

## Multiple recurrence and popular differences for polynomial patterns in rings of integers

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### Abstract

We demonstrate that the phenomenon of popular differences (aka the phenomenon of large intersections) holds for natural families of polynomial patterns in rings of integers of number fields. If  $K$  is a number field with ring of integers  $\mathcal{O}_K$  and  $E \subseteq \mathcal{O}_K$  has positive upper Banach density  $d^*(E) = \delta > 0$ , we show, *inter alia*:

- (1) if  $p(x) \in K[x]$  is an *intersective* polynomial (i.e.,  $p$  has a root modulo  $m$  for every  $m \in \mathcal{O}_K$ ) with  $p(\mathcal{O}_K) \subseteq \mathcal{O}_K$  and  $r, s \in \mathcal{O}_K$  are distinct and nonzero, then for any  $\varepsilon > 0$ , there is a syndetic set  $S \subseteq \mathcal{O}_K$  such that for any  $n \in S$ ,

$$d^* (\{x \in \mathcal{O}_K : \{x, x + rp(n), x + sp(n)\} \subseteq E\}) > \delta^3 - \varepsilon.$$

Moreover, if  $s/r \in \mathbb{Q}$ , then there are syndetically many  $n \in \mathcal{O}_K$  such that

$$d^* (\{x \in \mathcal{O}_K : \{x, x + rp(n), x + sp(n), x + (r + s)p(n)\} \subseteq E\}) > \delta^4 - \varepsilon;$$

- (2) if  $\{p_1, \dots, p_k\} \subseteq K[x]$  is a *jointly intersective* family (i.e.,  $p_1, \dots, p_k$  have a common root modulo  $m$  for every  $m \in \mathcal{O}_K$ ) of linearly independent polynomials with  $p_i(\mathcal{O}_K) \subseteq \mathcal{O}_K$ , then there are syndetically many  $n \in \mathcal{O}_K$  such that

$$d^* (\{x \in \mathcal{O}_K : \{x, x + p_1(n), \dots, x + p_k(n)\} \subseteq E\}) > \delta^{k+1} - \varepsilon.$$

These two results generalise and extend previous work of Frantzikinakis and Kra [21] and Frantzikinakis [19] on polynomial configurations in  $\mathbb{Z}$  and build upon recent work of the authors and Best [2] on linear patterns in general abelian groups. The above combinatorial results follow from multiple recurrence results in ergodic theory via a version of Furstenberg's correspondence principle. The ergodic-theoretic recurrence theorems require a sharpening of existing tools for handling polynomial multiple ergodic averages. A key advancement made in this paper is a new result on the equidistribution of polynomial orbits in nilmanifolds, which can be seen as a far-reaching generalisation of Weyl's equidistribution theorem for polynomials of several variables:

- (3) let  $d, k, l \in \mathbb{N}$ . Let  $(X, \mathcal{B}, \mu, T_1, \dots, T_l)$  be an ergodic, connected  $\mathbb{Z}^l$ -nilsystem. Let  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l\} \subseteq \mathbb{Q}[x_1, \dots, x_d]$  be a family of polynomials such that  $p_{i,j}(\mathbb{Z}^d) \subseteq \mathbb{Z}$  and  $\{1\} \cup \{p_{i,j}\}$  is linearly independent over  $\mathbb{Q}$ . Then the  $\mathbb{Z}^d$ -sequence

$(\prod_{j=1}^l T_j^{p_{1j}(n)} x, \dots, \prod_{j=1}^l T_j^{p_{kj}(n)} x)_{n \in \mathbb{Z}^d}$  is well-distributed in  $X^k$  for every  $x$  in a co-meager set of full measure.

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1. Introduction

1.1. Background and main results

Let  $(X, \mathcal{B}, \mu, T)$  be an invertible probability measure-preserving system. A classical result of Khintchine [31] says that for any  $A \in \mathcal{B}$ ,

$$\lim_{N \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A) \geq \mu(A)^2.$$

As a consequence, for any  $\varepsilon > 0$ , the set

$$R = \left\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon \right\}$$

is syndetic, meaning that it has bounded gaps (equivalently, finitely many translates of  $R$  cover  $\mathbb{Z}$ ). Furstenberg showed in [22] that for any  $A \in \mathcal{B}$  and any  $k \in \mathbb{N}$ ,

$$\liminf_{N \rightarrow \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > 0,$$

from which it follows that

$$\left\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap \dots \cap T^{-kn}A) > c \right\}$$

is syndetic for some  $c > 0$ . One may ask, for these longer expressions, if  $c$  can be made arbitrarily close to  $\mu(A)^{k+1}$ . (By considering weakly mixing systems, it is clear that  $c$  cannot exceed  $\mu(A)^{k+1}$  in general.) A somewhat surprising answer was given in [6]:

THEOREM 1.1 ([6, theorems 1.2 and 1.3]).

(1) For any ergodic invertible probability measure-preserving system  $(X, \mathcal{B}, \mu, T)$ , any  $\varepsilon > 0$ , and any  $A \in \mathcal{B}$ , the set

$$\left\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap T^{-2n}A) > \mu(A)^3 - \varepsilon \right\} \tag{1.1}$$

is syndetic.

(2) For any ergodic invertible probability measure-preserving system  $(X, \mathcal{B}, \mu, T)$ , any  $\varepsilon > 0$ , and any  $A \in \mathcal{B}$ , the set

$$\left\{ n \in \mathbb{Z} : \mu(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A) > \mu(A)^4 - \varepsilon \right\}$$

is syndetic.

(3) There exists an ergodic system  $(X, \mathcal{B}, \mu, T)$  with the following property: for any integer  $l \geq 1$ , there is a set  $A = A(l) \in \mathcal{B}$  of positive measure such that

$$\mu(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A \cap T^{-4n}A) \leq \frac{1}{2} \mu(A)^l$$

for every integer  $n \neq 0$ .

In the terminology of [2], Theorem 1.1 shows that the families  $\{n, 2n\}$  and  $\{n, 2n, 3n\}$  have the large intersections property, while  $\{n, 2n, \dots, kn\}$  does not have the large intersections property for  $k \geq 4$ . The combinatorial content, via Furstenberg’s correspondence principle, is that, for arithmetic progression of length 3 and 4, one can find a “popular” common difference: if  $E \subseteq \mathbb{Z}$  has positive upper Banach density  $d^*(E) = \delta > 0$  and  $\varepsilon > 0$ , then there exists (syndetically many)  $n \neq 0$  such that

$$d^* (\{x \in \mathbb{Z} : \{x, x + n, x + 2n\} \subseteq E\}) > \delta^3 - \varepsilon,$$

and there exists (syndetically many)  $m \neq 0$  such that

$$d^* (\{x \in \mathbb{Z} : \{x, x + m, x + 2m, x + 3m\} \subseteq E\}) > \delta^4 - \varepsilon.$$

A natural question to ask is whether various extensions of Szemerédi’s theorem also admit large intersections variants.

The polynomial Szemerédi theorem of the second author and Leibman [7] extends Furstenberg’s result to polynomial configurations. We say that a polynomial  $p(x) \in \mathbb{Q}[x]$  is integer-valued if  $p(\mathbb{Z}) \subseteq \mathbb{Z}$ .

**THEOREM 1.2** ([7, special case of theorem A]). *Let  $p_1, \dots, p_k \in \mathbb{Q}[x]$  be integer-valued polynomials with zero constant term. Then for any invertible probability measure-preserving system  $(X, \mathcal{B}, \mu, T)$  and any  $A \in \mathcal{B}$  with  $\mu(A) > 0$ , there exists  $c > 0$  such that the set*

$$R := \left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > c \right\} \tag{1.2}$$

has positive lower density, i.e.  $\liminf_{N \rightarrow \infty} |R \cap \{1, \dots, N\}|/N > 0$ .

The conclusion of Theorem 1.2 was strengthened in [12, theorem 0.1], where it was shown that  $R$  is syndetic for some  $c > 0$  depending on  $A$ .

There is a wider variety of combinatorial configurations in play when polynomials are introduced, and there is not yet a full classification of which families of polynomials have the large intersections property. However, large intersections variants of the polynomial Szemerédi theorem are known for two natural classes of polynomial configurations: independent polynomials and polynomials that are integer multiples of a fixed polynomial (for  $k = 2, 3$ ). This is summarised by the following two results, which we seek to extend in this paper:

**THEOREM 1.3** ([21, theorem 1.3]). *Let  $p_1, \dots, p_k \in \mathbb{Q}[x]$  be linearly independent integer-valued polynomials with zero constant term. Then for any invertible probability measure-preserving system, any  $A \in \mathcal{B}$ , and any  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > \mu(A)^{k+1} - \varepsilon \right\}$$

is syndetic.

THEOREM 1.4 ([19, theorem C]). *Let  $p \in \mathbb{Q}[x]$  be an integer-valued polynomial with zero constant term, and let  $a, b \in \mathbb{Z}$  be nonzero and distinct. Then for any ergodic invertible probability measure-preserving system, any  $A \in \mathcal{B}$ , and any  $\varepsilon > 0$ , the sets*

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-ap(n)}A \cap T^{-bp(n)}A \right) > \mu(A)^3 - \varepsilon \right\}$$

and

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-ap(n)}A \cap T^{-bp(n)}A \cap T^{-(a+b)p(n)}A \right) > \mu(A)^4 - \varepsilon \right\}$$

are syndetic.

We have so far stated all results about polynomial multiple recurrence only for polynomials with zero constant term. The essential feature of such families of polynomials is that they avoid “local obstructions.” To be precise, we say that a family of polynomials  $\{p_1, \dots, p_k\}$  is *jointly intersective* if for every  $m \in \mathbb{N}$ , there exists  $n \in \mathbb{Z}$  such that  $p_i(n) \in m\mathbb{Z}$  for every  $i = 1, \dots, k$ . If a family of polynomials is not jointly intersective, then the set appearing in (1.2) will be trivial for some rotations on finitely many points. In [11], it was shown that there are no other obstacles to multiple recurrence:

THEOREM 1.5 ([11, theorem 1.1]). *For a family of integer-valued polynomials  $\mathcal{P} = \{p_1, \dots, p_k\} \subseteq \mathbb{Q}[x]$ , the following are equivalent:*

- (i)  $\mathcal{P}$  is jointly intersective;
- (ii) for any probability measure-preserving system  $(X, \mathcal{B}, \mu, T)$  and any  $A \in \mathcal{B}$ , there exists  $c > 0$  such that

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > c \right\}$$

is syndetic.

The proofs of Theorems 1.3 and 1.4 can also be easily modified to apply to families of jointly intersective polynomials.

The polynomial Szemerédi theorem is in fact known for polynomials of several variables with zero constant term (see [7, theorem A] for the result with positive lower density and [13, theorem 0.7] for syndeticity). For polynomials arising from rings of integers, the polynomial Szemerédi theorem holds for all jointly intersective polynomials. We now make this result precise. Fix a number field  $K$  and denote by  $\mathcal{O}_K$  its ring of integers. By an  $\mathcal{O}_K$ -system, we will mean a quadruple  $(X, \mathcal{B}, \mu, T)$ , where  $T$  is a measure-preserving action of  $(\mathcal{O}_K, +)$  on a probability space  $(X, \mathcal{B}, \mu)$ .

*Definition 1.6.* A family of  $\mathcal{O}_K$ -valued polynomials  $\{p_1, \dots, p_k\}$  is *jointly intersective* if for every finite index subgroup  $\Lambda \subseteq (\mathcal{O}_K, +)$ , there exists  $\xi \in \mathcal{O}_K$  such that  $\{p_1(\xi), \dots, p_k(\xi)\} \subseteq \Lambda$ .

Recall that in an abelian group  $G$ , a set  $E \subseteq G$  is *syndetic* if finitely many translates of  $E$  cover  $G$ . That is,  $G = \bigcup_{i=1}^m (E + g_i)$  for some  $g_1, \dots, g_m \in G$ .

THEOREM 1.7 ([14, theorem 1.6]). *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p_1, \dots, p_k \in \mathcal{O}_K[x]$  be jointly intersective polynomials. For any  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and*

any  $A \in \mathcal{B}$ , there exists  $c > 0$  such that the set

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > c \right\} \tag{1.3}$$

is syndetic.

It is therefore natural to ask whether Khintchine-type recurrence theorems hold for polynomial configurations in rings of integers. That is, under what conditions on the polynomials  $\{p_1, \dots, p_k\}$  can the constant  $c$  in (1.3) be made arbitrarily close to  $\mu(A)^{k+1}$ ?

In this paper, we provide an answer to this question in natural and important cases by proving extensions of Theorems 1.3 and 1.4.

**THEOREM A.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. Then for any measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > \mu(A)^{k+1} - \varepsilon \right\} \tag{1.4}$$

is syndetic.

**THEOREM B.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be an  $\mathcal{O}_K$ -valued intersective polynomial. Let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \right) > \mu(A)^3 - \varepsilon \right\} \tag{1.5}$$

is syndetic.

Moreover, if  $s/r \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \cap T^{-(r+s)p(n)}A \right) > \mu(A)^4 - \varepsilon \right\} \tag{1.6}$$

is syndetic.

Note that for a pair of polynomials  $\{p, q\} \subseteq K[x] \setminus \{0\}$ , either  $p$  and  $q$  are linearly independent over  $K$  or  $q = cp$  for some  $c \in K$ . Thus, we have the following immediate consequence of Theorems A and B together:

**COROLLARY 1.8.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $\{p, q\} \subseteq K[x]$  is a jointly intersective pair of  $\mathcal{O}_K$ -valued polynomials. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ , any  $A \in \mathcal{B}$ , and any  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p(n)}A \cap T^{-q(n)}A \right) > \mu(A)^3 - \varepsilon \right\}$$

is syndetic.

Theorem A shows that for independent families of any size, we can achieve Khintchine-type results. In contrast, Theorem B only demonstrates a Khintchine-type result for configurations of length three or four and requires ergodicity of the system (for counterexamples in the non-ergodic case, see [2, section 11.1]). Moreover, for length four, we have made additional assumptions, which we discuss below. To complete the picture, we now

address what happens for patterns of length five and longer. For concreteness, let us consider general polynomial families of the form  $\{a_1p, \dots, a_kp\}$ , where  $a_i \in \mathcal{O}_K$  and  $p(x) \in K[x]$  is  $\mathcal{O}_K$ -valued. In the simplest case when  $K = \mathbb{Q}$  and  $a_i = i$ , a combinatorial construction of Ruzsa rules out Khintchine-type results when  $k \geq 4$  (see item 3 of Theorem 1.1 above). In [2, corollary 12.14], this was generalised to any number field  $K$  and any integers  $a_i \in \mathbb{Z}$  for  $k \geq 4$ . Furthermore, [2, proposition 12.13] gives a combinatorial criterion for checking the case  $k = 4$  for any coefficients  $a_i \in \mathcal{O}_K$ . We do not know how to prove the requisite combinatorial result, but we believe that Khintchine-type results will fail for any non-trivial family  $\{a_1p, \dots, a_kp\}$  with  $k \geq 4$ .

Now we turn to the other conditions imposed for the patterns of length four appearing in Theorem B. The strategy of proof in Theorem B is to reduce to the linear case  $p(n) = n$  and then apply knowledge about linear patterns. General Khintchine-type results for linear patterns appear in [2] (subsequently improved in [1, 3]), where a similar distinction is made between patterns of length three and of length four:

**THEOREM 1.9** ([2, theorems 1.10 and 1.11]). *Let  $(G, +)$  be a countable discrete abelian group. Let  $(X, \mathcal{B}, \mu, (T_g)_{g \in G})$  be an ergodic measure-preserving  $G$ -system. Let  $A \in \mathcal{B}$  and  $\varepsilon > 0$ .*

- (1) *suppose  $\varphi, \psi : G \rightarrow G$  are homomorphisms such that the subgroups  $\varphi(G)$ ,  $\psi(G)$ , and  $(\psi - \varphi)(G)$  have finite index in  $G$ . Then*

$$\left\{ g \in G : \mu \left( A \cap T_{\varphi(g)}^{-1}A \cap T_{\psi(g)}^{-1}A \right) > \mu(A)^3 - \varepsilon \right\}$$

*is syndetic in  $G$ .*

- (2) *suppose  $r, s \in \mathbb{Z}$  are distinct and nonzero such that the subgroups  $rG$ ,  $sG$ ,  $(r + s)G$ , and  $(s - r)G$  have finite index in  $G$ . Then*

$$\left\{ g \in G : \mu \left( A \cap T_{rg}^{-1}A \cap T_{sg}^{-1}A \cap T_{(r+s)g}^{-1}A \right) > \mu(A)^4 - \varepsilon \right\}$$

*is syndetic in  $G$ .*

The second half of Theorem 1.9 was also proved independently in [37, theorem 1.3]. By absorbing a constant into the polynomial  $p$  in Theorem B, imposing the condition  $\frac{s}{r} \in \mathbb{Q}$  is equivalent to assuming  $r, s \in \mathbb{Z}$ , so our assumptions allow us to apply Theorem 1.9 in the linear case  $p(n) = n$ .

In [16], it was shown that, for a related finitary problem, there are automorphisms  $\varphi$  and  $\psi$  such that  $\varphi + \psi$  and  $\psi - \varphi$  are also automorphisms but for which a Khintchine-type result fails:

**THEOREM 1.10** ([16, theorem 1.3]). *There is an absolute constant  $c > 0$  such that the following holds. If  $\alpha \in (0, c)$ , then for all sufficiently large  $n$  (depending on  $\alpha$ ), there is a set  $A \subseteq (\mathbb{F}_5^n)^2$  with  $|A| \geq \alpha \cdot 5^{2n}$  such that*

$$|A \cap A - (a, b) \cap A - (b, -a) \cap A - (a + b, b - a)| \leq (1 - c)\alpha^4 \cdot 5^{2n}$$

*for all  $(a, b) \in (\mathbb{F}_5^n)^2 \setminus \{(0, 0)\}$ .*

The authors of [16] explain the failure of large intersections in Theorem 1.10 as a consequence of an eigenvalue condition. Namely, for the corresponding matrices

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

the eigenvalues of  $M_1 M_2^{-1}$  are negatives of each other. They also show that in the absence of such an eigenvalue condition, a Khintchine-type result holds for patterns

$$\{x, x + M_1 y, x + M_2 y, x + (M_1 + M_2)y\}$$

(see [16, theorem 1.2]).

In our context of rings of integers, we can translate the eigenvalue condition into an algebraic criterion. Recall that two algebraic numbers  $\alpha, \beta \in K$  are *conjugate* (over  $\mathbb{Q}$ ) if they have the same minimal polynomial (over  $\mathbb{Q}$ ). Equivalently, there is a field automorphism  $\varphi : K \rightarrow K$  such that  $\varphi(\alpha) = \beta$ . If we denote by  $M_\alpha$  the  $\mathbb{Q}$ -linear map  $M_\alpha x = \alpha x$  on the  $\mathbb{Q}$ -vector space  $K$ , then the eigenvalues of  $M_\alpha$  are exactly the conjugates of  $\alpha$  (this follows from, e.g., [18, theorem 5.9], which gives a formula for the characteristic polynomial of  $M_\alpha$ ). We therefore make the following conjecture:

CONJECTURE 1.11. *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. The following are equivalent:*

- (i) *for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ , any  $A \in \mathcal{B}$ , any  $\varepsilon > 0$ , and any  $\mathcal{O}_K$ -valued intersective polynomial  $p \in K[x]$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)} A \cap T^{-sp(n)} A \cap T^{-(r+s)p(n)} A \right) > \mu(A)^4 - \varepsilon \right\}$$

*is syndetic;*

- (ii) *no two conjugates of  $s/r$  over  $\mathbb{Q}$  are negatives of each other.*

### 1.2. Method

In order to prove Khintchine-type recurrence results such as Theorem A and Theorem B, it is natural to consider associated multiple ergodic averages. The appropriate averaging schemes in rings of integers are those arising from *Følner sequences*. A *Følner sequence* in  $(\mathcal{O}_K, +)$  is a sequence of subsets  $(\Phi_N)_{N \in \mathbb{N}}$  of  $\mathcal{O}_K$  such that, for every  $n \in \mathcal{O}_K$ ,

$$\frac{|(\Phi_N + n) \Delta \Phi_N|}{|\Phi_N|} \rightarrow 0.$$

Examples of Følner sequences include boxes in  $\mathcal{O}_K \cong \mathbb{Z}^d$  with increasing side lengths. We say that a sequence  $(u_n)_{n \in \mathcal{O}_K}$  has *uniform Cesàro limit*  $u$ , denoted  $\text{UC-}\lim_{n \in \mathcal{O}_K} u_n = u$ , if

$$\frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} u_n \rightarrow u$$

for every Følner sequence  $(\Phi_N)_{N \in \mathbb{N}}$  in  $(\mathcal{O}_K, +)$ . The usefulness of uniform Cesàro limits in proving Khintchine-type theorems comes from the following routine fact (for a proof, see [2, lemma 1.9]):

PROPOSITION 1.12. *A set  $S \subseteq \mathcal{O}_K$  is syndetic if and only if for any Følner sequence  $(\Phi_N)_{N \in \mathbb{N}}$  in  $(\mathcal{O}_K, +)$ , one has  $\bigcup_{N \in \mathbb{N}} \Phi_N \cap S \neq \emptyset$ .*

Rather than computing the multiple ergodic averages

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i \tag{1.7}$$

directly for an arbitrary  $\mathcal{O}_K$ -system, we reduce to computing the averages (1.7) in simpler classes of systems. To be precise, we say a system  $\mathbf{Y} = (Y, \mathcal{D}, \nu, S)$  is a *factor* of  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$  if there are full measure subsets  $X_0 \subseteq X$  and  $Y_0 \subseteq Y$  and a measure-preserving map  $\pi : X_0 \rightarrow Y_0$  such that  $S^n \pi(x) = \pi(T^n x)$  for every  $x \in X_0, n \in \mathcal{O}_K$ . There is a natural correspondence between the factor  $Y$  and the  $T$ -invariant sub- $\sigma$ -algebra  $\pi^{-1}(\mathcal{D})$ . This allows us to take conditional expectations, and in a standard abuse of notation, we write  $\mathbb{E}[f | Y] := \mathbb{E}[f | \pi^{-1}(\mathcal{D})]$ . The factor  $\mathbf{Y}$  is *characteristic* for a family of sequences  $\{a_1(n), \dots, a_k(n)\}, n \in \mathcal{O}_K$ , if for any  $f_1, \dots, f_k \in L^\infty(\mu)$ ,

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \left( \prod_{i=1}^k T^{a_i(n)} f_i - \prod_{i=1}^k T^{a_i(n)} \mathbb{E}[f_i | Y] \right) = 0$$

in  $L^2(\mu)$ .

The main family of factors that we will deal with is the family of *nilfactors*  $(\mathcal{Z}_r)_{r \in \mathbb{N}}$  (also called *Host–Kra factors* from the work of Host and Kra on  $\mathbb{Z}$ -actions [30]). Assume for this discussion that  $T$  is an ergodic action of  $\mathcal{O}_K$ . The factor  $\mathcal{Z}_r$  is defined to be the minimal factor that is characteristic for all families  $\{l_1 n, \dots, l_{r+1} n\}$  with  $l_1, \dots, l_{r+1} \in \mathcal{O}_K$  distinct and nonzero. For our purposes, it will suffice to discuss some general properties of nilfactors.

The tower of factors  $\mathcal{Z}_1 \subseteq \mathcal{Z}_2 \subseteq \dots$  is a sequence of compact extensions. The first factor,  $\mathcal{Z}_1$ , is the *Kronecker factor*, which is the smallest factor for which every eigenfunction is measurable. As a measure-preserving system, it is isomorphic to an action by rotations on a compact abelian group. The Kronecker factor contains a subfactor that will also be of interest, namely the *rational Kronecker factor*, denoted  $\mathcal{K}_{rat}$ , which is an inverse limit of finite rotational systems (for a more detailed discussion of the rational Kronecker factor, see Section 2.1).

The higher-level nilfactors also have the structure of (inverse limits of) “rotational” systems but on more complex algebraic objects. Let  $G$  be an  $r$ -step nilpotent Lie group and  $\Gamma < G$  a co-compact discrete subgroup. The quotient space  $X = G/\Gamma$  is called an  *$r$ -step nilmanifold*. An  *$r$ -step nilsystem* is a system  $(X, \mathcal{B}, \mu, T)$ , where  $X = G/\Gamma$  is an  $r$ -step nilmanifold,  $\mu$  is the Haar probability measure on  $X$ , and  $T$  is an  $(\mathcal{O}_K, +)$ -action by *niltranslations*, i.e. transformations of the form  $x \mapsto ax$  for some  $a \in G$ . The nilfactor  $\mathcal{Z}_r$  is an inverse limit of  $r$ -step nilsystems. For  $\mathbb{Z}$ -actions, this was established by Host and Kra in [30] and independently by Ziegler in [39]. For our generality of  $\mathcal{O}_K$ -systems, this follows from [27, theorem 4.1.2].



By careful application of the van der Corput differencing trick, one can reduce polynomial expressions to (potentially much longer) linear expressions. This works so long as the polynomials  $p_1, \dots, p_k$  are *essentially distinct*, meaning that  $p_j - p_i$  is non-constant for every  $i \neq j$ . Hence, for any family of essentially distinct polynomial sequences  $\{p_1(n), \dots, p_k(n)\}, n \in \mathcal{O}_K$ , there is a characteristic factor that is a nilfactor (but the step of the nilfactor may far exceed  $k - 1$  in general):

**THEOREM 1.13** (cf. [14, theorem 5.2]). *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $\{p_1, \dots, p_k\} \subseteq K[x]$  are non-constant and essentially distinct  $\mathcal{O}_K$ -valued polynomials. Then there is an  $r \in \mathbb{N}$  such that for any ergodic  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and any  $f_1, \dots, f_k \in L^\infty(\mu)$ ,*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} \mathbb{E}[f_i \mid \mathcal{Z}_r].$$

in  $L^2(\mu)$ .

For the specific configurations appearing in Theorem A (independent polynomials) and in Theorem B (multiples of a single polynomial), we can control the step of the characteristic nilfactors. In order to properly formulate our results, we need one more definition.

**Definition 1.14.** A family of polynomials  $\{p_1, \dots, p_k\} \subseteq K[x]$  is *independent* if for all  $(c_1, \dots, c_k) \in K^k \setminus \{0\}$ , the polynomial  $\sum_{i=1}^k c_i p_i$  is non-constant.

Note that the family  $\{p_1, \dots, p_k\}$  is independent if and only if  $\{1, p_1, \dots, p_k\}$  is linearly independent over  $K$ . Furthermore, a jointly intersective family  $\{p_1, \dots, p_k\}$  is independent if and only if it is linearly independent.

**THEOREM C.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $p_1, \dots, p_k \in K[x]$  are independent and  $\mathcal{O}_K$ -valued. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and any  $f_1, \dots, f_k \in L^\infty(\mu)$ ,*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} \mathbb{E}[f_i \mid \mathcal{K}_{rat}],$$

where the limits are taken in  $L^2(\mu)$ .

**THEOREM D.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be a non-constant  $\mathcal{O}_K$ -valued polynomial. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ , any  $l_1, \dots, l_k \in \mathcal{O}_K$  distinct and nonzero, and any  $f_1, \dots, f_k \in L^\infty(\mu)$ ,*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{l_i p(n)} f_i = UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{l_i p(n)} \mathbb{E}[f_i \mid \mathcal{Z}_{k-1}],$$

where the limits are taken in  $L^2(\mu)$ . Moreover, if  $T$  is totally ergodic, then this limit does not depend on the polynomial  $p$ .

We will prove Theorems C and D via equidistribution results for polynomial sequences in nilmanifolds, which are of independent interest (see Theorem 3.3 and Proposition 3.12

below). After several reductions, the main technical result in the proof of Theorem C is the following far-reaching generalisation of Weyl’s polynomial equidistribution theorem for families of independent polynomials in several variables:

**THEOREM 1.15** (Theorem 3.8). *Let  $d, l, k, m \in \mathbb{N}$ . Let  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l\} \subseteq \mathbb{Q}[x_1, \dots, x_d]$  be  $\mathbb{Z}$ -valued and independent over  $\mathbb{Q}$ . Let  $T_1, \dots, T_l : \mathbb{T}^m \rightarrow \mathbb{T}^m$  be commuting unipotent affine transformations generating an ergodic  $\mathbb{Z}^l$ -action. Then the polynomial sequence*

$$\left( \prod_{j=1}^l T_j^{p_{1,j}(n)} x, \dots, \prod_{j=1}^l T_j^{p_{k,j}(n)} x \right)_{n \in \mathbb{Z}^d}$$

*is well-distributed in  $\mathbb{T}^{mk}$  for all  $x$  in a co-meager set of full measure.*

The upshot of Theorem C is that we may compute multiple ergodic averages for independent polynomials by studying the corresponding averages in a finite rotational system, where computations are much easier to carry out. Similarly, Theorem D says that in order to compute multiple ergodic averages for  $k$  distinct multiples of a fixed polynomial, we can make use of the algebraic structure of a  $(k - 1)$ -step nilsystem.

From here, we can follow a standard technique to deduce the corresponding Khintchine-type results. The assumption that the families of polynomials under consideration are jointly intersective, together with a standard approximation argument, allows us to reduce to the case that the action  $T$  is *totally ergodic*, i.e. that  $\mathcal{K}_{rat}$  is trivial. For independent polynomials, Theorem C guarantees that

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \mu \left( A \cap T^{-p_1(n)} A \cap \dots \cap T^{-p_k(n)} A \right) = \mu(A)^{k+1}$$

for totally ergodic  $T$ , from which Theorem A immediately follows. The details of this argument are carried out in Section 4.1. When all of the polynomials involved are multiples of a fixed polynomial and  $T$  is totally ergodic, Theorem D says that the relevant multiple ergodic average can be reduced to a linear average (corresponding to  $p(n) = n$ ). We are therefore able to capitalise on Khintchine-type results for linear averages (see Theorem 1.9 above) and extend them to the polynomial configurations we consider in Theorem B. The full details of this argument appear in Section 4.2.

### 1.3. Notions of largeness

Syndeticity is just one of many notions of largeness that naturally appear in ergodic theory and combinatorics. While it is useful in quantifying the size of subsets, it does not have all of the properties that one may desire. To illustrate one shortcoming of syndeticity, we return to Szemerédi’s theorem. Consider the family of sets  $\mathcal{R}_k := \{R_k(\mathbf{X}, A) : \mathbf{X} = (X, \mathcal{B}, \mu, T) \text{ mps}, A \in \mathcal{B}, \mu(A) > 0\}$ , where

$$R_k(\mathbf{X}, A) := \left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-n} A \cap \dots \cap T^{-kn} A \right) > 0 \right\}.$$

The family  $\mathcal{R}_k$  has the *filter property*: for any  $R, S \in \mathcal{R}_k$ , we have  $R \cap S \neq \emptyset$ . Indeed, given two measure-preserving systems  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$  and  $\mathbf{Y} = (Y, \mathcal{D}, \nu, S)$ , one can form the

product system  $\mathbf{X} \times \mathbf{Y}$  and easily verify that

$$R_k(\mathbf{X}, A) \cap R_k(\mathbf{Y}, B) = R_k(\mathbf{X} \times \mathbf{Y}, A \times B) \in \mathcal{R}_k.$$

One may hope that there is a different notion of largeness that captures this filter property. To discuss one such notion, we introduce the class of *IP sets*.

Let  $(x_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{O}_K$ . The *finite sum set* associated to  $(x_n)_{n \in \mathbb{N}}$  is the set

$$FS((x_n)_{n \in \mathbb{N}}) := \left\{ \sum_{n \in F} x_n : F \subseteq \mathbb{N} \text{ is finite and nonempty} \right\}.$$

We say that  $A \subseteq \mathcal{O}_K$  is an *IP set* if  $A \supseteq FS((x_n)_{n \in \mathbb{N}})$  for some infinite sequence  $(x_n)_{n \in \mathbb{N}}$ . A theorem of Hindman [28] asserts that IP sets are *partition regular*:

**THEOREM 1.16** (Hindman’s Theorem [28, theorem 3.1]). *Let  $A$  be an IP set. If  $A$  is finitely partitioned  $A = \bigcup_{i=1}^r C_i$ , then for some  $i_0 \in \{1, \dots, r\}$ ,  $C_{i_0}$  is an IP set.*

A set  $E$  is  $IP^*$  if  $E \cap A \neq \emptyset$  for every IP set  $A$ . It follows from Theorem 1.16 that  $IP^*$  sets have the filter property. From this point of view, the IP polynomial Szemerédi theorem is more satisfactory:

**THEOREM 1.17** ([13, theorem 0.7]). *Let  $p_1, \dots, p_k \in \mathbb{Q}[x]$  be integer-valued polynomials with zero constant term. Then for any ergodic invertible probability measure-preserving system and any  $A \in \mathcal{B}$ , the set*

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-p_1(n)} A \cap \dots \cap T^{-p_k(n)} A \right) > 0 \right\}$$

is  $IP^*$ .

*Remark 1.18.* For the linear pattern  $p_i(n) = in$ , Theorem 1.17 follows from [23, theorem A].

When bounding the size of the intersections from below, the filter property is no longer a straightforward consequence from considering product systems. Furthermore,  $IP^*$  turns out to be too strong of a notion of largeness. (Indeed, in a skew-product system on the torus  $\mathbb{T}^2$ , one can find a set  $A$  for which the set (1.1) fails to be  $IP^*$  for small  $\varepsilon > 0$ .) However, we can use a slightly weaker notion that retains the filter property. Define the *upper Banach density* of a set  $E \subseteq \mathcal{O}_K$  by

$$d^*(E) := \sup \left\{ \limsup_{N \rightarrow \infty} \frac{|E \cap \Phi_N|}{|\Phi_N|} : (\Phi_N)_{N \in \mathbb{N}} \text{ is a Følner sequence in } \mathcal{O}_K \right\}.$$

We say that  $E$  is *almost  $IP^*$* , or  $AIP^*$  for short, if  $E$  can be written as  $E = A \setminus B$ , where  $A$  is an  $IP^*$  set and  $B$  is a set with  $d^*(B) = 0$ . In [14], it was shown that the set (1.3) in Theorem 1.7 is in fact a shift of an  $AIP^*$  set.

In a similar vein, Theorem 1.3 was strengthened in [9]. There, the notion of largeness used is the even stronger notion of  $AVIP^*_0$ , which we define in Section 5.

**THEOREM 1.19** ([9, theorem 4.2]). *Let  $p_1, \dots, p_k \in \mathbb{Q}[x]$  be linearly independent integer-valued polynomials with zero constant term. Then for any ergodic invertible*

probability measure-preserving system, any  $A \in \mathcal{B}$ , and any  $\varepsilon > 0$ , the set

$$\left\{ n \in \mathbb{Z} : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > \mu(A)^{k+1} - \varepsilon \right\}$$

is  $AVIP_0^*$ .

In Section 5, we similarly strengthen the conclusions of Theorem A and Theorem B. In particular, we show that, if  $T$  is ergodic, then the sets (1.4), (1.5) and (1.6) are shifts of  $AVIP_0^*$  sets (see Theorem 5.6 and Theorem 5.7).

1.4. Combinatorial applications

We deduce several combinatorial facts from the ergodic-theoretic theorems above. For some of the combinatorial results, we have stronger finitary versions. For other combinatorial facts derived from ergodic-theoretic results under the assumption of ergodicity, we cannot easily deduce finitary consequences. This distinction arises from subtleties in Furstenberg’s correspondence principle, which we discuss in more detail below. The first version of Furstenberg’s correspondence principle that we will use is as follows:

**THEOREM 1.20** ([4, theorem 4.17]). *Fix a Følner sequence  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$  in  $\mathcal{O}_K$ . Suppose  $E \subseteq \mathcal{O}_K$  has positive upper density along  $\Phi$ , i.e.  $\bar{d}_\Phi(E) := \limsup_{N \rightarrow \infty} |E \cap \Phi_N|/|\Phi_N| > 0$ . Then there exists an  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and a set  $A \in \mathcal{B}$  with  $\mu(A) = \bar{d}_\Phi(E)$  such that, for any  $k \in \mathbb{N}$  and any  $n_1, \dots, n_k \in \mathcal{O}_K$ , one has*

$$\bar{d}_\Phi \left( \bigcap_{i=1}^k (E - n_i) \right) \geq \mu \left( \bigcap_{i=1}^k T^{-n_i}A \right). \tag{1.8}$$

Applying Theorem 1.20 directly alongside Theorem A, we get the following:

**THEOREM 1.21.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. Fix a Følner sequence  $\Phi = (\Phi_N)_{N \in \mathbb{N}}$  and suppose  $E \subseteq \mathcal{O}_K$  satisfies  $\bar{d}_\Phi(E) > 0$ . Then for any  $\varepsilon > 0$ ,*

$$\left\{ n \in \mathcal{O}_K : \bar{d}_\Phi (E \cap (E - p_1(n)) \cap \dots \cap (E - p_k(n))) > \bar{d}_\Phi(E)^{k+1} - \varepsilon \right\}$$

is syndetic.

Taking the natural Følner sequence  $\Phi_N = \{1, \dots, N\}^d$  under the isomorphism  $\mathcal{O}_K \cong \mathbb{Z}^d$ , we deduce a related finitary result:

**COROLLARY 1.22.** *Let  $K$  be a degree  $d$  number field with ring of integers  $\mathcal{O}_K \cong \mathbb{Z}^d$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. For any  $\delta, \varepsilon > 0$ , there exists  $N = N(\delta, \varepsilon) \in \mathbb{N}$  such that: if  $A \subseteq \{1, \dots, N\}^d$  with  $|A| > \delta N^d$ , then  $A$  contains at least  $(\delta^{k+1} - \varepsilon)N^d$  configurations of the form  $\{x, x + p_1(n), \dots, x + p_k(n)\}$  for some  $n \neq 0$ .*

*Proof.* Let  $\delta, \varepsilon > 0$ , and suppose no such  $N$  exists. That is, for some sequence  $N_m \rightarrow \infty$ , we can find sets  $A_m \subseteq \{1, \dots, N\}^d$  with  $|A_m| > \delta N_m^d$  such that

$$|A_m \cap (A_m - p_1(n)) \cap \dots \cap (A_m - p_k(n))| \leq (\delta^{k+1} - \varepsilon) N_m^d$$

for every  $n \neq 0$ .

By passing to a subsequence if necessary, we may assume

$$\lim_{m \rightarrow \infty} \frac{|(A_{m,i_1} - n_1) \cap \dots \cap (A_{m,i_r} - n_r) \cap \{1, \dots, N_m\}^d|}{N_m^d}$$

exists for all  $r \in \mathbb{N}$ ,  $n_1, \dots, n_r \in \mathcal{O}_K$ , and  $i_1, \dots, i_r \in \{0, 1\}$ , where  $A_{m,0} = A_m$  and  $A_{m,1} = \mathcal{O}_K \setminus A_m$ . Then we may define a measure on  $X = \{0, 1\}^{\mathcal{O}_K}$  by letting

$$\mu(\{x \in X : x_{n_1} = i_1, \dots, x_{n_r} = i_r\}) = \lim_{m \rightarrow \infty} \frac{|(A_{m,i_1} - n_1) \cap \dots \cap (A_{m,i_r} - n_r) \cap \{1, \dots, N_m\}^d|}{N_m^d}$$

and extending using Kolmogorov’s extension theorem. Note that the shift map  $(T^n x)(m) = x(n + m)$  preserves the measure  $\mu$ . Taking  $A = \{x \in X : x_0 = 1\}$ , we have

$$\mu(A) = \lim_{m \rightarrow \infty} \frac{|A_m|}{N_m^d} \geq \delta,$$

and on the other hand,

$$\begin{aligned} \mu\left(A \cap T^{-p_1(n)} A \cap \dots \cap T^{-p_k(n)} A\right) &= \lim_{m \rightarrow \infty} \frac{|A_m \cap (A_m - p_1(n)) \cap \dots \cap (A_m - p_k(n))|}{N_m^d} \\ &\leq \delta^{k+1} - \varepsilon \end{aligned}$$

for  $n \neq 0$ . This contradicts Theorem A.

Note that the system in Theorem 1.20 may not be ergodic. To obtain an inequality similar to (1.8) while ensuring that the system is ergodic, one needs to allow for replacing the density along  $\Phi$  by the density along some other Følner sequence depending on the choice of translates  $n_1, \dots, n_k$ . (An example due to Hindman [29] can be used to show that, for certain sets  $E$ , the measure-preserving system in the conclusion of Theorem 1.20 is necessarily non-ergodic; see the discussion following theorem 1.3 in [5] for more detail.) Using the notion of upper Banach density, we can formulate an ergodic version of Furstenberg’s correspondence principle:

**THEOREM 1.23.** *Suppose  $E \subseteq \mathcal{O}_K$  has positive upper Banach density. Then there exists an ergodic  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and a set  $A \in \mathcal{B}$  with  $\mu(A) = d^*(E)$  such that, for any  $k \in \mathbb{N}$  and any  $n_1, \dots, n_k \in \mathcal{O}_K$ , one has*

$$d^*\left(\bigcap_{i=1}^k (E - n_i)\right) \geq \mu\left(\bigcap_{i=1}^k T^{-n_i} A\right).$$

For  $\mathbb{Z}$ -actions, Theorem 1.23 appears in [6, proposition 3.1], utilising an observation of Emmanuel Lesigne based on the original argument of Furstenberg. For a general version in amenable groups (a class containing all countable abelian groups), see [5, theorem 2.8].

As a consequence, we obtain the following combinatorial version of Theorem 5.7:

**THEOREM 1.24.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be an  $\mathcal{O}_K$ -valued intersective polynomial. Let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. Then for any set*

$E \subseteq \mathcal{O}_K$  with  $d^*(E) > 0$  and any  $\varepsilon > 0$ , the set

$$\left\{ n \in \mathcal{O}_K : d^*(E \cap (E - rp(n)) \cap (E - sp(n))) > d^*(E)^3 - \varepsilon \right\}$$

is  $AVIP_{0,+}^*$  (in particular, it is syndetic).

Moreover, if  $\frac{s}{r} \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : d^*(E \cap (E - rp(n)) \cap (E - sp(n)) \cap (E - (r + s)p(n))) > d^*(E)^4 - \varepsilon \right\}$$

is  $AVIP_{0,+}^*$  (in particular, it is syndetic).

As discussed above, the ergodicity assumption in Theorems B and 5.7 precludes us from easily deducing finitary results along the lines of Corollary 1.22. Nevertheless, we suspect that a finitary analogue holds, which we formulate below:

CONJECTURE 1.25. *Let  $K$  be a degree  $d$  number field with ring of integers  $\mathcal{O}_K \cong \mathbb{Z}^d$ . Suppose  $p(x) \in K[x]$  is an  $\mathcal{O}_K$ -valued intersective polynomial.*

- (1) *let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. For any  $\delta, \varepsilon > 0$ , there exists  $N = N(\varepsilon, \delta) \in \mathbb{N}$  such that: if  $A \subseteq \{1, \dots, N\}^d$  with  $|A| > \delta N^d$ , then  $A$  contains at least  $(\delta^3 - \varepsilon)N^d$  configurations of the form  $\{x, x + rp(n), x + sp(n)\}$  for some  $n \neq 0$ .*
- (2) *let  $r, s \in \mathcal{O}_K$  be distinct and nonzero such that  $s/r \in \mathbb{Q}$  (or more generally, no two conjugates of  $\frac{s}{r}$  are negatives of each other). For any  $\delta, \varepsilon > 0$ , there exists  $N = N(\varepsilon, \delta) \in \mathbb{N}$  such that: if  $A \subseteq \{1, \dots, N\}^d$  with  $|A| > \delta N^d$ , then  $A$  contains at least  $(\delta^4 - \varepsilon)N^d$  configurations of the form  $\{x, x + rp(n), x + sp(n), x + rp(n) + sp(n)\}$  for some  $n \neq 0$ .*

In the simplest case when  $K = \mathbb{Q}$  and  $p(n) = n$ , Conjecture 1.25 was posed as a question in [6] and verified in [25, theorem 1.10] and [26, theorem 1.12]. For more general linear patterns (the case  $p(n) = n$ ), closely related finitary results were recently established in [16, 32].

### 1.5. Outline of the paper

The structure of the paper is as follows. In Section 2, we collect several useful facts that will be used repeatedly in the proofs of the main theorems. Section 3 is devoted to proving Theorems C and D on characteristic factors corresponding to the polynomial configurations of interest via equidistribution results on nilmanifolds. Using the knowledge of characteristic factors, we prove Khintchine-type results (Theorems A and B in Section 4. Finally, Section 5 handles the refinements of our Khintchine-type theorems to conclude stronger combinatorial properties about the abundance of combinatorial configurations.

## 2. Preliminaries

### 2.1. Rational Kronecker factor

Recall that the Kronecker factor for an ergodic measure-preserving system is spanned by eigenfunctions. As suggested by the name, the rational Kronecker factor will be spanned by eigenfunctions with rational eigenvalues. To make this precise, we need to define what it means for an eigenvalue (a group character) to be rational.

Since the additive group structure for the ring of integers in a degree  $d$  extension of  $\mathbb{Q}$  is  $\mathbb{Z}^d$ , we say that a character  $\chi : \mathbb{Z}^d \rightarrow \mathbb{T}$  is *rational* if there is an element  $(q_1, \dots, q_d) \in \mathbb{Q}^d$

such that

$$\chi(n_1, \dots, n_d) = e^{2\pi i(q_1 n_1 + \dots + q_d n_d)}. \tag{2.1}$$

For notational convenience, we will let  $e : \mathbb{R} \rightarrow \mathbb{T}$  be the function  $e(x) = e^{2\pi i x}$  so that we can write equation (2.1) in the more compact form

$$\chi(n) = e(q \cdot n)$$

for the usual dot product  $\cdot$  on  $\mathbb{R}^d$ .

The property of rational characters that we will utilise later on is periodicity. Given a number field  $K$  with ring of integers  $\mathcal{O}_K$ , we say that a character  $\chi : \mathcal{O}_K \rightarrow \mathbb{T}$  is *periodic*, with *period*  $p \in \mathcal{O}_K$ , if for all  $n, m \in \mathcal{O}_K$ , we have

$$\chi(n + mp) = \chi(n).$$

To translate this back into language where rationality makes sense, take an integral basis  $\{b_1, \dots, b_d\}$  so that  $(\mathcal{O}_K, +) \cong \bigoplus_{i=1}^d \mathbb{Z} \cdot b_i$ . Since  $\widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$ , there is an element  $\alpha \in \mathbb{T}^d$  so that

$$\chi \left( \sum_{i=1}^d n_i b_i \right) = e(n \cdot \alpha)$$

for  $n \in \mathbb{Z}^d$ . We can then say that  $\chi : \mathcal{O}_K \rightarrow \mathbb{T}$  is *rational* if  $\alpha \in \mathbb{Q}^d / \mathbb{Z}^d$ . We now show that rationality and periodicity coincide:

LEMMA 2.1. *A character  $\chi : \mathcal{O}_K \rightarrow \mathbb{T}$  is rational if and only if it is periodic.*

*Proof.* Suppose  $\chi$  is rational, say  $\chi(n) = e(n \cdot q)$  with  $q = (q_1, \dots, q_d) \in \mathbb{Q}^d$ . Choose  $D \in \mathbb{Z}$  so that  $Dq_i \in \mathbb{Z}$  for every  $i = 1, \dots, d$ , and set  $p := D \left( \sum_{i=1}^d b_i \right)$ . We claim that  $p$  is a period for  $\chi$ . Indeed, given  $m = \sum_{i=1}^d m_i b_i \in \mathcal{O}_K$ , we have

$$mp = D \left( \sum_{i,j} m_i b_i b_j \right) = \sum_k \left( \sum_{i,j} m_i c_{i,j,k} \right) D b_k,$$

where  $c_{i,j,k} \in \mathbb{Z}$  so that  $b_i b_j = \sum_k c_{i,j,k} b_k$ . Hence,  $mp \cdot q = \sum_k \left( \sum_{i,j} m_i c_{i,j,k} \right) D q_k \in \mathbb{Z}$ , so  $\chi(mp) = e(mp \cdot q) = 1$  for every  $m \in \mathcal{O}_K$ .

Conversely, suppose  $\chi$  is periodic with period  $p \in \mathcal{O}_K$ . Let  $\alpha \in \mathbb{T}^d$  such that  $\chi(n) = e(n \cdot \alpha)$ . Since  $\{b_1, \dots, b_d\}$  is a  $\mathbb{Q}$ -basis for  $K$ , we can write  $1/p = \sum_{i=1}^d a_i b_i$  for some  $a_1, \dots, a_d \in \mathbb{Q}$ . Let  $D \in \mathbb{Z}$  such that  $Da_i \in \mathbb{Z}$  for every  $i = 1, \dots, d$ . Then  $D/p$  is a  $\mathbb{Z}$ -linear combination of basis elements, so  $\frac{D}{p} \in \mathcal{O}_K$ . Now let  $m_i = Dp/b_i \in \mathcal{O}_K$ . Since  $\chi$  is  $p$ -periodic, we have

$$1 = \chi(m_i p) = \chi(D b_i) = e(D \alpha_i).$$

That is,  $D \alpha_i \in \mathbb{Z}$ , so  $\alpha_i \in \mathbb{Q}$  for all  $i = 1, \dots, d$ . Thus,  $\chi$  is rational.

Now we can give our definition:

*Definition 2.2.* Let  $K$  be a number field, and let  $\mathcal{O}_K$  be its ring of integers. Let  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$  be an ergodic  $\mathcal{O}_K$ -system. The *rational Kronecker factor* of  $\mathbf{X}$ , denoted by  $\mathcal{K}_{rat}(\mathbf{X})$ , is the factor generated by the algebra

$$\overline{\text{span}\{f \in L^2(\mu) : T^n f = \chi(n)f \text{ for some rational character } \chi : \mathcal{O}_K \rightarrow \mathbb{T}\}}.$$

Building on Lemma 2.1, we can characterize total ergodicity in several equivalent ways:

**PROPOSITION 2.3.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$  be an ergodic  $\mathcal{O}_K$ -system. The following are equivalent:*

- (i) *the rational Kronecker factor  $\mathcal{K}_{rat}(\mathbf{X})$  is trivial;*
- (ii) *for every  $r \in \mathcal{O}_K \setminus \{0\}$ ,  $(T^{rn})_{n \in \mathcal{O}_K}$  is ergodic;*
- (iii) *for every finite index subgroup  $\Lambda \subseteq (\mathcal{O}_K, +)$ , the action  $(T^n)_{n \in \Lambda}$  is ergodic.*

*Proof.* Since  $r\mathcal{O}_K$  has finite index in  $\mathcal{O}_K$ , we trivially have the implication (iii)  $\implies$  (ii). We will show (i)  $\implies$  (iii) and (ii)  $\implies$  (i).

(i)  $\implies$  (iii). Suppose  $\mathcal{K}_{rat}(\mathbf{X})$  is trivial. Let  $\Lambda \subseteq (\mathcal{O}_K, +)$  be a finite index subgroup, and suppose  $T^n f = f$  for every  $n \in \Lambda$ . Then the orbit  $\{T^n f : n \in \mathcal{O}_K\}$  is finite: it consists of the elements  $T^m f$  for  $m$  in a finite set  $F$  satisfying  $\Lambda + F = \mathcal{O}_K$ . In particular, the orbit is (pre-)compact, so  $f$  is a linear combination of eigenfunctions,  $f = \sum_i c_i f_i$ , with  $T^n f_i = \chi_i(n) f_i$  for some characters  $\chi_i : \mathcal{O}_K \rightarrow \mathbb{T}$ . Since  $T^n f = f$  for  $n \in \Lambda$ , we have  $\chi_i(n) = 1$  for  $n \in \Lambda$ . Therefore,  $\chi_i$  takes only the finitely many values  $\chi_i(m)$ ,  $m \in F$ . It follows that  $\chi_i$  is rational. But  $\mathcal{K}_{rat}(\mathbf{X})$  is trivial, so in fact  $\chi_i = 1$ . Hence,  $T^n f = f$  for every  $n \in \mathcal{O}_K$ . Since  $(T^n)_{n \in \mathcal{O}_K}$  is ergodic, we have that  $f$  is a constant function. Thus,  $(T^n)_{n \in \Lambda}$  is ergodic.

(ii)  $\implies$  (i). We prove the contrapositive. Suppose  $\mathcal{K}_{rat}(\mathbf{X})$  is not trivial. Then there is a non-constant function  $f \in L^2(\mu)$  and a rational character  $\chi : \mathcal{O}_K \rightarrow \mathbb{T}$  such that  $T^n f = \chi(n)f$  for  $n \in \mathcal{O}_K$ . By Lemma 2.1,  $\chi$  is periodic, say with period  $p$ . That is,  $\chi(n + pm) = \chi(n)$  for  $n, m \in \mathcal{O}_K$ . But then  $T^{pn} f = \chi(pn)f = f$  for every  $n \in \mathcal{O}_K$ . Hence,  $(T^{pn})_{n \in \mathbb{N}}$  is not ergodic, so (ii) fails.

### 2.2. Nilsystems

**PROPOSITION 2.4.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic  $\mathcal{O}_K$ -nilsystem. Then  $T$  is totally ergodic if and only if  $X$  is connected.*

*Proof.* Write  $X = G/\Gamma$ . Let  $a : \mathcal{O}_K \rightarrow G$  be a homomorphism so that  $T^n x = a(n) \cdot x$  for  $n \in \mathcal{O}_K$  and  $x \in X$ . Let  $x_0$  denote the image of the identity element in  $X$ .

Suppose  $T$  is totally ergodic, and let  $X_0$  be the connected component of  $x_0$ . Since  $X$  is compact, it is a disjoint union of finitely many translates of  $X_0$ , say  $X = \bigcup_{i=0}^{k-1} X_i$  with  $X_i = g_i X_0$ . Hence,  $G$  permutes the components  $X_0, \dots, X_{k-1}$ , giving a homomorphism  $\varphi : G \rightarrow S_k$ , where  $S_k$  is the symmetric group on  $k$  symbols. This in turn gives a homomorphism  $\omega = \varphi \circ a : \mathcal{O}_K \rightarrow S_k$ . Let  $\Omega = \ker \omega \subseteq \mathcal{O}_K$ . Since  $S_k$  is a finite group,  $\Omega$  has finite index in  $\mathcal{O}_K$ . Therefore,  $(T^n)_{n \in \Omega}$  acts ergodically on  $X$  (see Proposition 2.3(iii)). In particular,  $\overline{\{a(n)x_0 : n \in \Omega\}} = X$ . But for  $n \in \Omega$ , we have  $a(n)X_i = X_i$ , so  $a(n)x_0 \in X_0$ . Thus,  $X = X_0$ .



Conversely, suppose  $X$  is connected, and let  $r \in \mathcal{O}_K \setminus \{0\}$ . The group  $r\mathcal{O}_K$  has finite index in  $\mathcal{O}_K$ , so let  $s_0, \dots, s_{k-1} \in \mathcal{O}_K$  such that  $\bigcup_{i=0}^{k-1} (r\mathcal{O}_K + s_i) = \mathcal{O}_K$ . Let  $Y := \overline{\{a(rn)x_0 : n \in \mathcal{O}_K\}}$ . Then by ergodicity of  $T$ , we have  $X = \bigcup_{i=0}^{k-1} a(s_i)Y$ . We claim that for  $0 \leq i, j \leq k-1$ , the sets  $a(s_i)Y$  and  $a(s_j)Y$  are either disjoint or identical. Indeed, suppose  $x \in a(s_i)Y \cap a(s_j)Y$ . Then there are sequences  $(n_t)_{t \in \mathbb{N}}$  and  $(m_t)_{t \in \mathbb{N}}$  in  $\mathcal{O}_K$  such that

$$a(s_i) \lim_{t \rightarrow \infty} a(rn_t)x_0 = a(s_j) \lim_{t \rightarrow \infty} a(rm_t)x_0 = x.$$

Let  $(\gamma_t)_{t \in \mathbb{N}}$  and  $(\delta_t)_{t \in \mathbb{N}}$  be sequences in  $\Gamma$  and  $g \in G$  with  $g\Gamma = x$  so that

$$a(s_i) \lim_{t \rightarrow \infty} a(rn_t)\gamma_t = a(s_j) \lim_{t \rightarrow \infty} a(rm_t)\delta_t = g.$$

Then

$$\begin{aligned} a(s_j - s_i) \lim_{t \rightarrow \infty} a(r(m_t - n_t))x_0 &= \lim_{t \rightarrow \infty} \gamma_t (a(s_i)a(rm_t)\gamma_t)^{-1} a(s_j)a(rm_t)\delta_t \Gamma \\ &= \lim_{t \rightarrow \infty} \gamma_t g^{-1} g \Gamma = x_0. \end{aligned}$$

It follows that  $a(s_j - s_i)Y = Y$ , so multiplying by  $a(s_i)$ , we get  $a(s_i)Y = a(s_j)Y$ .

But then we have written  $X$  as a finite disjoint union of closed sets. Since  $X$  is connected, we must have  $a(s_i)Y = X$  for every  $i = 0, \dots, k-1$ . In particular,  $Y = X$ , so  $(T^n)_{n \in \mathcal{O}_K}$  is ergodic.

### 2.3. Weyl systems

The results in this paper depend critically on understanding polynomial orbits in *Weyl systems*. Following [10], we call a topological dynamical system  $(X, T)$  a *Weyl system* if  $X$  is a compact abelian Lie group and  $T$  is a  $\mathbb{Z}^d$ -action by unipotent affine transformations. In proving our multiple recurrence results, we will focus our attention on *connected Weyl systems*, that is Weyl systems where  $X$  is connected (and hence a torus).

The main result on polynomial orbits is the following:

**PROPOSITION 2.5.** (cf. [10, proposition 3.2]). *Let  $(X, T)$  be a  $\mathbb{Z}^d$ -Weyl system and  $p_1, \dots, p_m : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$  polynomials. Then for every  $x \in X$ ,  $Y := \overline{\{(T^{p_1(n)}x, \dots, T^{p_m(n)}x) : n \in \mathbb{Z}^d\}}$  is a union of finitely many subtori  $(Y_w)_{w \in W}$  of  $X^m$ . Moreover, there is a homomorphism  $\omega : \mathbb{Z}^d \rightarrow W$  such that the sequence  $(T^{p_1(n)}x, \dots, T^{p_m(n)}x)_{n \in \omega^{-1}(w)}$  is well-distributed in  $Y_w$  for each  $w \in W$ .*

This can be seen via a multivariable version of Weyl’s theorem on polynomial equidistribution in tori (see the explanation of [10, proposition 3.2]) or as a special case of a more general result due to Leibman:

**THEOREM 2.6.** ([35, theorem B\*]). *Let  $X = G/\Gamma$  be a nilmanifold. Let  $g : \mathbb{Z}^d \rightarrow G$  be a polynomial map, and let  $x \in X$ . There is a connected closed subgroup  $H \subseteq G$ , a homomorphism  $\omega : \mathbb{Z}^d \rightarrow W$  onto a finite group  $W$ , and a set  $\{x_w : w \in W\} \subseteq X$  such that the sets  $Y_w := Hx_w$ ,  $w \in W$ , are closed in  $X$  and  $(g(n)x)_{n \in \omega^{-1}(w)}$  is well-distributed in  $Y_w$  for every  $w \in W$ .*

As a consequence, we can deduce a simple criterion for checking that a polynomial sequence is well-distributed in a torus. First we need some notation. For a sequence

$u : \mathbb{Z}^d \rightarrow \mathbb{T}^m$  with polynomial coordinates  $u(n) = (u_1(n), \dots, u_m(n))$ , we write

$$\text{span}(u) := \text{span}_{\mathbb{R}} \left\{ (u_1(x), \dots, u_m(x)) : x \in \mathbb{R}^d \right\}.$$

**COROLLARY 2.7.** *Let  $\alpha_1, \dots, \alpha_r$  be rationally independent irrational elements of  $\mathbb{T}$ . Let  $u_1, \dots, u_r : \mathbb{Z}^l \rightarrow \mathbb{Z}^m$  be polynomials with zero constant term. Then the sequence*

$$(u_1(n)\alpha_1 + \dots + u_r(n)\alpha_r)_{n \in \mathbb{Z}^l}$$

*is well-distributed in the subtorus  $\text{span}(u_1) + \dots + \text{span}(u_r) \pmod{1}$  of  $\mathbb{T}^m$ .*

*Proof.* The case  $d = 1$  is handled by [10, corollary 3.3]. The same proof works for general  $d \in \mathbb{N}$ .

#### 2.4. Properties of polynomials

**Definition 2.8.** The polynomials  $p_1, \dots, p_m \in \mathbb{Q}[x_1, \dots, x_d]$  are algebraically independent (over  $\mathbb{Q}$ ) if, for every nonzero  $f \in \mathbb{Q}[x_1, \dots, x_m]$ , the polynomial  $f(p_1, \dots, p_m)$  is nonzero.

**PROPOSITION 2.9.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be a nonconstant  $\mathcal{O}_K$ -valued polynomial. Fix an integral basis  $\{b_1, \dots, b_d\} \subseteq \mathcal{O}_K$ , and let  $p_1, \dots, p_d \in \mathbb{Q}[x_1, \dots, x_d]$  be  $\mathbb{Z}$ -valued polynomials so that*

$$p \left( \sum_{i=1}^d n_i b_i \right) = \sum_{i=1}^d p_i(n_1, \dots, n_d) b_i.$$

*Then the polynomials  $p_1, \dots, p_d$  are algebraically independent (over  $\mathbb{Q}$ ).*

*Proof.* By [34, chapter I, 11.4], it suffices to check that the Jacobian matrix

$$J := \begin{pmatrix} \frac{\partial p_1}{\partial x_1} & \dots & \frac{\partial p_d}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial p_1}{\partial x_d} & \dots & \frac{\partial p_d}{\partial x_d} \end{pmatrix}$$

has full rank.

But the  $i$ th row of the Jacobian matrix is given by

$$\frac{\partial p}{\partial x_i}(x) = \lim_{h \rightarrow 0} \frac{p(x + hb_i) - p(x)}{h} = \left( \lim_{h \rightarrow 0} \frac{p(x + hb_i) - p(x)}{hb_i} \right) b_i = p'(x)b_i.$$

Since  $p$  is nonconstant,  $p'(x) \neq 0$ . Moreover,  $\{b_1, \dots, b_d\}$  is linearly independent over  $\mathbb{Q}$ , so the rows of  $J$  are linearly independent. Therefore,  $J$  has full rank, so  $p_1, \dots, p_d$  are algebraically independent.

**PROPOSITION 2.10.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ , and let  $\{b_1, \dots, b_d\}$  be an integral basis in  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be an independent family of polynomials (over  $K$ ). For each  $i = 1, \dots, k$ , let  $p_{i,1}, \dots, p_{i,d} \in \mathbb{Q}[x_1, \dots, x_d]$  be the*

coordinate polynomials so that

$$p_i \left( \sum_{j=1}^d x_j b_j \right) = \sum_{j=1}^d p_{i,j}(x_1, \dots, x_d) b_j.$$

Then the family  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d\} \subseteq \mathbb{Q}[x_1, \dots, x_d]$  is independent (over  $\mathbb{Q}$ ). That is, for any  $(c_{i,j})_{1 \leq i \leq k, 1 \leq j \leq d} \in \mathbb{Q}^{kd} \setminus \{0\}$ , the polynomial  $\sum_{i=1}^k \sum_{j=1}^d c_{i,j} p_{i,j}$  is nonconstant.

*Proof.* First, since  $\{b_1, \dots, b_d\}$  is linearly independent over  $\mathbb{Q}$ , the family  $\{b_j p_i : 1 \leq i \leq k, 1 \leq j \leq d\} \subseteq K[x]$  is independent over  $\mathbb{Q}$ .

Now let  $q_{i,j} \in \mathbb{Q}[x_1, \dots, x_d]$  be the  $b_1$ -coordinate of  $b_j p_i$ . That is,

$$b_j p_i \left( \sum_{l=1}^d x_l b_l \right) = q_{i,j}(x_1, \dots, x_d) b_1 + r_{i,j}(x_1, \dots, x_d),$$

where  $r_{i,j}(x_1, \dots, x_d) \in \text{span}_{\mathbb{Q}}\{b_2, \dots, b_d\}$ . We claim that  $\{q_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d\}$  is independent over  $\mathbb{Q}$ . Suppose not. Then for some  $(c_{i,j})_{1 \leq i \leq k, 1 \leq j \leq d} \in \mathbb{Q}^{kd} \setminus \{0\}$  and some  $c \in \mathbb{Q}$ , we have

$$\sum_{i=1}^k \sum_{j=1}^d c_{i,j} q_{i,j}(x_1, \dots, x_d) = c.$$

Then

$$Q \left( \sum_{l=1}^d x_l b_l \right) := \sum_{i=1}^k \sum_{j=1}^d c_{i,j} b_j p_i \left( \sum_{l=1}^d x_l b_l \right) = c b_1 + \sum_{i=1}^k \sum_{j=1}^d c_{i,j} r_{i,j}(x_1, \dots, x_d).$$

Hence, for the polynomial function  $f \left( \sum_{l=1}^d x_l b_l \right) := x_1 - c$ , we have  $f(Q) = 0$ . By Proposition 2.9, it follows that  $Q$  is constant. But  $\{b_j p_i : 1 \leq i \leq k, 1 \leq j \leq d\}$  is independent over  $\mathbb{Q}$ , so this is a contradiction.

For  $1 \leq j, l, m \leq d$ , let  $a_{j,l,m} \in \mathbb{Z}$  so that  $b_j b_l = \sum_{m=1}^d a_{j,l,m} b_m$ . By direct computation, we have

$$b_j p_i \left( \sum_{l=1}^d x_l b_l \right) = \sum_{l=1}^d p_{i,l}(x_1, \dots, x_d) \sum_{m=1}^d a_{j,l,m} b_m = \sum_{m=1}^d \left( \sum_{l=1}^d a_{j,l,m} p_{i,l}(x_1, \dots, x_d) \right) b_m.$$

Thus,

$$q_{i,j}(x_1, \dots, x_d) = \sum_{l=1}^d a_{j,l,1} p_{i,l}(x_1, \dots, x_d) \in \text{span}_{\mathbb{Q}}\{p_{i,1}, \dots, p_{i,d}\}.$$

Therefore,  $\text{span}(\{1\} \cup \{q_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d\}) \subseteq \text{span}(\{1\} \cup \{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d\})$ . It follows that  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq d\}$  is independent over  $\mathbb{Q}$ .

LEMMA 2.11. Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be jointly intersective  $\mathcal{O}_K$ -valued polynomials. Let  $r \in \mathcal{O}_K \setminus \{0\}$ . Then there exists  $\xi \in \mathcal{O}_K$

and  $D \in \mathcal{O}_K \setminus \{0\}$  such that

$$p_i(\xi + D\mathcal{O}_K) \subseteq r\mathcal{O}_K$$

for  $i = 1, \dots, k$ .

*Proof.* The subgroup  $r\mathcal{O}_K$  has finite index in  $\mathcal{O}_K$ , so there exists  $\xi \in \mathcal{O}_K$  such that  $p_i(\xi) \in r\mathcal{O}_K$  for  $i = 1, \dots, k$ .

Fix  $1 \leq i \leq k$ . Now write  $p_i(x) = a_mx^m + \dots + a_1x + a_0$  with  $a_0, a_1, \dots, a_m \in K$ . Since  $p_i$  is  $\mathcal{O}_K$ -valued, we have  $a_0 = p_i(0) \in \mathcal{O}_K$ . Let  $D_i \in \mathcal{O}_K$  so that  $D_ia_j \in \mathcal{O}_K$  for all  $j = 1, \dots, m$ . We claim  $p_i(\xi + D_ir\mathcal{O}_K) \subseteq r\mathcal{O}_K$ . Indeed, for  $n \in \mathcal{O}_K$ , we have

$$\begin{aligned} p(\xi + D_in) &= p_i(\xi) + \sum_{j=1}^m \sum_{l=1}^j a_j \binom{j}{l} (D_in)^l \xi^{j-l} \\ &= p_i(\xi) + r \cdot \sum_{j=1}^m \left( D_ia_j \sum_{l=1}^j \binom{j}{l} D^{l-1} n^{l-1} \xi^{j-l} \right) \in r\mathcal{O}_K. \end{aligned}$$

Taking  $D = r \cdot \text{lcm}(D_1, \dots, D_k)$  completes the proof.

### 2.5. Eligible collections

Theorem C and Theorem D each establish characteristic factors for certain polynomial configurations in ergodic systems. In both cases, it is significantly easier to deal with totally ergodic systems. The notion of *eligible* collections, introduced by Frantzikinakis in [19] for  $\mathbb{Z}$ -valued polynomials, can be utilized to reduce the ergodic case to the simpler case in which the system is totally ergodic.

*Definition 2.12.* Let  $\mathcal{P}$  be a collection of families of  $k$   $\mathcal{O}_K$ -valued polynomials. We say that  $\mathcal{P}$  is *eligible* if for any  $\{p_1, \dots, p_k\} \in \mathcal{P}$ , we have:

- (i)  $\{p_1(n) - p_1(0), \dots, p_k(n) - p_k(0)\} \in \mathcal{P}$ ;
- (ii)  $\{p_1(rn + s), \dots, p_k(rn + s)\} \in \mathcal{P}$  for any  $r, s \in \mathcal{O}_K$  with  $r \neq 0$ ;
- (iii)  $\{cp_1(n), \dots, cp_k(n)\} \in \mathcal{P}$  for any  $c \in K \setminus \{0\}$  such that  $cp_i$  is  $\mathcal{O}_K$ -valued for  $i = 1, \dots, k$ .

**PROPOSITION 2.13** (cf. [19, proposition 4.1]). *Let  $\mathcal{P}$  be eligible. Suppose that for some  $m \in \mathbb{N}$ , the nilfactor  $\mathcal{Z}_m$  is characteristic for every  $P \in \mathcal{P}$  in totally ergodic systems. Then  $\mathcal{Z}_m$  is characteristic for every  $P \in \mathcal{P}$  in ergodic systems.*

*Proof.* Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic  $\mathcal{O}_K$ -system. Let  $f_1, \dots, f_k \in L^\infty(\mu)$ , and suppose  $\mathbb{E}[f_i | \mathcal{Z}_m] = 0$  for some  $i = 1, \dots, k$ . Without loss of generality,  $i = 1$ . We want to show

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = 0$$

in  $L^2(\mu)$ . Shifting by the constant terms and using property (i), we may assume  $p_i(0) = 0$ .

By Theorem 1.13, we may assume that  $X = G/\Gamma$  is a nilmanifold and  $T$  acts by nil-translations  $T^n x = a(n)x$  with  $a(n) \in G$ . We claim that there exists  $r \in \mathcal{O}_K$  such that the (finitely many) ergodic components of the action  $(T^m)_{m \in \mathcal{O}_K}$  are totally ergodic. Let  $\mathcal{Z}$  be the Kronecker factor of  $(X, T)$ . This is an action by rotations on an abelian Lie group of the form  $\mathbb{Z}_{a_1} \times \cdots \times \mathbb{Z}_{a_d} \times \mathbb{T}^c$  with  $a_1, \dots, a_d \in \mathbb{N}, c \in \mathbb{N} \cup \{0\}$ . Set  $a := \prod_{j=1}^d a_j \in \mathbb{N}$ . Letting  $r = a(b_1 + \cdots + b_d)$ , where  $\{b_1, \dots, b_d\}$  is an integral basis for  $\mathcal{O}_K$ , we then have that the ergodic components of  $(T^m)_{m \in \mathcal{O}_K}$  are totally ergodic (we have trivialized the rational component of the Kronecker factor).

Since  $p_i(0) = 0$  for each  $i = 1, \dots, k$ , there exist  $D \in \mathcal{O}_K \setminus \{0\}$  so that the polynomials  $q_i(n) := r^{-1}p_i(Dn)$  are  $\mathcal{O}_K$ -valued by Lemma 2.11. By properties (ii) and (iii),  $\{q_1, \dots, q_k\} \in \mathcal{P}$ .

Now,  $(T^m)_{m \in \mathcal{O}_K}$  has finitely many ergodic components and  $\mathcal{Z}_m(T^m) \subseteq \mathcal{Z}_m(T^n)$ , so  $\mathbb{E}[f_1 | \mathcal{Z}_m^{(j)}] = 0$ , where  $\mathcal{Z}_m^{(j)}$  is the nilfactor for the  $j$ th ergodic component of  $T^m$ . Summing over the finitely many ergodic components of  $T^m$ , we thus have

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(Dn)} f_i = \text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{r \cdot q_i(n)} f_i = 0.$$

Note that by the proof of Lemma 2.11,  $p_i(Dn + s) \equiv p_i(s) \pmod{r\mathcal{O}_K}$ . Hence,  $q_i^{(s)}(n) := r^{-1}(p_i(Dn + s) - p_i(s))$  is  $\mathcal{O}_K$ -valued. Moreover, since  $\mathcal{P}$  is eligible, we have  $\{q_1^{(s)}, \dots, q_k^{(s)}\} \in \mathcal{P}$ . By assumption,  $\mathbb{E}[f_i | \mathcal{Z}_m] = 0$  for some  $i = 1, \dots, k$ . It follows that  $\mathbb{E}[T^{p_i(s)} f_i | \mathcal{Z}_m] = 0$ , since  $\mathcal{Z}_m$  is  $T$ -invariant. Thus, by the argument in the previous paragraph, we have

$$\begin{aligned} \text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(Dn+s)} f_i &= \text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(Dn+s) - p_i(s)} (T^{p_i(s)} f_i) \\ &= \text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{r \cdot q_i^{(s)}(n)} (T^{p_i(s)} f_i) = 0 \end{aligned}$$

for  $s \in \mathcal{O}_K/r\mathcal{O}_K$ . This completes the proof.

### 3. Characteristic factors

#### 3.1. Proof of Theorem C

We want to prove that the rational Kronecker factor,  $\mathcal{K}_{rat}$ , is characteristic for the average

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i$$

when  $p_1, \dots, p_k \in K[x]$  are independent  $\mathcal{O}_K$ -valued polynomials.

We will first prove a special case:

**THEOREM 3.1.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $p_1, \dots, p_k \in K[x]$  are independent  $\mathcal{O}_K$ -valued polynomials. If  $(X, \mathcal{B}, \mu, T)$  is a totally ergodic  $\mathcal{O}_K$ -system*

and  $f_1, \dots, f_k \in L^\infty(\mu)$ , then

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = \prod_{i=1}^k \int_X f_i d\mu.$$

*Remark 3.2.* After a previous version of this paper appeared on *arXiv*, Best and Ferré Moragues reproved Theorem 3.1 using a different method (see [17, theorem 1.6]).

3.1.1. *Reduction to Weyl systems*

Recall that a sequence  $(x_n)_{n \in \mathcal{O}_K}$  in a compact topological space  $X$  is *well-distributed* with respect to a probability measure  $\mu$  on  $X$  if  $UC\text{-}\lim_{n \in \mathcal{O}_K} \delta_{x_n} = \mu$  in the weak- $*$  topology. That is, for any continuous function  $f : X \rightarrow \mathbb{C}$  and any Følner sequence  $(\Phi_N)_{N \in \mathbb{N}}$  in  $(\mathcal{O}_K, +)$ , one has

$$\frac{1}{|\Phi_N|} \sum_{n \in \Phi_N} f(x_n) \rightarrow \int_X f d\mu.$$

By Theorem 1.13, Theorem 3.1 is equivalent to the following equidistribution result:

**THEOREM 3.3.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $(X, \mathcal{B}, \mu, T)$  be a totally ergodic  $\mathcal{O}_K$ -nilsystem. Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be independent  $\mathcal{O}_K$ -valued polynomials. Then for almost every  $x \in X$ , the sequence  $(T^{p_1(n)}x, \dots, T^{p_k(n)}x)_{n \in \mathcal{O}_K}$  is well-distributed in  $X^k$ .*

Having reduced to an equidistribution result on nilmanifolds, we can now make several more reductions. First, by Proposition 2.4, the nilmanifold in Theorem 3.3 is necessarily connected, since it admits a totally ergodic action by niltranslations. Next, by Proposition 2.10, we may expand the polynomials  $p_1, \dots, p_k$  in coordinates with respect to an integral basis in order to obtain an independent family of  $\mathbb{Z}$ -valued polynomials. Hence, Theorem 3.3 follows from:

**THEOREM 3.4.** *Let  $d, k, l \in \mathbb{N}$ . Let  $(X, \mathcal{B}, \mu, T)$  be an ergodic, connected  $\mathbb{Z}^l$ -nilsystem. Let  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l\} \subseteq \mathbb{Q}[x_1, \dots, x_d]$  be a family of independent  $\mathbb{Z}$ -valued polynomials. Then the sequence*

$$\left( \prod_{j=1}^l T_j^{p_{1,j}(n)} x, \dots, \prod_{j=1}^l T_j^{p_{k,j}(n)} x \right)_{n \in \mathbb{Z}^d}$$

*is well-distributed in  $X^k$  for every  $x$  in a co-meager set of full measure.*

Now we will reduce from a connected nilsystem to the case that  $(X, T)$  is a Weyl system, i.e.  $X$  is a finite-dimensional torus and  $T$  acts by unipotent affine transformations. Let  $G_0$  be the connected component of the identity in  $G$ ,  $Z = X/[G_0, G_0]$ , and  $\pi : G \rightarrow Z$  the projection map. The following result of Leibman shows that we can reduce to studying orbits in  $Z$ :

**THEOREM 3.5** ([35, theorem C]). *Let  $X = G/\Gamma$  be a connected nilmanifold,  $x \in X$ , and  $g : \mathbb{Z}^d \rightarrow G$  a polynomial map. The following are equivalent:*

- (i) the orbit  $\{g(n)x : n \in \mathbb{Z}^d\}$  is dense in  $X$ ;
- (ii)  $\{g(n)\pi(x) : n \in \mathbb{Z}^d\}$  is dense in  $Z$ ;
- (iii)  $(g(n)x)_{n \in \mathbb{Z}^d}$  is well-distributed in  $X$ ;
- (iv)  $(g(n)\pi(x))_{n \in \mathbb{Z}^d}$  is well-distributed in  $Z$ .

LEMMA 3.6 (cf. [20, proposition 2.1]). *Without loss of generality,  $G_0$  is abelian.*

*Proof.* Use Theorem 3.5 to reduce to the projection onto  $Z$ . Now,  $(G/[G_0, G_0])_0$  is an abelian group, and a factor of a totally ergodic system is totally ergodic.

LEMMA 3.7 (cf. [20, propositions 3.1 and 3.2]). *Without loss of generality,  $(X, T)$  is a connected Weyl system.*

*Proof.* To reduce from a connected nilsystem such that  $G_0$  is abelian to a connected Weyl system, see [20, proposition 3.1]. The isomorphism between a niltranslation and a unipotent affine transformation does not depend on the element of  $G$  defining the niltranslation, so the result still holds for  $d$  commuting niltranslations.

We have therefore reduced Theorem 3.1 to the following result about well-distribution of polynomial orbits for unipotent affine actions on tori:

THEOREM 3.8. *Let  $d, l, k, m \in \mathbb{N}$ . Let  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l\} \subseteq \mathbb{Q}[x_1, \dots, x_d]$  be  $\mathbb{Z}$ -valued and independent over  $\mathbb{Q}$ . Let  $T_1, \dots, T_l : \mathbb{T}^m \rightarrow \mathbb{T}^m$  be commuting unipotent affine transformations generating an ergodic  $\mathbb{Z}^l$ -action. Then the polynomial sequence*

$$\left( \prod_{j=1}^l T_j^{p_{1,j}(n)} x, \dots, \prod_{j=1}^l T_j^{p_{k,j}(n)} x \right)_{n \in \mathbb{Z}^d}$$

*is well-distributed in  $\mathbb{T}^{mk}$  for all  $x$  in a co-meager set of full measure.*

### 3.1.2. Equidistribution of $\mathbb{Z}^l$ -polynomial sequences

In order to prove Theorem 3.8, we will use two classic results in equidistribution. The first is a multivariable version of Weyl’s polynomial equidistribution theorem.

LEMMA 3.9 (cf. [38, Satz 20]). *Fix  $d \in \mathbb{N}$ . Let  $p(x_1, \dots, x_d) \in \mathbb{R}[x_1, \dots, x_d]$ . If at least one coefficient of  $p$  other than the constant term is irrational, then the  $\mathbb{Z}^d$ -sequence  $(p(n_1, \dots, n_d))_{n \in \mathbb{Z}^d}$  is well-distributed mod 1.*

Remark 3.10. Weyl proved Lemma 3.9 in the case  $d = 2$ , with indications of how to prove the case of general  $d$ , for Følner sequences that are increasing dilations of a fixed set. Lemma 3.9 in its full generality is today an easy exercise with the help of an appropriate variant of the van der Corput trick (see, e.g. [13, lemma A6]).

The next lemma allows one to reduce equidistribution in a multidimensional torus to equidistribution in the circle.

LEMMA 3.11. Fix  $d, m \in \mathbb{N}$ . A  $\mathbb{Z}^d$ -sequence  $u: \mathbb{Z}^d \rightarrow \mathbb{T}^m$  is well-distributed in  $\mathbb{T}^m$  if and only if for every  $c \in \mathbb{Z}^m \setminus \{0\}$ , the sequence  $c \cdot u(n) = c_1 u_1(n) + \dots + c_m u_m(n)$  is well-distributed in  $\mathbb{T}$ .

*Proof.* See [33, theorem 6.3] for the case  $d = 1$ . The same argument works for general  $d \in \mathbb{N}$ .

With these two lemmas at hand, we are now ready to prove Theorem 3.8.

*Proof of Theorem 3.8.* For each  $j = 1, \dots, l$ , we can write  $T_j x = A_j x + \alpha_j$  for some nilpotent  $(m \times m)$ -matrix  $A_j$  with integer entries and a vector  $\alpha_j \in \mathbb{T}^m$ . Since the matrices  $A_1, \dots, A_l$  commute, they are simultaneously triangularisable. That is, there is a matrix  $P$  with rational entries and lower-triangular matrices  $B_j$  such that  $A_j P = P B_j$ . Multiplying  $P$  by a common denominator of its entries, we may assume that  $P$  has integer entries. Then  $P$  is well-defined as a surjective endomorphism of  $\mathbb{T}^m$ . (One can show that in general,  $P$  cannot be assumed to be an automorphism.) Let  $\beta_j \in \mathbb{T}^m$  such that  $P \beta_j = \alpha_j$ , and set  $S_j x := B_j x + \beta_j$ . Then we have  $T_j P x = A_j P x + \alpha_j = P B_j x + P \beta_j = P S_j x$ . That is,  $T$  is a factor of  $S$  with factor map  $P: \mathbb{T}^m \rightarrow \mathbb{T}^m$ .

Now we check that  $S$  is an ergodic  $\mathbb{Z}^l$ -action on  $\mathbb{T}^m$ . For  $n = (n_1, \dots, n_l) \in \mathbb{Z}^l$ , let  $S^n := \prod_{j=1}^l S_j^{n_j}$  and  $T^n := \prod_{j=1}^l T_j^{n_j}$ . Suppose  $A \subseteq \mathbb{T}^m$  is  $S$ -invariant. That is,  $S^n A = A$  for every  $n \in \mathbb{Z}^l$ . Applying the factor map  $P: \mathbb{T}^m \rightarrow \mathbb{T}^m$ , we have  $PA = PS^n A = T^n PA$ , so  $PA$  is a  $T$ -invariant set. But  $T$  is ergodic by assumption, so  $\mu(PA) \in \{0, 1\}$ . If  $\mu(PA) = 0$ , then  $\mu(A) \leq \mu(P^{-1}PA) = \mu(A) = 0$ . On the other hand, if  $\mu(PA) = 1$ , then  $\mu(A) \geq 1/|\det(P)|$ .

Assume  $A$  is an  $S$ -invariant set of minimal positive measure so that  $S|_A: A \rightarrow A$  is ergodic. Let  $x \in A$  be a generic point for  $S|_A$ . Then by Proposition 2.5,  $A$  differs from the set  $Y = \overline{\{S^n x: n \in \mathbb{Z}^l\}}$  by a null-set, and  $Y$  is a subtorus of  $\mathbb{T}^m$ . But  $\mu(Y) = \mu(A) > 0$ , so  $Y = \mathbb{T}^m$ . Thus, for any  $S$ -invariant set  $A$  of positive measure, we have  $\mu(A) = 1$ . Therefore,  $S$  is ergodic.

The above argument shows that, without loss of generality, we may assume that the transformations  $T_j$  are of the form

$$T_j x = \left( x_1 + \alpha_1^{(j)}, x_2 + a_{2,1}^{(j)} x_1 + \alpha_2^{(j)}, \dots, x_m + \sum_{r=1}^{m-1} a_{m,r}^{(j)} x_r + \alpha_m^{(j)} \right)$$

for  $x = (x_1, \dots, x_m) \in \mathbb{T}^m$ .

Let  $u_x(n) := (T^{p_1(n)} x, \dots, T^{p_k(n)} x)$  for  $x \in \mathbb{T}^m$  and  $n \in \mathbb{Z}^d$ , where  $T^{p_i(n)}$  denotes the transformation  $\prod_{j=1}^l T_j^{p_{ij}(n)}$ . We now break the proof into two cases, depending on the coefficients  $a_{s,r}^{(j)}$ .

First consider the case  $a_{s,r}^{(j)} = 0$  for all  $1 \leq j \leq l, 2 \leq s \leq m$ , and  $1 \leq r \leq s - 1$ . That is,

$$T_j x = \left( x_1 + \alpha_1^{(j)}, x_2 + \alpha_2^{(j)}, \dots, x_m + \alpha_m^{(j)} \right) = x + \alpha^{(j)}$$

is a toral rotation for  $j = 1, \dots, l$ . Let  $\alpha: \mathbb{Z}^l \rightarrow \mathbb{T}^m$  be the homomorphism  $\alpha(n_1, \dots, n_l) = \sum_{j=1}^l n_j \alpha^{(j)}$ . Then for any  $x \in \mathbb{T}^m$ , we have

$$u_x(n) = x + (\alpha(p_1(n)), \dots, \alpha(p_k(n))) = x + u_0(n).$$



Now let  $(c_{i,s})_{1 \leq i \leq k, 1 \leq s \leq m} \in \mathbb{Z}^{mk} \setminus \{0\}$ . By Lemma 3.11, it suffices to show

$$c \cdot u_0(n) = \sum_{i=1}^k c_i \cdot \alpha(p_i(n))$$

is well-distributed in  $\mathbb{T}$ . For each  $i = 1, \dots, k$  and  $n \in \mathbb{Z}^l$ , we have

$$c_i \cdot \alpha(n) = \sum_{s=1}^m c_{i,s} \sum_{j=1}^l n_j \alpha_s^{(j)} = \sum_{j=1}^l n_j \sum_{s=1}^m c_{i,s} \alpha_s^{(j)}.$$

Letting  $\beta_i = \left( \sum_{s=1}^m c_{i,s} \alpha_s^{(1)}, \dots, \sum_{s=1}^m c_{i,s} \alpha_s^{(l)} \right) \in \mathbb{T}^l$ , we therefore have

$$c_i \cdot \alpha(n) = \beta_i \cdot n.$$

Thus,

$$c \cdot u_0(n) = \sum_{i=1}^k \beta_i \cdot p_i(n) = \sum_{i=1}^k \sum_{j=1}^l \beta_{i,j} p_{i,j}(n),$$

which is well-distributed by Lemma 3.9.

Now suppose  $a_{s,r}^{(j)} \neq 0$  for some  $1 \leq j \leq l, 2 \leq s \leq m$ , and  $1 \leq r \leq s - 1$ . Let  $x = (x_1, \dots, x_m)$  so that  $\{1, x_1, \dots, x_m\}$  is linearly independent over  $\mathbb{Q} \left( \left\{ \alpha_s^{(j)} : 1 \leq j \leq l, 1 \leq s \leq m \right\} \right)$ .

Put  $r_0 := \max \left\{ 1 \leq r \leq m - 1 : a_{s,r}^{(j)} \neq 0 \text{ for some } 1 \leq j \leq l \text{ and } r + 1 \leq s \leq m \right\}$ , and let  $S := \left\{ r_0 + 1 \leq s \leq m : a_{s,r_0}^{(j)} \neq 0 \text{ for some } 1 \leq j \leq l \right\}$ . For  $s \in S$ , let

$$v_s := \left( a_{s,r_0}^{(1)}, \dots, a_{s,r_0}^{(l)} \right) \in \mathbb{Z}^l \setminus \{0\}.$$

We claim that without loss of generality,  $\{v_s : s \in S\}$  is linearly independent over  $\mathbb{Q}$ . Indeed, suppose  $\sum_{s \in S} c_s v_s = 0$  for some  $(c_s)_{s \in S} \in \mathbb{Z}^S \setminus \{0\}$ . Taking  $s_0 := \max\{s \in S : c_s \neq 0\}$ , we can perform a change of variables

$$\tilde{x}_{s_0} := \sum_{s \in S} c_s x_s.$$

In the new coordinates, this gives  $\tilde{v}_{s_0} = 0$ , so  $\tilde{S} = S \setminus \{s_0\}$  and linear dependence is removed. Moreover, this change of variables is  $|c_{s_0}|$ -to-one, so well-distribution in the new coordinates implies well-distribution in the original system, since orbit closures of polynomial sequences must be finite unions of subtori (see Proposition 2.5).

Assume now that  $\{v_s : s \in S\}$  is linearly independent over  $\mathbb{Q}$ . Let  $(c_{i,s})_{1 \leq i \leq k, 1 \leq s \leq m} \in \mathbb{Z}^{mk} \setminus \{0\}$ . If  $c_{i,s} = 0$  for  $1 \leq i \leq k$  and  $s \in S$ , then we can reduce to the lower-dimensional torus consisting of those coordinates not in the set  $S$ . Thus, we may assume  $c_{i,s} \neq 0$  for some  $1 \leq i \leq k$  and  $s \in S$ . Now expand

$$c \cdot u_x(n) = \sum_{s=1}^m P_s(n) x_s + R(n),$$

where  $P_s$  is  $\mathbb{Z}$ -valued for each  $s = 1, \dots, m$  and  $R(n)$  is linearly independent from  $\{x_1, \dots, x_S\}$  over  $\mathbb{Q}$  for every  $n \in \mathbb{Z}^d$ . By the restriction on the coordinates of  $x$ , we can compute

$$\begin{aligned} P_{r_0}(n) &= \sum_{i=1}^k c_{i,r_0} + \sum_{i=1}^k \sum_{s \in S} c_{i,s} (v_s \cdot p_i(n)) \\ &= \sum_{i=1}^k c_{i,r_0} + \sum_{i=1}^k \sum_{j=1}^l \sum_{s \in S} c_{i,s} \alpha_{s,r_0}^{(j)} p_{i,j}(n) \\ &= \text{const.} + \sum_{i=1}^k \sum_{j=1}^l d_{i,j} p_{i,j}(n), \end{aligned}$$

where

$$(d_{i,1}, \dots, d_{i,l}) = \sum_{s \in S} c_{i,s} v_s$$

for  $1 \leq i \leq k$ . By assumption,  $(c_{i,s})_{1 \leq i \leq k, s \in S} \neq 0$ . Since  $\{v_s : s \in S\}$  is linearly independent over  $\mathbb{Q}$ , this implies that  $d_{i,j} \neq 0$  for some  $1 \leq i \leq k$  and  $1 \leq j \leq l$ . Therefore,  $P_{r_0}(n)$  is nonconstant, since  $\{p_{i,j} : 1 \leq i \leq k, 1 \leq j \leq l\}$  is an independent family. It follows that the polynomial  $c \cdot u_x(n)$  has at least one irrational coefficient other than the constant term, so  $c \cdot u_x(n)$  is well-distributed in  $\mathbb{T}$  by Lemma 3.9.

We have shown that  $c \cdot u_x(n)$  is well-distributed in  $\mathbb{T}$  for every  $c \in \mathbb{Z}^{mk} \setminus \{0\}$ . By Lemma 3.11, it follows that  $u_x(n)$  is well-distributed in  $\mathbb{T}^{mk}$  as desired.

Let  $E$  be the set of exceptional points  $x \in \mathbb{T}^m$  such that  $(u_x(n))_{n \in \mathbb{Z}^d}$  is not well-distributed in  $\mathbb{T}^{mk}$ . The above argument show that if  $x = (x_1, \dots, x_m) \in E$ , then

$$c_0 + \sum_{r=1}^m c_r x_r = 0 \tag{3.1}$$

for some coefficients

$$(c_r)_{r=0}^m \in \mathbb{Q} \left( \left\{ \alpha_s^{(j)} : 1 \leq j \leq l, 1 \leq s \leq m \right\} \right)^{m+1} \setminus \{0\}.$$

For each such choice of coefficients  $(c_r)_{r=0}^m$ , the equation (3.1) defines a subtorus of dimension  $m - 1$ . Hence,  $E$  is contained in a countable union of  $(m - 1)$ -dimensional subtori. In particular,  $E$  is both a set of measure zero and meager in  $\mathbb{T}^m$ .

3.1.3. *The general case*

Theorem 3.1 says the for totally ergodic systems, the trivial factor is characteristic for independent polynomials  $\{p_1, \dots, p_k\}$ . In particular, the Kronecker factor  $\mathcal{Z} = \mathcal{Z}_1$  is characteristic. Now, the collection of all families of independent  $\mathcal{O}_K$ -valued polynomials is clearly eligible, so by Proposition 2.13, the Kronecker factor is characteristic for independent polynomials in any ergodic system.

In order to prove Theorem C, it remains only to reduce from the Kronecker factor to the rational Kronecker factor. We want to prove: if  $\mathbb{E}[f_i | \mathcal{K}_{rat}] = 0$  for some  $i = 1, \dots, k$ , then  $\text{UC-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = 0$ . Since the Kronecker factor is spanned by eigenfunctions,

we may assume that  $f_i$  is an eigenfunction with eigenvalue  $\alpha_i \in \mathbb{T}^d$  for  $i = 1, \dots, k$ . That is,  $T^n f_i = e(n \cdot \alpha_i) f_i$ . The condition that  $\mathbb{E}[f_i | \mathcal{K}_{rat}] = 0$  means that  $\alpha_i \notin \mathbb{Q}^d$  for some  $i = 1, \dots, k$ . Expanding the multiple ergodic average, we have

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{p_i(n)} f_i = \text{UC-} \lim_{n \in \mathcal{O}_K} e \left( \sum_{i=1}^k \sum_{j=1}^d p_{i,j}(n) \alpha_{i,j} \right) \prod_{i=1}^k f_i. \tag{3.2}$$

Since  $\alpha_{i,j} \notin \mathbb{Q}$  for some  $1 \leq i \leq k$  and  $1 \leq j \leq d$ , the polynomial  $\sum_{i=1}^k \sum_{j=1}^d p_{i,j}(n) \alpha_{i,j}$  has an irrational coefficient other than the constant term. Thus, by Lemma 3.9, the average (3.2) is equal to 0 as desired.

### 3.2. Proof of Theorem D

We follow the approach of Frantzikinakis (see [19, theorem A]), modifying as necessary to upgrade to our multidimensional setting.

The polynomials  $l_1 p(n), \dots, l_k p(n)$  are essentially distinct, so the characteristic factor for the averages

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{l_i p(n)} f_i \tag{3.3}$$

is a nilfactor,  $\mathcal{Z}_r$ , for some  $r \in \mathbb{Z}$ , by Theorem 1.13. The content of Theorem D is thus to show that  $r = k - 1$ .

We do this in several steps. First, we will show that for totally ergodic systems, the limit (3.3) does not depend on  $p$ . As a consequence,  $\mathcal{Z}_{k-1}$  is characteristic for totally ergodic systems, since it is characteristic when  $p(n) = n$ . We then apply Proposition 2.13 to conclude that  $\mathcal{Z}_{k-1}$  is characteristic in any ergodic system.

#### 3.2.1. Totally ergodic systems

For this section, we will assume that  $(X, \mathcal{B}, \mu, T)$  is a totally ergodic  $\mathcal{O}_K$ -system, and we set out to prove that the limit (3.3) is independent of the choice of polynomial  $p$ .

By Theorem 1.13 and a standard approximation argument, we may further assume that  $X = G/\Gamma$  is a nilmanifold and  $T$  is an action by niltranslations  $T^n x = a(n)x$  with  $a(n) \in G$ . It therefore suffices to show that the orbits  $\{(T^{l_1 p(n)} x, \dots, T^{l_k p(n)} x) : n \in \mathcal{O}_K\}$  and  $\{(T^{l_1 n} x, \dots, T^{l_k n} x) : n \in \mathcal{O}_K\}$  are equidistributed for almost every  $x \in X$ . Equivalently, letting  $g(n) := (a(l_1 n), \dots, a(l_k n)) \in G^k$  and  $\tilde{x} = (x, \dots, x) \in X^k$ , we want to show that  $\{g(p(n))\tilde{x} : n \in \mathcal{O}_K\}$  and  $\{g(n)\tilde{x} : n \in \mathcal{O}_K\}$  are equidistributed for almost every  $x \in X$ . Now, by Theorem 3.5, it is enough to show that these sequences have the same closure in  $X^k$ .

By Proposition 2.9, any nonconstant polynomial  $p \in K[x]$  has algebraically independent coordinates, so we will prove a related result about  $\mathbb{Z}^d$ -valued polynomials with algebraically independent coordinates:

**PROPOSITION 3.12** (cf. [19, Proposition 2.7]). *Let  $X = G/\Gamma$  be a nilmanifold,  $g : \mathbb{Z}^l \rightarrow G$  a polynomial sequence, and  $x \in X$ . Suppose  $p : \mathbb{Z}^d \rightarrow \mathbb{Z}^l$  is a polynomial with algebraically independent coordinates. If  $Y := \overline{\{g(n)x : n \in \mathbb{Z}^l\}}$  is connected, then  $\overline{\{g(p(n))x : n \in \mathbb{Z}^d\}} = Y$ .*

*Proof.* By Theorem 2.6,  $Y$  is a subnilmanifold  $H/\Delta$ . Now by Theorem 3.5, we may replace  $H$  by  $H/[H_0, H_0]$  and assume that  $H_0$  is abelian. As in Lemma 3.7, we may further

reduce to the case that  $Y = \mathbb{T}^m$  and  $g(n)x = T_1^{p_1(n)} \cdots T_k^{p_k(n)}x$  with  $T_i$  unipotent affine actions. The coordinates of  $\{g(n)x : n \in \mathbb{Z}^l\}$  are real polynomials in  $n \in \mathbb{Z}^l$ , so it remains to show: if  $u : \mathbb{Z}^l \rightarrow \mathbb{T}^m$  is a sequence with polynomial coordinates and  $\overline{\{u(n) : n \in \mathbb{Z}^l\}} = \mathbb{T}^m$ , then  $\overline{\{u(p(n)) : n \in \mathbb{Z}^d\}} = \mathbb{T}^m$  for every polynomial  $p : \mathbb{Z}^d \rightarrow \mathbb{Z}^l$  with algebraically independent coordinates.

The polynomial  $u(n)$  can be decomposed as  $u(n) = u(0) + u_0(n)q + u_1(n)\alpha_1 + \cdots + u_r(n)\alpha_r$ , where  $q \in \mathbb{Q}$ ,  $\alpha_1, \dots, \alpha_r \in \mathbb{R}$  are linearly independent irrational numbers and  $u_i : \mathbb{Z}^l \rightarrow \mathbb{Z}^m$  are polynomials with  $u_i(0) = 0$ . By Corollary 2.7, the orbit  $\{u(n) : n \in \mathbb{Z}^l\}$  is dense in  $\mathbb{T}^m$  if and only if

$$\text{span}(u_1) + \cdots + \text{span}(u_r) = \mathbb{R}^m.$$

Thus, it suffices to prove  $\text{span}(u_i \circ p) = \text{span}(u_i)$  for each  $i = 1, \dots, r$ .

Fix  $1 \leq i \leq r$ . Suppose the coordinates of  $u_i \circ p$  satisfy a linear relation  $\sum_{j=1}^m c_j u_{i,j}(p(n)) = 0$  for some  $c_1, \dots, c_m \in \mathbb{Z}$ , where  $u_i = (u_{i,1}, \dots, u_{i,m})$  with  $u_{i,j} : \mathbb{Z}^l \rightarrow \mathbb{Z}$ . Let  $v : \mathbb{Z}^l \rightarrow \mathbb{Z}$  be the polynomial  $v(n) := \sum_{j=1}^m c_j u_{i,j}(n)$ . Then  $v \circ p = 0$ . But the coordinates of  $p$  are algebraically independent, so we must have  $v = 0$ . That is, the coordinates of  $u_i$  satisfies the same linear relation. Therefore,  $\text{span}(u_i \circ p) = \text{span}(u_i)$  as desired.

It remains only to show that  $Y := \overline{\{g(n)\tilde{x} : n \in \mathcal{O}_K\}}$  is connected. This is where we use that the system is totally ergodic. Let  $Y_w = Hx_w$ ,  $w \in W$ , as in Theorem 2.6. Since  $W$  is a finite group,  $\omega^{-1}(0) \subseteq \mathbb{Z}^d$  has finite index in  $\mathbb{Z}^d$ . Because  $T$  is totally ergodic, we therefore have  $Y_0 = Y$ , so  $Y$  is indeed connected.

In summary, we have shown the following:

**THEOREM 3.13.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $(X, \mathcal{B}, \mu, T)$  be a totally ergodic  $\mathcal{O}_K$ -system. Let  $p(x) \in K[x]$  be a non-constant  $\mathcal{O}_K$ -valued polynomial. Let  $l_1, \dots, l_k \in \mathcal{O}_K$  be distinct and nonzero. Then*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{l_i p(n)} f_i = UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^k T^{l_i n} f_i.$$

3.2.2. *General case*

Now we prove Theorem D. Letting  $l := \gcd(l_1, \dots, l_k)$  and replacing  $l_1, \dots, l_k$  by  $l'_i := l_i/l$  and  $p$  by  $lp$ , we may assume without loss of generality that  $l = 1$ . By [27, theorem 4.1.2], the characteristic factor for  $\{l_1 n, \dots, l_k n\}$  is  $\mathcal{Z}_{k-1}$ . Thus, by Theorem 3.13,  $\mathcal{Z}_{k-1}$  is characteristic for  $\{l_1 p(n), \dots, l_k p(n)\}$  in the case of totally ergodic systems. It is easily checked that the collection

$$\mathcal{P} := \{\{l_1 p(n), \dots, l_k p(n)\} : p(x) \in K[x] \text{ is nonconstant and } \mathcal{O}_K\text{-valued}\}$$

is eligible (see Definition 2.12) under the assumption that  $l = \gcd(l_1, \dots, l_k) = 1$ . Hence, Theorem D follows by Proposition 2.13.

4. *Large intersections*

Having established characteristic factors for the polynomial multiple ergodic averages of interest, we now move to deducing the related Khintchine-type theorems.

4.1. Proof of Theorem A

We want to prove Theorem A, restated here for the convenience of the reader:

**THEOREM A.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. Then for any measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p_1(n)}A \cap \dots \cap T^{-p_k(n)}A \right) > \mu(A)^{k+1} - \varepsilon \right\} \tag{1.4}$$

is syndetic.

We will prove the following stronger statement:

**THEOREM 4.1.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $\{p_1, \dots, p_k\} \subseteq K[x]$  is a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. Then for any measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , there exist  $\xi \in \mathcal{O}_K$  and  $D \in \mathcal{O}_K \setminus \{0\}$  such that*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \mu \left( A \cap T^{-p_1(\xi+Dn)}A \cap \dots \cap T^{-p_k(\xi+Dn)}A \right) > \mu(A)^{k+1} - \varepsilon.$$

Assuming Theorem 4.1, the set

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p_1(\xi+Dn)}A \cap \dots \cap T^{-p_k(\xi+Dn)}A \right) > \mu(A)^{k+1} - \varepsilon \right\}$$

is syndetic by Proposition 1.12. Since  $\xi + D\mathcal{O}_K$  is syndetic in  $\mathcal{O}_K$ , Theorem 4.1 follows immediately.

*Proof of Theorem 4.1.* First assume that  $T$  is ergodic. The rational Kronecker factor  $\mathcal{K}_{rat}$  is the inverse limit of the periodic factors  $\mathcal{K}_r := \{f \in L^2(\mu) : T^r f = f \text{ for all } n \in \mathcal{O}_K\}$ ,  $r \in \mathcal{O}_K$ . Note that  $\mathcal{K}_r \subseteq \mathcal{K}_s$  if  $r | s$  in  $\mathcal{O}_K$ . Thus, we may approximate  $\mathcal{K}_{rat}$  by  $\mathcal{K}_r$  for some  $r \in \mathcal{O}_K$ . To be precise, there exists  $r \in \mathcal{O}_K$  such that

$$\left\| \mathbb{E}[\mathbb{1}_A | \mathcal{K}_{rat}] - \mathbb{E}[\mathbb{1}_A | \mathcal{K}_r] \right\|_1 < \frac{\varepsilon}{k+1}.$$

Now, the system  $(X, \mathcal{B}, \mu, (T^m)_{m \in \mathcal{O}_K})$  has finitely many ergodic components. In fact, for some  $m \leq [\mathcal{O}_K : r\mathcal{O}_K]$ ,  $X$  can be partitioned into  $m$  disjoint sets  $X_1, \dots, X_m \in \mathcal{B}$  with  $\mu(X_j) = 1/m$  such that  $\mu(X_j \Delta T^{-m}X_j) = 0$  and  $(X, \mathcal{B}, \mu_j, (T^m)_{m \in \mathcal{O}_K})$  is ergodic, where  $\mu_j(B) = m \cdot \mu(B \cap X_j)$ .

By Lemma 2.11, let  $\xi \in \mathcal{O}_K$  and  $D \in \mathcal{O}_K \setminus \{0\}$  such that  $p_i(\xi + D\mathcal{O}_K) \subseteq r\mathcal{O}_K$  for  $i = 1, \dots, k$ . For each  $i = 1, \dots, k$ , let  $q_i(x) \in K[x]$  be the  $\mathcal{O}_K$ -valued polynomial  $q_i(x) := r^{-1}p_i(\xi + Dx)$ . Then by Theorem C,

$$\begin{aligned} & UC\text{-}\lim_{n \in \mathcal{O}_K} \mu \left( A \cap T^{-p_1(\xi+Dn)}A \cap \dots \cap T^{-p_k(\xi+Dn)}A \right) \\ &= UC\text{-}\lim_{n \in \mathcal{O}_K} \frac{1}{m} \sum_{j=1}^m \mu_j \left( A \cap (T^r)^{-q_1(n)}A \cap \dots \cap (T^r)^{-q_k(n)}A \right) \\ &= UC\text{-}\lim_{n \in \mathcal{O}_K} \frac{1}{m} \sum_{j=1}^m \int_X \mathbb{E}[\mathbb{1}_A | \mathcal{K}_{rat}] \prod_{i=1}^k (T^r)^{q_i(n)} \mathbb{E}[\mathbb{1}_A | \mathcal{K}_{rat}] d\mu_j \end{aligned}$$

$$\begin{aligned}
 &= \text{UC-} \lim_{n \in \mathcal{O}_K} \int_X \mathbb{E} [\mathbb{1}_A | \mathcal{K}_{rat}] \prod_{i=1}^k T^{rqi(n)} \mathbb{E} [\mathbb{1}_A | \mathcal{K}_{rat}] d\mu \\
 &> \text{UC-} \lim_{n \in \mathcal{O}_K} \int_X \mathbb{E} [\mathbb{1}_A | \mathcal{K}_r] \prod_{i=1}^k T^{rqi(n)} \mathbb{E} [\mathbb{1}_A | \mathcal{K}_r] d\mu - \varepsilon \\
 &= \int_X (\mathbb{E} [\mathbb{1}_A | \mathcal{K}_r])^{k+1} d\mu - \varepsilon \\
 &\geq \left( \int_X \mathbb{E} [\mathbb{1}_A | \mathcal{K}_r] d\mu \right)^{k+1} - \varepsilon \\
 &= \mu(A)^{k+1} - \varepsilon.
 \end{aligned}$$

Now suppose  $T$  is not ergodic. Let  $\mu = \int_{\Omega} \mu_{\omega} d\rho(\omega)$  be the ergodic decomposition. For each  $\omega \in \Omega$ , let  $r_{\omega} \in \mathcal{O}_K$  be minimal (with respect to divisibility) so that

$$\|\mathbb{E} [\mathbb{1}_A | \mathcal{K}_{rat}(\mu_{\omega})] - \mathbb{E} [\mathbb{1}_A | \mathcal{K}_{r_{\omega}}](\mu_{\omega})\|_{L^1(\mu_{\omega})} < \frac{\varepsilon}{2(k+1)}.$$

The function  $\omega \mapsto r_{\omega}$  is measurable, so we may define  $\Omega_r := \{\omega \in \Omega : r_{\omega} | r\}$  and let  $\mu_r := \int_{\Omega_r} \mu_{\omega} d\rho(\omega)$ . Then let  $r \in \mathcal{O}_K$  so that  $\rho(\Omega \setminus \Omega_r) < \varepsilon/2$ .

Note that in the proof of the ergodic case, the numbers  $\xi$  and  $D$  depend only on  $r$  and not on  $\mu$ . Thus, for every  $\omega \in \Omega_r$ , we have

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \mu_{\omega} \left( A \cap T^{-p_1(\xi+Dn)} A \cap \dots \cap T^{-p_k(\xi+Dn)} A \right) > \mu_{\omega}(A)^{k+1} - \frac{\varepsilon}{2}.$$

Now we integrate over  $\Omega$ :

$$\begin{aligned}
 &\text{UC-} \lim_{n \in \mathcal{O}_K} \mu \left( A \cap T^{-p_1(\xi+Dn)} A \cap \dots \cap T^{-p_k(\xi+Dn)} A \right) \\
 &\geq \text{UC-} \lim_{n \in \mathcal{O}_K} \int_{\Omega_r} \mu_{\omega} \left( A \cap T^{-p_1(\xi+Dn)} A \cap \dots \cap T^{-p_k(\xi+Dn)} A \right) d\rho(\omega) \\
 &> \int_{\Omega_r} \left( \mu_{\omega}(A)^{k+1} - \frac{\varepsilon}{2} \right) d\rho(\omega) \\
 &\geq \int_{\Omega_r} \mu_{\omega}(A)^{k+1} d\rho(\omega) - \frac{\varepsilon}{2} \\
 &> \int_{\Omega} \mu_{\omega}(A)^{k+1} d\rho(\omega) - \varepsilon \\
 &\geq \left( \int_{\Omega} \mu_{\omega}(A) d\rho(\omega) \right)^{k+1} - \varepsilon \\
 &= \mu(A)^{k+1} - \varepsilon.
 \end{aligned}$$

4.2. Proof of Theorem B

Now we turn to proving Theorem B, restated below:

**THEOREM B.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be an  $\mathcal{O}_K$ -valued intersective polynomial. Let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. Then for any ergodic*

measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \right) > \mu(A)^3 - \varepsilon \right\} \tag{1.5}$$

is syndetic.

Moreover, if  $s/r \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \cap T^{-(r+s)p(n)}A \right) > \mu(A)^4 - \varepsilon \right\} \tag{1.6}$$

is syndetic.

First we will prove the special case when  $T$  is totally ergodic. In this case, by applying Theorem D, we can compute limits explicitly:

**THEOREM 4.2.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\mathbf{X} = (X, \mathcal{B}, \mu, T)$  be a totally ergodic  $\mathcal{O}_K$ -system. Let  $Z$  be a compact abelian group and  $\alpha : (\mathcal{O}_K, +) \rightarrow Z$  a homomorphism such that the Kronecker factor of  $\mathbf{X}$  is isomorphic to the system  $\mathbf{Z} = (Z, \mathcal{B}_Z, \mu_Z, S)$ , where  $\mathcal{B}_Z$  is the Borel  $\sigma$ -algebra,  $\mu_Z$  is the Haar probability measure, and  $S$  acts by rotations  $S^n z = z + \alpha_n$  for  $n \in \mathcal{O}_K$ .*

- (1) *Let  $r, s \in \mathcal{O}_K$  distinct and nonzero,  $p(x) \in K[x]$  an  $\mathcal{O}_K$ -valued polynomial, and  $f_1, f_2 \in L^\infty(\mu)$ . Then*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} T^{rp(n)}f_1(x) \cdot T^{sp(n)}f_2(x) = \int_{Z^2} \tilde{f}_1(z+u)\tilde{f}_2(z+v) \, d\nu(u, v) \tag{4.1}$$

in  $L^2(\mu)$ , where  $x \mapsto z$  is the factor map,  $\tilde{f} = \mathbb{E}[f | Z]$ , and  $\nu$  is the Haar measure on the subgroup  $\overline{\{(\alpha_m, \alpha_{sm}) : n \in \mathcal{O}_K\}} \subseteq Z^2$ .

- (2) *Let  $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$  be coprime, and put  $a_3 = a_1 + a_2$ . There is a compact abelian group  $H$  such that the nilfactor  $(X, \mathcal{Z}_2, \mu, T)$  is isomorphic to a skew-product system  $\mathbf{Z} \times_\sigma H$ , and there exists a function  $\psi : Z^2 \rightarrow H$  such that  $\psi(0, \cdot) = 0$  and  $t \mapsto \psi(t, \cdot)$  is continuous as a function from  $Z$  to the space  $\mathcal{M}(Z, H)$  of measurable functions  $Z \rightarrow H$  in the topology of convergence in measure, and integers  $b_1, b_2, b_3 \in \mathbb{Z}$  such that: for any  $\mathcal{O}_K$ -valued polynomial  $p(x) \in K[x]$  and any  $f_1, f_2, f_3 \in L^\infty(\mu)$ , we have*

$$UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^3 T^{a_i p(n)} f_i(x) = \int_{Z \times H^2} \prod_{i=1}^3 \tilde{f}_i(z + a_i t, h + a_i u + a_i^2 v + b_i \psi(t, z)) \, dt \, du \, dv \tag{4.2}$$

in  $L^2(\mu)$ , where  $\tilde{f} = \mathbb{E}[f | \mathcal{Z}_2]$ .

*Proof.* Since the system  $\mathbf{X}$  is totally ergodic, the limits

$$UC\text{-}\lim_{n \in \mathcal{O}_K} T^{rp(n)}f_1 \cdot T^{sp(n)}f_2 \quad \text{and} \quad UC\text{-}\lim_{n \in \mathcal{O}_K} \prod_{i=1}^3 T^{a_i p(n)}f_i$$

are independent of the choice of the polynomial  $p$  by Theorem D. Thus, we may assume without loss of generality that  $p(n) = n$ . The identity (4.1) is then a special case of [2, theorem 3.1], and (4.2) is a special case of [2, theorem 7.1].

COROLLARY 4.3. *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $X = (X, \mathcal{B}, \mu, T)$  be a totally ergodic  $\mathcal{O}_K$ -system with Kronecker factor  $(Z, \alpha)$ .*

- (1) *Let  $r, s \in \mathcal{O}_K$  distinct and nonzero,  $p(x) \in K[x]$  an  $\mathcal{O}_K$ -valued polynomial, and  $f_0, f_1, f_2 \in L^\infty(\mu)$ . Then for any continuous function  $\eta : Z^2 \rightarrow \mathbb{C}$ , we have*

$$\begin{aligned} UC\text{-} \lim_{n \in \mathcal{O}_K} \eta(\alpha_{rp(n)}, \alpha_{sp(n)}) \int_X f_0 \cdot T^{rp(n)} f_1 \cdot T^{sp(n)} f_2 \, d\mu \\ = \int_{Z^3} \eta(u, v) \tilde{f}_0(z) \tilde{f}_1(z + u) \tilde{f}_2(z + v) \, dz \, d\nu(u, v) \end{aligned} \tag{4.3}$$

in  $L^2(\mu)$ , where  $x \mapsto z$  is the factor map,  $\tilde{f} = \mathbb{E}[f \mid Z]$ , and  $\nu$  is the Haar measure on the subgroup  $\overline{\{(\alpha_m, \alpha_{sn}) : n \in \mathcal{O}_K\}} \subseteq Z^2$ .

- (2) *Let  $a_1, a_2 \in \mathbb{Z} \setminus \{0\}$  be coprime, and put  $a_0 = 0, a_3 = a_1 + a_2$ . Let  $H, \psi$ , and  $b_i$  be as in Theorem 4.2(2). Let  $p(x) \in K[x]$  be an  $\mathcal{O}_K$ -valued polynomial, and let  $f_0, f_1, f_2, f_3 \in L^\infty(\mu)$ . Then for any continuous function  $\eta : Z \rightarrow \mathbb{C}$ ,*

$$\begin{aligned} UC\text{-} \lim_{n \in \mathcal{O}_K} \eta(\alpha_{p(n)}) \int_X \prod_{i=0}^3 T^{a_i p(n)} f_i \, d\mu \\ = \int_{Z^2 \times H^3} \eta(t) \prod_{i=0}^3 \tilde{f}_i(z + a_i t, h + a_i u + a_i^2 v + b_i \psi(t, z)) \, dz \, dt \, dh \, du \, dv \end{aligned} \tag{4.4}$$

in  $L^2(\mu)$ , where  $\tilde{f} = \mathbb{E}[f \mid Z_2]$ .

*Proof.*

- (1) Since  $Z^2$  is a compact abelian group, we may assume by the Stone–Weierstrass theorem that  $\eta(u, v) = \lambda_1(u)\lambda_2(v)$  for  $u, v \in Z$ , where  $\lambda_1, \lambda_2 \in \widehat{Z}$ . Defining

$$g_0(x) = \overline{\lambda_1(z)\lambda_2(z)} f_0(x)$$

and

$$g_i(x) = \lambda_i(z) f_i(x)$$

for  $i = 1, 2$ , the formula (4.3) then follows by applying (4.1) to the functions  $g_1, g_2$  and integrating against  $g_0$ .

- (2) Again, without loss of generality, we may assume  $\eta = \lambda \in \widehat{Z}$ . Since  $\gcd(a_1, a_2) = 1$ , there are integers  $c_1, c_2 \in \mathbb{Z}$  so that  $c_1 a_1 + c_2 a_2 = 1$ . Let  $c_3 = 0$  and  $c_0 = -(c_1 + c_2)$  so that

$$\sum_{i=0}^3 c_i = 0 \quad \text{and} \quad \sum_{i=0}^3 c_i a_i = 1.$$

Then define  $g_i(x) := \lambda(c_i z) f_i(x)$  for  $i = 0, 1, 2, 3$ . Applying the formula (4.2) for the functions  $g_1, g_2, g_3$  and integrating against  $g_0$  produces the desired formula (4.1).



PROPOSITION 4.4. Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $(X, \mathcal{B}, \mu, T)$  be a totally ergodic  $\mathcal{O}_K$ -system,  $r, s \in \mathcal{O}_K$  distinct and nonzero, and  $p(x) \in K[x]$  an  $\mathcal{O}_K$ -valued polynomial. Then for any  $A \in \mathcal{B}$  with  $\mu(A) > 0$  and any  $\varepsilon > 0$ , the set

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \right) > \mu(A)^3 - \varepsilon \right\}$$

is syndetic.

Moreover, if  $s/r \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \cap T^{-(r+s)p(n)}A \right) > \mu(A)^4 - \varepsilon \right\}$$

is syndetic.

Remark 4.5. We do not assume that the polynomial  $p$  is intersective in Proposition 4.4. This is because, in the totally ergodic setting, there are no “local obstructions” that need to be avoided. In order to extend to the ergodic setting, however, we will have to restrict to intersective polynomials.

Proof of Proposition 4.4. We adapt the method from [19].

First we prove the double recurrence result. Using the formula (4.3) with  $f_i = \mathbb{1}_A$  and choosing  $\eta$  supported on a small neighborhood of 0, it suffices to show

$$\int_Z (\mathbb{E} [\mathbb{1}_A | \mathcal{Z}])^3 dz \geq \mu(A)^3.$$

But this follows immediately from Jensen’s inequality, so

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \right) > \mu(A)^3 - \varepsilon \right\}$$

is syndetic.

Now we move to triple recurrence. Since  $\frac{s}{r} \in \mathbb{Q}$ , we can write  $r = a_1k$  and  $s = a_2k$  for some coprime  $a_1, a_2 \in \mathbb{Z}$  and some  $k \in K$ . Let  $q(n) = kp(n)$ . Note that  $a_1q(n) = rp(n)$  and  $a_2q(n) = sp(n)$  are  $\mathcal{O}_K$ -valued. Therefore,  $q$  is itself  $\mathcal{O}_K$ -valued, since  $\gcd(a_1, a_2) = 1$ . Hence, without loss of generality, we will assume that  $r$  and  $s$  are coprime integers.

Now put  $a_0 = 0, a_1 = r, a_2 = s$ , and  $a_3 = r + s$ . Applying formula (4.4) with  $f_i = \mathbb{1}_A$  and choosing the function  $\eta$  to be supported on a small neighbourhood of 0, we want to show

$$\int_{Z \times H^3} \prod_{i=0}^3 \mathbb{E} [\mathbb{1}_A | \mathcal{Z}_2] (z, h + a_iu + a_i^2v) dh du dv dz \geq \mu(A)^4. \tag{4.5}$$

Fix  $z \in Z$ , and let  $F_z : H \rightarrow [0, 1]$  be the function  $F_z(x) = \mathbb{E} [\mathbb{1}_A | \mathcal{Z}_2] (z, x)$ . Now we perform several changes of variables. First, take  $h = a_3x$ :

$$\begin{aligned} & \int_{H^3} \prod_{i=0}^3 F_z(h + a_iu + a_i^2v) dh du dv \\ &= \int_{H^3} F_z(a_3x)F_z(a_3(x + u + a_3v)) F_z(a_3x + a_1u + a_1^2v)F_z(a_3x + a_2u + a_2^2v) du dx dv. \end{aligned}$$

Next,  $x + u + a_3v = y$ :

$$= \int_{H^3} F_z(a_3x)F_z(a_3y)F_z(a_2x + a_1y - a_1a_2v)F_z(a_1x + a_2y - a_1a_2v) dv dx dy.$$

Now,  $a_1(x + y) - a_1a_2v = w$ :

$$\begin{aligned} &= \int_{H^3} F_z(a_3x)F_z(a_3y)F_z((a_2 - a_1)x + w)F_z((a_2 - a_1)y + w) dx dy dw \\ &= \int_H \left( \int_H F_z(a_3x)F_z((a_2 - a_1)x + w) dx \right)^2 dw \end{aligned}$$

Apply Jensen’s inequality:

$$\geq \left( \int_{H^2} F_z(a_3x)F_z((a_2 - a_1)x + w) dw dx \right)^2$$

Finally, let  $w + (a_2 - a_1)x = u$  and  $a_3x = t$ :

$$\begin{aligned} &= \left( \int_H F_z(t) dt \right)^2 \left( \int_H F_z(u) du \right)^2 \\ &= \left( \int_H F_z dm_H \right)^4. \end{aligned}$$

Thus, applying Jensen’s inequality one more time, we have

$$\begin{aligned} &\int_{Z \times H^3} \prod_{i=0}^3 \mathbb{E} [\mathbb{1}_A \mid \mathcal{Z}_2] (z, h + a_iu + a_i^2v) dh du dv dz \\ &= \int_Z \left( \int_{H^3} \prod_{i=0}^3 F_z(h + a_iu + a_i^2v) dh du dv \right) dz \\ &\geq \int_Z \left( \int_H F_z dm_H \right)^4 dz \\ &\geq \left( \int_Z \int_H F_z dm_H dz \right)^4 \\ &= \mu(A)^4. \end{aligned}$$

That is, the inequality (4.5) holds, so the set

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)}A \cap T^{-sp(n)}A \cap T^{-(r+s)p(n)}A \right) > \mu(A)^4 - \varepsilon \right\}$$

is syndetic.

We have proved Theorem B in the case when  $T$  is totally ergodic. We will now extend this to the general case that  $T$  is simply ergodic. Theorem D still applies, so by a standard approximation argument, we may assume without loss of generality that  $T$  acts by niltranslations. The Kronecker factor is then a group of the form  $\mathbb{Z}_{a_1} \times \dots \times \mathbb{Z}_{a_d} \times \mathbb{T}^c$ . As in the proof of Proposition 2.13, we can therefore find  $k \in \mathcal{O}_K$  such that the Kronecker factor of  $(T^{kn})_{n \in \mathcal{O}_K}$  is connected, and hence each of the finitely many ergodic components of  $(T^{kn})_{n \in \mathcal{O}_K}$  is totally

ergodic by Proposition 2.4. Let  $X_1, \dots, X_m$  the atoms of the  $(T^{kn})_{n \in \mathcal{O}_K}$ -invariant  $\sigma$ -algebra, and let  $\mu_j(B) := m \cdot \mu(B \cap X_j)$  so that  $\mu$  has ergodic decomposition  $\mu = (1/m) \sum_{j=1}^m \mu_j$  for the action  $(T^{kn})_{n \in \mathcal{O}_K}$ .

By Lemma 2.11, let  $\xi \in \mathcal{O}_K$  and  $D \in \mathcal{O}_K \setminus \{0\}$  so that  $p(\xi + D\mathcal{O}_K) \subseteq k\mathcal{O}_K$ . Let  $q(x) \in K[x]$  be the  $\mathcal{O}_K$ -valued polynomial  $q(n) = k^{-1}p(\xi + Dn)$  for every  $n \in \mathcal{O}_K$ . Following the argument in the proof of Proposition 4.4, we can choose a continuous function  $\eta$  concentrated on a sufficiently small neighborhood of 0 in  $Z^2$  with  $\int_{Z^2} \eta \, d\nu = 1$  so that

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \eta(\alpha_{rq(n)}, \alpha_{sq(n)}) \mu_j \left( A \cap T^{-krq(n)} A \cap T^{-ksq(n)} A \right) \geq \mu_j(A)^3$$

for  $j = 1, \dots, m$ . Summing over  $j = 1, \dots, m$  and applying Jensen's inequality, we get

$$\text{UC-} \lim_{n \in \mathcal{O}_K} \eta(\alpha_{rq(n)}, \alpha_{sq(n)}) \mu \left( A \cap T^{-krq(n)} A \cap T^{-ksq(n)} A \right) \geq \mu(A)^3$$

from which it follows that

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-krq(n)} A \cap T^{-ksq(n)} A \right) > \mu(A)^3 - \varepsilon \right\}$$

is syndetic in  $\mathcal{O}_K$ .

A similar argument with the ergodic decomposition can be used to show that, if  $\frac{s}{r} \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-krq(n)} A \cap T^{-ksq(n)} A \cap T^{-k(r+s)q(n)} A \right) > \mu(A)^4 - \varepsilon \right\}$$

is also syndetic.

Thus, the sets

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)} A \cap T^{-sp(n)} A \right) > \mu(A)^3 - \varepsilon \right\}$$

and (if  $s/r \in \mathbb{Q}$ )

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)} A \cap T^{-sp(n)} A \cap T^{-(r+s)p(n)} A \right) > \mu(A)^4 - \varepsilon \right\}$$

are relatively syndetic in  $\xi + D\mathcal{O}_K$ . But  $\xi + D\mathcal{O}_K$  is syndetic in  $\mathcal{O}_K$ , so we are done.

### 5. Refinements

#### 5.1. Polynomial IP sets

Recall that a set  $E \subseteq \mathcal{O}_K$  is  $\text{IP}^*$  if it intersects every finite sum set

$$FS((x_n)_{n \in \mathbb{N}}) := \left\{ \sum_{n \in F} x_n : F \subseteq \mathbb{N} \text{ is finite and nonempty} \right\},$$

where  $(x_n)_{n \in \mathbb{N}}$  is a sequence of distinct elements of  $\mathcal{O}_K$ . Similarly, we say  $E$  is  $\text{IP}_r^*$  if it intersects every finite sum set of the form

$$FS(x_1, \dots, x_r) := \left\{ \sum_{k=1}^s x_{n_k} : 1 \leq s \leq r, n_1 < n_2 < \dots < n_s \right\},$$

where  $x_1, \dots, x_r \in \mathcal{O}_K$  are distinct and nonzero. Finally,  $E$  is called an  $\text{IP}_0^*$  set if  $E$  is  $\text{IP}_r^*$  for some  $r \in \mathbb{N}$ . Clearly, every  $\text{IP}_0^*$  set is also  $\text{IP}^*$ , but the converse is not true.

Now we will define polynomial generalisations of IP and IP<sub>0</sub> sets. For a set  $S$ , let  $\mathcal{F}(S)$  denote the semigroup of finite subsets of  $S$  with the union operation.

*Definition 5.1.* Let  $(H, +)$  be an abelian group, and let  $\varphi : \mathcal{F}(S) \rightarrow H$ .

- (1) We say that  $\varphi$  is *linear* if  $\varphi(\alpha \cup \beta) = \varphi(\alpha) + \varphi(\beta)$  whenever  $\alpha \cap \beta = \emptyset$ .
- (2) For  $\beta \in \mathcal{F}(S)$ , the  $\beta$ -*derivate* of  $\varphi$  is the function  $D_\beta \varphi : \mathcal{F}(S \setminus \beta) \rightarrow H$  given by  $D_\beta \varphi(\alpha) = \varphi(\alpha \cup \beta) - \varphi(\alpha)$ .
- (3) We say  $\varphi$  is a *polynomial of degree  $\leq d$*  if for any disjoint sets  $\beta_0, \dots, \beta_d \in \mathcal{F}(S)$ , one has  $D_{\beta_0} D_{\beta_1} \dots D_{\beta_d} \varphi = 0$ .

Note that an IP set has the form  $\{\varphi(\alpha) : \alpha \in \mathcal{F}(\mathbb{N}), \alpha \neq \emptyset\}$  for a linear mapping  $\varphi : \mathcal{F}(S) \rightarrow \mathcal{O}_K$  with  $\varphi(\emptyset) = 0$ . For a polynomial mapping  $\varphi : \mathcal{F}(S) \rightarrow \mathcal{O}_K$ , we call the corresponding set  $\{\varphi(\alpha) : \alpha \in \mathcal{F}(\mathbb{N}), \alpha \neq \emptyset\}$  a *VIP set*. Similarly, if  $\varphi : \mathcal{F}(\{1, \dots, r\}) \rightarrow \mathcal{O}_K$  is a polynomial mapping of degree  $\leq d$  with  $\varphi(\emptyset) = 0$ , we say that  $\{\varphi(\alpha) : \alpha \in \mathcal{F}(\{1, \dots, r\}), \alpha \neq \emptyset\}$  is  $VIP_{d,r}$ . A set  $E \subseteq \mathcal{O}_K$  is  $VIP^*$  if it intersects every VIP set, and  $E$  is  $VIP_{d,r}^*$  if it intersects every  $VIP_{d,r}$  set. Finally,  $E$  is  $VIP_0^*$  if for any  $d \in \mathbb{N}$ ,  $E$  is  $VIP_{d,r}^*$  for some  $r \in \mathbb{N}$ .

As we will see below,  $VIP_0^*$  is an appropriate notion of largeness for nilsequences. However, for a multi-correlation sequence, which differs from a nilsequence by a nullsequence (see Theorem 5.4 below), we need the slightly weaker notion of  $AVIP_0^*$ . A set  $E$  is *almost-VIP<sub>0</sub><sup>\*</sup>*, or  $AVIP_0^*$  for short, if there is a  $VIP_0^*$  set  $A$  such that  $d^*(A \setminus E) = 0$ .

For any notion of largeness discussed so far, we use the added decoration of  $+$  in the subscript to indicate a shift. In particular,  $(A)VIP_{0,+}^*$  means a shift of an  $(A)VIP_0^*$  set.

5.2. Recurrence in nilmanifolds

**THEOREM 5.2** ([8, theorem 0.6]). *Let  $(X, T)$  be a  $\mathbb{Z}^d$ -nilsystem. Then, for any  $x_0 \in X$  and any neighbourhood  $U$  of  $x_0$ , the set*

$$R_U(x_0) := \left\{ n \in \mathbb{Z}^d : T^n x_0 \in U \right\}$$

is a  $VIP_0^*$  set.

**COROLLARY 5.3.** *Let  $\varphi : \mathbb{Z}^d \rightarrow \mathbb{R}$  be a nilsequence. For any  $c < \sup \varphi$ , the set*

$$R := \left\{ n \in \mathbb{Z}^d : \varphi(n) > c \right\}$$

is  $VIP_{0,+}^*$ .

*Proof.* Let  $\varepsilon = \sup \varphi - c > 0$ . Then let  $(X, T)$  be a minimal nilsystem,  $x_0 \in X$ , and  $F \in C(X)$  such that  $\sup_{n \in \mathbb{Z}^d} |\varphi(n) - F(T^n x_0)| < \varepsilon/2$ . Note that  $\sup F > \sup \varphi - \varepsilon/2$ .

Let  $U := \{x \in X : F(x) > \sup \varphi - \varepsilon/2\}$ . Then  $U$  is a nonempty open set. Since  $(X, T)$  is minimal, we have  $T^m x_0 \in U$  for some  $m \in \mathbb{Z}^d$ . By Theorem 5.2,

$$S := \left\{ n \in \mathbb{Z}^d : T^n(T^m x_0) \in U \right\}$$

is  $VIP_0^*$ .

Suppose  $n \in S$ . Then

$$\varphi(n + m) > F(T^{n+m}x_0) - \frac{\varepsilon}{2} > \sup \varphi - \varepsilon = c$$

Therefore,  $R \supseteq S + m$  is  $\text{VIP}_{0,+}^*$ .

5.3. Nilsequence-nilsequence decomposition

Let  $r \in \mathbb{N}$ . A basic  $r$ -step nilsequence is a function  $\varphi(n) = F(T^n x_0)$ , where  $(X, \mathcal{B}, \mu, T)$  is an  $r$ -step nilsystem,  $F : X \rightarrow \mathbb{C}$  is a continuous function, and  $x_0 \in X$ . An  $r$ -step nilsequence is a uniform limit of basic  $r$ -step nilsequences. Knowing that a nilfactor is characteristic for polynomial multiple ergodic averages gives a decomposition of the corresponding multi-correlation sequences. Recall that a function  $\psi : \mathcal{O}_K \rightarrow \mathbb{C}$  is a nullsequence if  $\text{UC-}\lim_{n \in \mathcal{O}_K} |\psi(n)|^2 = 0$ .

**THEOREM 5.4.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p_1, \dots, p_k \in K[x]$  be non-constant, essentially distinct,  $\mathcal{O}_K$ -valued polynomials. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$  and any  $f_0, f_1, \dots, f_k \in L^\infty(\mu)$ , there is a decomposition*

$$a(n) := \int_X f_0 \cdot T^{p_1(n)} f_1 \cdot \dots \cdot T^{p_k(n)} f_k \, d\mu = \varphi(n) + \psi(n),$$

where  $\varphi$  is a nilsequence and  $\psi$  is a nullsequence.

*Proof.* First, by [14, theorem 5.2], there exists  $r \in \mathbb{N}$  such that

$$a(n) - \int_X \mathbb{E}[f_0 \mid \mathcal{Z}_r] \cdot T^{p_1(n)} \mathbb{E}[f_1 \mid \mathcal{Z}_r] \cdot \dots \cdot T^{p_k(n)} \mathbb{E}[f_k \mid \mathcal{Z}_r] \, d\mu$$

is a nullsequence, so we may assume that  $(X, \mathcal{B}, \mu, T)$  is a nilsystem.

Next, up to a uniform approximation in  $n$ , we may assume that  $f_0, f_1, \dots, f_k$  are continuous functions. Then by [36, theorem 1.3],  $a(n)$  is the sum of a (basic) nilsequence and a nullsequence. Taking a uniform limit gives the desired decomposition.

**PROPOSITION 5.5.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Suppose  $\varphi : \mathcal{O}_K \rightarrow \mathbb{C}$  is a nilsequence,  $\psi : \mathcal{O}_K \rightarrow \mathbb{C}$  is a nullsequence, and  $a(n) = \varphi(n) + \psi(n)$ . Suppose that for some  $c > 0$ , the set*

$$R(c) := \{n \in \mathcal{O}_K : a(n) > c\}$$

is syndetic. Then  $R(c')$  is  $\text{AVIP}_{0,+}^*$  for every  $c' < c$ .

*Proof.* Let  $c' < c$ . Then the set

$$E := \left\{ n \in \mathcal{O}_K : |\psi(n)| \geq \frac{c - c'}{2} \right\}$$

has upper Banach density  $d^*(E) = 0$ . Therefore,  $R(c) \setminus E$  is still syndetic; in particular, it is nonempty. But for  $n \in R(c) \setminus E$ , we have  $\varphi(n) > c - (c - c'/2) = (c + c'/2)$ . So, by Corollary 5.3,

$$S := \left\{ n \in \mathcal{O}_K : \varphi(n) > \frac{c + c'}{2} \right\}$$

is  $\text{VIP}_{0,+}^*$ . Finally, since  $(c + c')/2 - (c - c')/2 = c'$ , we have  $R(c') \supseteq S \setminus E$ , so  $R(c')$  is  $\text{AVIP}_{0,+}^*$ .

By Theorem 5.4, Proposition 5.5 applies to polynomial multi-correlation sequences in ergodic systems. We can therefore strengthen the conclusions of Theorems 4.1 and 4.2, respectively, under the assumption of ergodicity:

**THEOREM 5.6.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $\{p_1, \dots, p_k\} \subseteq K[x]$  be a jointly intersective family of linearly independent  $\mathcal{O}_K$ -valued polynomials. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-p_1(n)} A \cap \dots \cap T^{-p_k(n)} A \right) > \mu(A)^{k+1} - \varepsilon \right\}$$

is  $\text{AVIP}_{0,+}^*$ .

**THEOREM 5.7.** *Let  $K$  be a number field with ring of integers  $\mathcal{O}_K$ . Let  $p(x) \in K[x]$  be an  $\mathcal{O}_K$ -valued intersective polynomial. Let  $r, s \in \mathcal{O}_K$  be distinct and nonzero. Then for any ergodic measure-preserving  $\mathcal{O}_K$ -system  $(X, \mathcal{B}, \mu, T)$ ,  $A \in \mathcal{B}$ , and  $\varepsilon > 0$ , the set*

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)} A \cap T^{-sp(n)} A \right) > \mu(A)^3 - \varepsilon \right\}$$

is  $\text{AVIP}_{0,+}^*$ .

Moreover, if  $\frac{s}{r} \in \mathbb{Q}$ , then

$$\left\{ n \in \mathcal{O}_K : \mu \left( A \cap T^{-rp(n)} A \cap T^{-sp(n)} A \cap T^{-(r+s)p(n)} A \right) > \mu(A)^4 - \varepsilon \right\}$$

is  $\text{AVIP}_{0,+}^*$ .

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