{T_i}$ such that if P is a V-prime with $J_{T_i} \not\subseteq P$, then $S_i \not\subseteq P$. Let $S = S_0 \cup \ldots \cup S_n$. We claim that there is an $h(X) \in (1 : z)$ with $h(X)$ relatively prime to $f(X)$ in $K[X]$, and such that every element of S occurs as a coefficient of $h(X)$. For $s \in S$, we have $s \in S_i \subseteq J_{T_i}$ for some i, so that $s(f(X)/g_i)$ and $s(g(X)/g_i)$ are both in $R[X]$. Letting $h_s(X) = s(g(X)/g_i)$, clearly

$$h_s(X) \in (g(X) : f(X)) = (1 : z).$$

Now let

$$h'(X) = \sum X^{e_s} h_s(X),$$

the sum taken over $s \in S$, and the e_s chosen to increase rapidly enough (with respect to some ordering of S) that every $s \in S$ appears as a coefficient of $h'(X)$. Obviously $h'(X) \in (1 : z)$. Now select a nonunit $r \in R$. (If R is a field, $R[X]$ is a distinguished domain.) Let $e = \deg g(X) + 1$ and $m = 1, 2, 3, \ldots$. Let

$$h_m(X) = X^e h'(X) + r^m g(X) \in (1 : z).$$

Every $s \in S$ appears as a coefficient of $h_m(X)$, by the choice of e. As $f(X)$ has only finitely many irreducible factors in $K[X]$, and as $h_m(X) - h_k(X) = (r^m - r^k)g(X)$ is relatively prime to $f(X)$ for $m \neq k$, clearly some $h_m(X)$ is relatively prime to $f(X)$. Let $h(X)$ be that $h_m(X)$, proving our claim.

We have $h(X) \in (1 : z)$. We will show that $h(X)$ is not a zero divisor modulo $(1 : z^{-1})$, which will complete the proof. Suppose that $u(X) \in R[X]$ and $h(X)u(X) \in (1 : z^{-1})$. We will show $u(X) \in (1 : z^{-1})$. Since both $h(X)$ and $g(X)$ are relatively prime to $f(X)$ in $K[X]$, $h(X)u(X) \in (1 : z^{-1})$ clearly implies that $u(X) = f(X)\alpha(X)$ for some $\alpha(X) \in K[X]$. We have

$$h(X) f(X)\alpha(X) = h(X)u(X) \in (1 : z^{-1})$$

and we want $f(X)\alpha(X) \in (1 : z^{-1})$. Since $z^{-1} = g(X)/f(X)$ we may restate this: We have $h(X)g(X)\alpha(X) \in R[X]$ and we want $g(X)\alpha(X) \in R[X]$. Using the comment immediately preceding this theorem, clearly it is enough to assume that $\alpha(X) = \alpha$ is a constant in K.

We are now assuming that $\alpha \in K$ and that $h(X)g(X)\alpha \in R[X]$. We wish to show that $g(X)\alpha \in R[X]$. Suppose not. Recalling that $g_0, \ldots, g_n$ are the nonzero coefficients of $g(X)$, $g(X)\alpha \notin R[X]$ implies that $(1 : \alpha g_0) \cap \ldots \cap (1 : \alpha g_n)$ is a proper ideal of R. Let P be a prime minimal over this intersection. Then P is minimal over some $(1 : \alpha g_k)$ and since U-primes are never minimal over ideals of this form, P is a V-prime. Let Q be the largest prime properly contained in P. Then

$$(1 : \alpha g_0) \cap \ldots \cap (1 : \alpha g_n) \nsubseteq Q$$

so that

$$\{\alpha g_0, \ldots, \alpha g_n\} \subseteq R_Q,$$

while by the choice of P,

$$\{\alpha g_0, \ldots, \alpha g_n\} \nsubseteq R_P.$$

Since $\alpha f(X) = u(X) \in R[X]$, if $f(X) = f_m X^m + \ldots + f_0$, then

$$\{\alpha f_0, \ldots, \alpha f_m, \alpha g_0, \ldots, \alpha g_n\} \subseteq R_Q$$

while

$$\{\alpha f_0, \ldots, \alpha f_m, \alpha g_0, \ldots, \alpha g_n\} \nsubseteq R_P.$$

By Lemma 3.1, one member of $\{\alpha f_0, \ldots, \alpha f_m, \alpha g_0, \ldots, \alpha g_n\}$ divides all of the other members in R_P. Since some $\alpha g_k \notin R_P$ while all $\alpha f_l \in R_P$, clearly $\alpha g_k/\alpha f_l \notin R_P$. Thus, in fact, for some αg_i, we have

$$\{\alpha f_0/\alpha g_i, \ldots, \alpha f_m/\alpha g_i, \alpha g_0/\alpha g_i, \ldots, \alpha g_n/\alpha g_i\} \subseteq R_P.$$

That is, the set T_i, defined at the start of this proof, is in R_P. Thus $J_{T_i} \nsubseteq P$ and so for some $s \in S_i \subseteq S$, $s \notin P$. However s is a coefficient of $h(X)$. Since

$$h(X)g(X)\alpha \in R[X] \subseteq R_P[X],$$

and since R_P is integrally closed, any coefficient of $h(X)$ multiplied by any coefficient of $g(X)\alpha$ is in R_P. Since s is a unit in R_P, we see that $\alpha g_0, \ldots, \alpha g_n$ are all in R_P, which is a contradiction. This completes the proof.

Remark. If every prime of R is either a U-prime or a V-prime, then the same is true of $R[X]$. The proof is arduous, and we spare the reader.

Question 3. If $R[X]$ is a distinguished domain, is R a distinguished domain?

If the answer to Question 2 is yes, then the answer to Question 3 is also yes. To see this, it is not hard to see that if $R[X]$ is a distinguished domain then every nonzero prime of R is either a U-prime or a V-prime, with V-primes in R extending to V-primes in $R[X]$. Now for $0 \neq z \in K$,

$$(R[X] : zR[X]) = (1 : z)R[X]$$

and by Proposition 2.3 there are $g_1(X)$, $g_2(X) \in (1 : z)R[X]$ with $\{g_1(X), g_2(X)\}$ not contained in any V-prime of $R[X]$ which does not contain $(1 : z)R[X]$. Let $b_1, \ldots, b_n$ be the coefficients of $g_1(X)$ and $g_2(X)$. Then $\{b_1, \ldots, b_n\}$ is not in any V-prime of R which does not contain $(1 : z)$. Thus the truth of Question 2 implies the truth of Question 3.

Question 4. Suppose that R has only finitely many maximal ideals and that R_M is a distinguished domain for each maximal M. Is R a distinguished domain?

If we do not restrict this question to finitely many maximals, the answer is no, as is shown by the example in the next section. If the answer to Question 3 is yes, then the answer to Question 4 is yes. This follows immediately from our next proposition.

PROPOSITION 3.3. *Let R have only finitely maximal ideals and let R_M be a distinguished domain for each maximal M. Then $R[X]$ is a distinguished domain.*

Proof. Let $0 \neq z \in K$. By Proposition 2.3, for each maximal M there are $b_{1M}, b_{2M} \in (1 : z)$ such that if P is a V-prime contained in M with $(1 : z) \not\subseteq P$, then $\{b_{1M}, b_{2M}\} \not\subseteq P$. Let $S' = \cup \{b_{1M}, b_{2M}\}$ over all maximal M. Then S' is a finite set contained in $(1 : z)$, and if P is a V-prime with $(1 : z) \not\subseteq P$ then $S' \not\subseteq P$. We now see that our present hypothesis on R implies that the conclusion of Corollary 2.4 holds. Now the argument used in proving Theorem 3.2 only used three facts: R was integrally closed, every nonzero prime of R was either a U-prime or a V-prime, and the conclusion of Corollary 2.4 held. Each of these facts is true for our present R, and so as in Theorem 3.2, $R[X]$ is a distinguished domain.

4. A counterexample. We now present an example of a locally distinguished domain which is not a distinguished domain. Being locally distinguished, every prime of this domain is either a U-prime or a V-prime.

Example. Let F be a field and let X, Y, and $Z = \{Z_1, Z_2, \ldots\}$ be indeterminates. For $j = 1, 2, 3, \ldots$, let

$$R_j = F(Z - \{Z_j\}, X/Z_j, Y/Z_j)[Z_j]_{(Z_j)}.$$

Then R_j is a D.V.R. and the associated valuation is 1 at X and Y and is 0 at any polynomial in $Z - \{Z_j\}$. Now let

$$R_0 = F(Z)[X, Y]_{(X,Y)}.$$

For $j = 0, 1, 2, \ldots$, let N_j be the maximal ideal of R_j. Finally, let $R = \cap R_j, j = 0, 1, 2, \ldots$ and let $M_j = N_j \cap R$.

Claim. $R_{M_j} = R_j$. For $j = 0$, $Z \subseteq R - M_j$. Thus $F(Z)[X, Y] \subseteq R_{M_j}$ and the claim follows. For $j > 0$, $Z - \{Z_j\} \subseteq R - M_j$. Also X/Z_j and Y/Z_j are in $R - M_j$. Thus

$$F(Z - \{Z_j\}, X/Z_j, Y/Z_j)[Z_j] \subseteq R_{M_j}$$

and again the claim follows.

Claim. $M_j, j = 0, 1, 2, \ldots$, are exactly the maximal ideals of R. Let I be any ideal of R and suppose that $I \nsubseteq M_j$ for any j. Pick $f \in I - M_0$. Notice that $F[X, Y, Z] \subseteq R$. As $f \in R \subset R_0 = F[Z, X, Y]_{(X,Y)}$, write $f = g/h$ with $g, h \in F[Z, X, Y]$ and $h \notin (X, Y)F[Z, X, Y]$. Thus $h \in R - M_0$ and so $g = fh \in R - M_0$. However $g \in I$, since $f \in I$. We now have $g \in F[X, Y, Z]$ but $g \notin M_0$. Thus $g \notin (X, Y)F[X, Y, Z]$ and so we may write $g = g_1 + g_2$ with $g_1 \in (X, Y)F[X, Y, Z]$ and $g_2 \in F[Z]$ and with $g_2 \neq 0$. Suppose Z_j does not appear in g_2. Then g_2 is a unit in R_j. Since X and Y are nonunits in R_j, $g_1 + g_2$ is a unit in R_j. Thus $g = g_1 + g_2 \notin M_j$ for any j with Z_j not appearing in g_2. Therefore, let $M_{j_1}, \ldots, M_{j_n}$ be all of the M_j, $j > 0$, which contain g. Select $g' \in I - (M_{j_1} \cup \ldots \cup M_{j_n})$ (recall we are assuming that I is in no M_j). Let

$$k = X/Z_{j_1}Z_{j_2} \ldots Z_{j_n}.$$

Note that k is in all R_j, and is in N_j exactly when $j \notin \{j_1, \ldots, j_n\}$. We now easily see that $g + kg'$ is not in any N_j. Thus $g + kg'$ is a unit in R. However $g + kg' \in I$. Therefore $I = R$. Thus any proper ideal of R must be contained in some M_j, $j = 0, 1, \ldots$. This shows that the M_j include all the maximal ideals. However by the preceding claim, obviously there are no containment relations between two different M_j, and so all M_j must be maximal.

Claim. $M_0 \subseteq \cup M_j$, $j > 0$. Say $f \in M_0$. Write $f = g/h \in R_0$ with g, $h \in F[Z, X, Y]$ and $h \notin (X, Y)F[Z, X, Y]$. Since $f \in M_0$, clearly $g \in (X, Y)F[Z, X, Y]$. As $h \notin (X, Y)F[Z, X, Y]$, the argument in the previous claim shows that $h \notin M_j$ for all but finitely many j. However X, $Y \in M_j$ for all j, so that $g \in M_j$ for all j. Thus $f \in M_j$ for all but finitely many j.

Claim. R is locally a distinguished domain. This is obvious from the preceding.

Claim. R is not a distinguished domain. We will show that $(X : Y^2)$ consists of zero divisors modulo $(Y^2 : X)$. Since for $j > 0$, the valuation associated with R_j gives both X and Y value 1, $X/Y^2 \notin R_j$ and $Y^2/X \in R_j$. Thus $(Y^2 : X) \subseteq M_j$ and $(X : Y^2) \nsubseteq M_j$ for all $j > 0$. By Proposition 2.2, $\cup M_j, j > 0$, consists of zero divisors modulo $(Y^2 : X)$. By the preceding claim, M_0 consists of zero divisors modulo $(Y^2 : X)$. Since obviously $Y^2/X \notin R_0$, $(X : Y^2) \subseteq M_0$, and so $(X : Y^2)$ consists of zero divisors modulo $(Y^2 : X)$. This completes the example.

Remark. With R as in the above example, and T an indeterminate over R, it is not difficult to show that $R[T]$ is also not a distinguished domain. In fact with X and Y as above, $(XR[T] : Y^2R[T])$ consists of zero divisors modulo $(Y^2R[T] : XR[T])$. This is seen using the fact that any $f \in M_0$ is in all but finitely many M_j.

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