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Short Probabilistic Proof of the Brascamp–Lieb and Barthe Theorems

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Abstract. We give a short proof of the Brascamp–Lieb theorem, which asserts that a certain general form of Young's convolution inequality is saturated by Gaussian functions. The argument is inspired by Borell's stochastic proof of the Prékopa–Leindler inequality and applies also to the reversed Brascamp–Lieb inequality, due to Barthe.

1 Introduction

A Brascamp–Lieb datum on $\mathbb{R}^n$ is a finite sequence

$$(c_1, B_1), \dots, (c_m, B_m), \tag{1.1}$$

where c_i is a positive number and $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$ is linear and onto. The Brascamp–Lieb constant associated with this datum is the smallest real number C such that the inequality

$$\int_{\mathbb{R}^n} \prod_{i=1}^m (f_i \circ B_i)^{c_i} \, \mathrm{d}x \le C \prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} f_i \, \mathrm{d}x \Big)^{c_i} \tag{1.2}$$

holds for every set of non-negative integrable functions $f_i : \mathbb{R}^{n_i} \to \mathbb{R}$. The Brascamp–Lieb theorem [9, 13] asserts that (1.2) is saturated by Gaussian functions. In other words if (1.2) holds for every set of functions $f_1, \dots, f_m$ of the form

$$f_i(x) = \mathrm{e}^{-\langle A_i x, x\rangle/2},$$

where A_i is a symmetric positive definite matrix on $\mathbb{R}^{n_i}$, then (1.2) holds for every set of functions $f_1, \dots, f_m$.

The reversed Brascamp–Lieb constant associated with (1.1) is the smallest constant C_r such that for every set of nonnegative measurable functions $f_1, \dots, f_m, f$ satisfying

$$\prod_{i=1}^m f_i(x_i)^{c_i} \le f\Big(\sum_{i=1}^m c_i B_i^* x_i \Big) \tag{1.3}$$

for every $(x_1, \dots, x_m) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}$ we have

$$\prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} f_i \, \mathrm{d}x \Big)^{c_i} \le C_r \int_{\mathbb{R}^n} f \, \mathrm{d}x. \tag{1.4}$$

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It was shown by Barthe [1] that, again, Gaussian functions saturate the inequality. The original paper of Brascamp and Lieb [9] relies on symmetrization techniques. Barthe's argument uses optimal transport and works for both the direct and the reversed inequality. More recent proofs of the direct inequality [5, 6, 10, 11] all rely on semi-group techniques. There are also semi-group proofs of the reversed inequality, at least when the Brascamp–Lieb datum has the property

$$\begin{aligned} B_i B_i^* &= \mathrm{id}_{\mathbb{R}^{n_i}}, \quad \forall i \le m, \\ \sum_{i=1}^m c_i B_i^* B_i &= \mathrm{id}_{\mathbb{R}^n}, \end{aligned} \tag{1.5}$$

called *frame condition* in the sequel. This was achieved by Barthe and Cordero-Erausquin [2] in the rank 1 case (when all dimensions n_i equal 1) and Barthe and Huet [3] in any dimension.

The purpose of the present article is to give a short probabilistic proof of the Brascamp–Lieb and Barthe theorems. Our main tool will be a representation formula for the quantity

$$\ln\Big(\int e^{g(x)}\, \gamma(dx) \Big),$$

where γ is a Gaussian measure. Let us describe it briefly. Let $(\Omega, \mathcal{A}, \mathsf{P})$ be a probability space, let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration, and let $(W_t)_{t\in[0,T]}$ be a Brownian motion taking values in $\mathbb{R}^n$ (we fix a finite time horizon T). Assuming that the covariance matrix A of the random vector W_1 has full rank, we let $\mathbb{H}$ be the Cameron–Martin space associated with W, namely the Hilbert space of absolutely continuous paths $u\colon [0,T] \to \mathbb{R}^n$ starting from 0, equipped with the norm

$$\|u\|_{\mathbb{H}} = \Big(\int_0^T \langle A^{-1} \dot{u}_s, \dot{u}_s \rangle \, ds \Big)^{1/2}.$$

In the sequel we call a *drift* any adapted process U which belongs to $\mathbb{H}$ almost surely. The following formula is due to Boué and Dupuis [8] (see also [7, 12]).

Proposition 1.1 *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be measurable and bounded from below. Then*

$$\log\big(\mathsf{E} e^{g(W_T)} \big) = \sup\Big\{ \mathsf{E}\Big(g(W_T + U_T) - \frac{1}{2} \|U\|_{\mathbb{H}}^2 \Big) \Big\},$$

where the supremum is taken over all drifts U.

In [7], Borell rediscovered this formula and showed that it yields the Prékopa–Leindler inequality (a reversed form of Hölder's inequality) very easily. Later, Cordero and Maurey noticed that under the frame condition, both the direct and reversed Brascamp–Lieb inequalities could be recovered this way (this was not published but is explained in [12]). The purpose of this article, following Borell, Cordero, and Maurey, is to show that the Brascamp–Lieb and Barthe theorems in full generality are direct consequences of Proposition 1.1.

2 The Direct Inequality

Replace f_i by $x \mapsto f_i(x/\lambda)$ in inequality (1.2). The left-hand side of the inequality is multiplied by λ^n and the right-hand side by $\lambda^{\sum_{i=1}^m c_i n_i}$. Therefore, a necessary condition for C to be finite is

$$\sum_{i=1}^m c_i n_i = n.$$

This homogeneity condition will be assumed throughout the rest of the article.

Theorem 2.1 *Assume that there exists a positive definite matrix A satisfying*

$$A^{-1} = \sum_{i=1}^m c_i B_i^* (B_i A B_i^*)^{-1} B_i. \tag{2.1}$$

Then the Brascamp–Lieb constant is

$$C = \Big(\frac{\det(A)}{\prod_{i=1}^m \det(B_i A B_i^*)^{c_i}} \Big)^{1/2},$$

and there is equality in (1.2) for the Gaussian functions

$$f_i \colon x \in \mathbb{R}^{n_i} \mapsto \mathrm{e}^{-\langle (B_i A B_i^*)^{-1} x, x \rangle /2}, \quad i \le m. \tag{2.2}$$

Remark If the frame condition (1.5) holds, then $A = \mathrm{id}_{\mathbb{R}^n}$ satisfies (2.1) and the Brascamp–Lieb constant is 1.

Proof Because of (2.1), if the functions f_i are defined by (2.2), then

$$\prod_{i=1}^m \big(f_i(B_i x) \big)^{c_i} = \mathrm{e}^{-\langle A^{-1} x, x \rangle /2}.$$

The equality case follows easily (recall the homogeneity condition $\sum c_i n_i = n$).

Let us prove the inequality. Let $f_1, \dots, f_m$ be nonnegative integrable functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively and let

$$f \colon x \in \mathbb{R}^n \longmapsto \prod_{i=1}^m f_i(B_i x)^{c_i}.$$

Fix $\delta > 0$, let $g_i = \log(f_i + \delta)$ for every $i \le m$, and let

$$g(x) = \sum_{i=1}^m c_i g_i(B_i x).$$

The functions $(g_i)_{i \le m}, g$ are bounded from below. Fix a time horizon T, let $(W_t)_{t \ge T}$ be a Brownian motion on $\mathbb{R}^n$, starting from 0 and having covariance A, and let $\mathbb{H}$ be

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the associated Cameron–Martin space. By Proposition 1.1, given $\epsilon > 0$, there exists a drift U such that

$$\log\big(\mathsf{E}\mathrm{e}^{g(W_T)}\big) \leq \mathsf{E}\Big(g(W_T+U_T) - \frac{1}{2}\|U\|_{\mathbb{H}}^2\Big) + \epsilon = \sum_{i=1}^m c_i \mathsf{E} g_i(B_i W_T + B_i U_T) - \frac{1}{2}\mathsf{E}\|U\|_{\mathbb{H}}^2 + \epsilon. \tag{2.3}$$

The process $B_i W$ is a Brownian motion on $\mathbb{R}^{n_i}$ with covariance $B_i A B_i^*$. Set $A_i = B_i A B_i^*$ and let $\mathbb{H}_i$ be the Cameron–Martin space associated with $B_i W$. Equality (2.1) gives

$$\langle A^{-1}x, x\rangle = \sum_{i=1}^m c_i \langle A_i^{-1} B_i x, B_i x\rangle$$

for every $x \in \mathbb{R}^n$. This implies that

$$\|u\|_{\mathbb{H}}^2 = \sum_{i=1}^m c_i \|B_i u\|_{\mathbb{H}_i}^2$$

for every absolutely continuous path $u\colon [0,T] \to \mathbb{R}^n$, so that (2.3) becomes

$$\log\big(\mathsf{E}\mathrm{e}^{g(W_T)}\big) \leq \sum_{i=1}^m c_i \mathsf{E}\Big(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\Big) + \epsilon.$$

By Proposition 1.1 we have

$$\mathsf{E}\Big(g_i(B_i W_T + B_i U_T) - \frac{1}{2}\|B_i U\|_{\mathbb{H}_i}^2\Big) \leq \log\big(\mathsf{E}\mathrm{e}^{g_i(B_i W_T)}\big)$$

for every $i \leq m$. We obtain (dropping ϵ, which is arbitrary)

$$\log\big(\mathsf{E}\mathrm{e}^{g(W_T)}\big) \leq \sum_{i=1}^m c_i \log\big(\mathsf{E}\mathrm{e}^{g_i(B_i W_T)}\big). \tag{2.4}$$

Recall that $f \leq \mathrm{e}^g$ and observe that

$$\prod_{i=1}^m \big(\mathsf{E}(\mathrm{e}^{g_i}(B_i W_T))\big)^{c_i} \leq \prod_{i=1}^m \big(\mathsf{E} f_i(B_i W_T)\big)^{c_i} + O(\delta^c),$$

for some positive constant c. Inequality (2.4) becomes (dropping the $O(\delta^c)$ term)

$$\mathsf{E} f(W_T) \leq \prod_{i=1}^m \big(\mathsf{E} f_i(B_i W_T)\big)^{c_i}. \tag{2.5}$$

Since W_T is a centered Gaussian vector with covariance TA,

$$\mathsf{E} f(W_T) = \frac{1}{(2\pi T)^{n/2}\det(A)^{1/2}} \int_{\mathbb{R}^n} f(x)\mathrm{e}^{-\langle A^{-1}x,x\rangle/2T}\,\mathrm{d}x,$$

and there is a similar equality for $\mathsf{E} f_i(B_i W_T)$. Then it is easy to see that letting T tend to $+\infty$ in inequality (2.5) yields the result (recall that $\sum c_i n_i = n$). ∎

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Example (Optimal constant in Young's inequality) Young's convolution inequality asserts that if $p, q, r \geq 1$ and are linked by the equation

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{2.6}$$

then

$$\|F * G\|_r \leq \|F\|_p \|G\|_q,$$

for all $F \in L_p$ and $G \in L_q$. When either p, q, or r equals 1 or $+\infty$, the inequality is a consequence of Hölder's inequality and is easily seen to be sharp. On the other hand when p, q, r belong to the open interval $(1, +\infty)$, the best constant C in the inequality

$$\|F * G\|_r \leq C\|F\|_p \|G\|_q,$$

is actually smaller than 1. Let us compute it using the previous theorem. Observe that by duality C is the best constant in the inequality

$$\int_{\mathbb{R}^2} f^{c_1}(x+y) g^{c_2}(y) h^{c_3}(x)\, \mathrm{d}x\mathrm{d}y \leq C\Big(\int_{\mathbb{R}} f\Big)^{c_1}\Big(\int_{\mathbb{R}} g\Big)^{c_2}\Big(\int_{\mathbb{R}} h\Big)^{c_3}, \tag{2.7}$$

where

$$c_1 = \frac{1}{p}, \quad c_2 = \frac{1}{q}, \quad c_3 = 1 - \frac{1}{r}.$$

In other words, C is the Brascamp–Lieb constant in $\mathbb{R}^2$ associated with the data

$$(c_1, B_1), \quad (c_2, B_2), \quad (c_3, B_3),$$

where $B_1 = (1,1)$, $B_2 = (0,1)$ and $B_3 = (1,0)$. According to the previous result, we have to find a positive definite matrix A satisfying

$$A^{-1} = \sum_{i=1}^{3} c_i B_i^* (B_i A B_i^*)^{-1} B_i.$$

Letting $A = \left(\begin{smallmatrix} x & z \\ z & y \end{smallmatrix}\right)$, this equation turns out to be equivalent to

$$\begin{aligned} (1-c_2)xy + yz + c_2 z^2 &= 0, \\ (1-c_3)xy + xz + c_3 z^2 &= 0, \\ c_1 + c_2 + c_3 &= 2. \end{aligned}$$

The third equation is just the Young constraint (2.6). The first two equations admit two families of solutions: either (x, y, z) is a multiple of $(1, 1, -1)$ or (x, y, z) is a multiple of

$$\big(c_3(1-c_3), c_2(1-c_2), -(1-c_2)(1-c_3)\big).$$

The constraint $xy - z^2 > 0$ rules out the first solution. The second solution is fine, since c_1, c_2 and c_3 are assumed to belong to the open interval $(0,1)$. By Theorem 2.1, the best constant in (2.7) is

$$C = \Big(\frac{\det(A)}{\prod_{i=1}^{3} \det(B_i A B_i^*)^{c_i}} \Big)^{1/2} = \Big(\frac{(1-c_1)^{1-c_1}(1-c_2)^{1-c_2}(1-c_3)^{1-c_3}}{c_1^{c_1} c_2^{c_2} c_3^{c_3}} \Big)^{1/2}.$$

In terms of p, q, r, we have

$$C = \Big(\frac{p^{1/p}\, q^{1/q}\, {r'}^{1/r'}}{{p'}^{1/p'}\, {q'}^{1/q'}\, r^{1/r}} \Big)^{1/2},$$

where p', q', r' are the conjugate exponents of p, q, r, respectively. This is indeed the best constant in Young's inequality first obtained by Beckner [4].

3 The Reversed Inequality

Theorem 3.1 *Again assume that there is a matrix A satisfying* (2.1). *Then the reversed Brascamp–Lieb constant is*

$$C_r = \Big(\frac{\det(A)}{\prod_{i=1}^{m} \det(B_i A B_i^*)^{c_i}} \Big)^{1/2}.$$

There is equality in (1.4) *for the following Gaussian functions*

$$f_i\colon x \in \mathbb{R}^{n_i} \longmapsto \mathrm{e}^{-\langle B_i A B_i^* x, x\rangle/2}, \quad i \le m.$$
$$f\colon x \in \mathbb{R}^{n} \longmapsto \mathrm{e}^{-\langle A x, x\rangle/2}.$$

Remark Observe that under condition (2.1), the Brascamp–Lieb constant and the reversed constant are the same, but the extremizers differ.

We shall use the following elementary lemma.

Lemma 3.2 *Let $A_1, \dots, A_m$ be positive definite matrices on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$, respectively and let*

$$A = \Big(\sum_{i=1}^{m} c_i B_i^* A_i^{-1} B_i \Big)^{-1}.$$

Then for all $x \in \mathbb{R}^n$

$$\langle Ax, x\rangle = \inf\Big\{ \sum_{i=1}^{m} c_i \langle A_i x_i, x_i\rangle,\ \sum_{i=1}^{m} c_i B_i^* x_i = x \Big\}.$$

Proof Let $x_1, \dots, x_m$ and let

$$x = \sum_{i=1}^{m} c_i B_i^* x_i. \tag{3.1}$$

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Then by the Cauchy–Schwarz inequality (recall that the matrices A_i are positive definite),

$$\begin{aligned}\langle Ax, x\rangle &= \sum_{i=1}^m c_i \langle Ax, B_i^* x_i\rangle = \sum_{i=1}^m c_i \langle B_i Ax, x_i\rangle \\ &\le \Big(\sum_{i=1}^m c_i \langle A_i^{-1} B_i Ax, B_i Ax\rangle\Big)^{1/2} \Big(\sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle\Big)^{1/2} \\ &= \langle Ax, x\rangle^{1/2} \Big(\sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle\Big)^{1/2}.\end{aligned}$$

Besides, given $x \in \mathbb{R}^n$, set $x_i = A_i^{-1} B_i Ax$ for all $i \le m$. Then (3.1) holds and there is equality in the above Cauchy–Schwarz inequality. This concludes the proof. ∎

Proof of Theorem 3.1 The equality case is a straightforward consequence of the hypothesis (2.1) and Lemma 3.2; details are left to the reader.

Let us prove the inequality. There is no loss of generality assuming that the functions $f_1, \dots, f_m$ are bounded from above (otherwise replace f_i by $\max(f_i, k)$, let k tend to $+\infty$, and use monotone convergence). Fix $\delta > 0$ and let $g_i = \log(f_i + \delta)$ for every $i \le m$. By (1.3) and since the functions f_i are bounded from above, there exist positive constants c, C such that the function

$$g\colon x \in \mathbb{R}^n \longmapsto \log\big(f(x) + C\delta^c\big),$$

satisfies

$$\sum_{i=1}^m c_i g_i(x_i) \le g\Big(\sum_{i=1}^m c_i B_i^* x_i\Big) \tag{3.2}$$

for every $x_1, \dots, x_m$. Observe that the functions $(g_i)_{i\le m}, g$ are bounded from below. Let $(W_t)_{t\le T}$ be a Brownian motion on $\mathbb{R}^n$ having covariance matrix A. Set $A_i = B_i A B_i^*$, then $A_i^{-1} B_i W$ is a Brownian motion on $\mathbb{R}^{n_i}$ with covariance matrix

$$(A_i^{-1}B_i) A (A_i^{-1} B_i)^* = A_i^{-1}(B_i A B_i^*) A_i^{-1} = A_i^{-1}.$$

Let $\mathbb{H}_i$ be the associated Cameron–Martin space. By Proposition 1.1 there exists a ($\mathbb{R}^{n_i}$-valued) drift U_i such that

$$\log\big(\mathsf{E} e^{g_i(A_i^{-1} B_i W_T)}\big) \le \mathsf{E}\Big(g_i(A_i^{-1} B_i W_T + (U_i)_T) - \frac{1}{2}\|U_i\|_{\mathbb{H}_i}^2\Big) + \epsilon. \tag{3.3}$$

By (3.2) and (2.1)

$$\begin{aligned}\sum_{i=1}^m c_i g_i(A_i^{-1} B_i W_T + (U_i)_T) &\le g\Big(\sum_{i=1}^m c_i B_i^* (A_i^{-1} B_i W_T + (U_i)_T)\Big) \\ &= g\Big(A^{-1} W_T + \sum_{i=1}^m c_i B_i^* (U_i)_T\Big).\end{aligned}$$

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The Brownian motion $(A^{-1}W)_{t\leq T}$ has covariance matrix $A^{-1}A(A^{-1})^* = A^{-1}$. Let $\mathbb{H}$ be the associated Cameron–Martin space. Lemma 3.2 shows that

$$\Big\langle A\Big(\sum_{i=1}^m c_i B_i^* x_i\Big), \sum_{i=1}^m c_i B_i^* x_i\Big\rangle \leq \sum_{i=1}^m c_i \langle A_i x_i, x_i\rangle$$

for every $x_1,\dots,x_m$ in $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_m}$, respectively. Therefore,

$$\Big\|\sum_{i=1}^m c_i B_i^* u_i\Big\|_{\mathbb{H}}^2 \leq \sum_{i=1}^m c_i \|u_i\|_{\mathbb{H}_i}^2.$$

for every sequence of absolutely continuous paths $(u_i\colon [0,T]\to\mathbb{R}^{n_i})_{i\leq m}$. Thus multiplying (3.3) by c_i and summing over i yields

$$\sum_{i=1}^m c_i \log\big(\mathsf{E}e^{g_i(A_i^{-1}B_iW_T)}\big) \leq \\ \mathsf{E}\Big[g\Big(A^{-1}W_T + \sum_{i=1}^m c_i B_i^*(U_i)_T\Big) - \frac{1}{2}\Big\|\sum_{i=1}^m c_i B_i^* U_i\Big\|_{\mathbb{H}}^2\Big] + \sum_{i=1}^m c_i\epsilon.$$

Hence, using Proposition 1.1 again and dropping ϵ again,

$$\sum_{i=1}^m c_i \log\Big(\mathsf{E}e^{g_i(A_i^{-1}B_iW_T)}\Big)^{c_i} \leq \log\big(\mathsf{E}e^{g(A^{-1}W_T)}\big). \tag{3.4}$$

Recall that $f_i \leq e^{g_i}$ for every $i\leq m$ and that $e^g = f + C\delta^c$. Since δ is arbitrary, inequality (3.4) becomes

$$\prod_{i=1}^m \big(\mathsf{E}f_i(A_i^{-1}B_iW_T)\big)^{c_i} \leq \mathsf{E}f(A^{-1}W_T).$$

Again, letting T tend to $+\infty$ in this inequality yields the result. ∎

4 The Brascamp–Lieb and Barthe Theorems

So far we have seen that both the direct inequality and the reversed version are saturated by Gaussian functions when there exists a matrix A such that

$$A^{-1} = \sum_{i=1}^m c_i B_i^*(B_iAB_i^*)^{-1}B_i. \tag{4.1}$$

In this section, we briefly explain why this yields the Brascamp–Lieb and Barthe theorems.

Applying (1.2) to Gaussian functions gives

$$\prod_{i=1}^m \det(A_i)^{c_i} \leq C^2 \det\Big(\sum_{i=1}^m c_i B_i^* A_i B_i\Big) \tag{4.2}$$

for every sequence $A_1, \dots, A_m$ of positive definite matrices on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}$. Let C_g be the Gaussian Brascamp–Lieb constant, namely the best constant in the previous inequality. We have $C_g \le C$, and it turns out that applying (1.4) to Gaussian functions yields $C_g \le C_r$ (one has to apply Lemma 3.2 at some point).

It is known (see Carlen and Cordero [10]) that there is a dual formulation of (1.2) in terms of relative entropy. In the same way, there is a dual formulation of (4.2). For every positive matrix A on $\mathbb{R}^n$, one has

$$\log\det(A) = \inf_{B>0}\left\{ \operatorname{tr}(AB) - n - \log(\det(B)) \right\},$$

with equality when $B = A^{-1}$. Using this and the equality $\sum_{i=1}^m c_i n_i = n$, it is easily seen that C_g is also the best constant such that the inequality

$$\det(A) \le C_g^2 \prod_{i=1}^m \det(B_i A B_i^*)^{c_i} \tag{4.3}$$

holds for every positive definite matrix A on $\mathbb{R}^n$.

Example Assume that $m = n$, that $c_1 = \dots = c_n = 1$ and that $B_i(x) = x_i$ for $i \in [n]$. Inequality (4.2) trivially holds with constant 1 (and there is equality for every $A_1, \dots, A_n$). On the other hand, (4.3) becomes

$$\det(A) \le \prod_{i=1}^n a_{ii},$$

for every positive definite A, with equality when A is diagonal. This is Hadamard's inequality.

Lemma 4.1 *If A is extremal in* (4.3), *then A satisfies* (4.1).

Proof Compute the gradient of the map

$$A > 0 \mapsto \log\det(A) - \sum_{i=1}^m c_i \log\det(B_i A B_i^*). \qquad \blacksquare$$

Therefore, if the constant C_g is finite and if there is an extremizer A in (4.3), then A satisfies (4.1) and together with the results of the previous sections we get the Brascamp–Lieb and Barthe equalities $C = C_r = C_g$. Although it may happen that $C_g < +\infty$ and no Gaussian extremizer exists, there is a way to bypass this issue. For the Brascamp–Lieb theorem, there is an abstract argument showing that is it is enough to prove the equality $C = C_g$ when there is a Gaussian extremizer. This argument relies on:

(a) a criterion for having a Gaussian extremizer, due to Barthe [1] in the rank 1 case (namely when the dimensions n_i are all equal to 1) and Bennett, Carbery, Christ and Tao [6] in the general case;

(b) a multiplicativity property of C and C_g due to Carlen, Lieb, and Loss [11] in the rank 1 case and obtained in full generality in [6] again.

There is no point repeating this argument here, and we refer to [6, 11] instead. This settles the case of the $C = C_g$ equality. As for the $C = C_r$ equality, we observe that the above argument can be carried out verbatim once the mutliplicativity property of the reversed Brascamp–Lieb constant is established. This is the purpose of the rest of the article.

Definition 4.2 Given a proper subspace E of $\mathbb{R}^n$, for $i \leq m$, we let

$$\begin{aligned} B_{i,E} \colon E &\to B_i E & \qquad B_{i,E^\perp} \colon E^\perp &\to (B_i E)^\perp \\ x &\mapsto B_i x, & x &\mapsto q_i(B_i x), \end{aligned}$$

where q_i is the orthogonal projection onto $(B_i E)^\perp$. Observe that both $B_{i,E}$ and $B_{i,E^\perp}$ are onto. Now we let $C_{r,E}$ be the reversed Brascamp–Lieb constant on E associated with the datum

$$(c_1, B_{1,E}), \dots, (c_m, B_{m,E})$$

and $C_{r,E^\perp}$ be the reversed Brascamp–Lieb constant on $E^\perp$ associated with the datum

$$(c_1, B_{1,E^\perp}), \dots, (c_m, B_{m,E^\perp}).$$

Proposition 4.3 *Let E be a proper subspace of $\mathbb{R}^n$, and assume that E is critical, in the sense that*

$$\dim(E) = \sum_{i=1}^m c_i \dim(B_i E).$$

Then $C_r = C_{r,E} \times C_{r,E^\perp}$.

Bennett, Carbery, Christ and Tao proved the corresponding property for C and C_g, we adapt their argument to prove the multiplicativity of C_r.

Let us prove the inequality $C_r \leq C_{r,E} \times C_{r,E^\perp}$ first. This does not require E to be critical. Let $f_1, \dots, f_m, f$ be functions on $\mathbb{R}^{n_1}, \dots, \mathbb{R}^{n_m}, \mathbb{R}$ respectively, satisfying

$$\prod_{i=1}^m f_i(z_i)^{c_i} \leq f\Big(\sum_{i=1}^m c_i B_i^* z_i \Big)$$

for all $z_1, \dots, z_m$. Fix $(x_1, \dots, x_n) \in B_1 E \times \cdots \times B_m E$. Since $(B_{i,E^\perp})^* y_i = B_i^* y_i$ for every $y_i \in (B_i E)^\perp$, applying the reversed Brascamp–Lieb inequality on $E^\perp$ to the functions

$$y \in (B_i E)^\perp \mapsto f_i(x_i + y), \ i \leq m$$

yields

$$\begin{aligned} \prod_{i=1}^m \Big(\int_{(B_i E)^\perp} f_i(x_i + y_i) \, \mathrm{d}y_i \Big)^{c_i} &\leq C_{r,E^\perp} \int_{E^\perp} f\Big(\sum_{i=1}^m c_i B_i^* x_i + y \Big) \, \mathrm{d}y \\ &= C_{r,E^\perp} \int_{E^\perp} f\Big(\sum_{i=1}^m c_i (B_{i,E})^* x_i + y \Big) \, \mathrm{d}y. \end{aligned}$$

For the latter equality, observe that $(B_{i,E})^* x_i = p(B_i^* x_i)$, where p is the orthogonal projection with range E and use the translation invariance of the Lebesgue measure. Applying the reversed Brascamp–Lieb inequality (this time on E) and using Fubini's theorem we get

$$\prod_{i=1}^{m}\Big(\int_{\mathbb{R}^{n_i}} f_i \,\mathrm{d}x\Big)^{c_i} \le C_{r,E}C_{r,E^\perp}\int_{\mathbb{R}^n} f\,\mathrm{d}x,$$

which is the result.
We start the proof of the inequality $C_{r,E}C_{rE^\perp} \le C_r$ with a couple of simple observations.

Lemma 4.4 *Upper semi-continuous functions having compact support saturate the reversed Brascamp–Lieb inequality.*

Proof The regularity of the Lebesgue measure implies that given a nonnegative integrable function f_i on $\mathbb{R}^{n_i}$ and $\epsilon > 0$ there exists a nonnegative linear combination of indicators of compact sets g_i satisfying

$$g_i \le f_i \quad \text{and} \quad \int_{\mathbb{R}^{n_i}} f_i\,\mathrm{d}x \le (1+\epsilon)\int_{\mathbb{R}^{n_i}} g_i\,\mathrm{d}x.$$

The lemma follows easily. ∎

The proof of the following lemma is left to the reader.

Lemma 4.5 *If $f_1,\dots,f_m$ are compactly supported and upper semi-continuous on $\mathbb{R}^{n_1},\dots,\mathbb{R}^{n_m}$ respectively, then the function f defined on $\mathbb{R}^n$ by*

$$f(x) = \sup\Big\{\prod_{i=1}^{m} f_i(x_i)^{c_i},\ \sum_{i=1}^{m} c_i B_i^* x_i = x\Big\}$$

is compactly supported and upper semi-continuous as well.

Remark If the Brascamp–Lieb datum happens to be *degenerate*, in the sense that the map $(x_1,\dots,x_m) \mapsto \sum_{i=1}^{m} B_i^* x_i$ is not onto, then the Brascamp–Lieb constants are easily seen to be $+\infty$. Still the previous lemma remains valid, provided the convention $\sup\varnothing = 0$ is adopted.

Let us prove that $C_{r,E} \times C_{r,E^\perp} \le C_r$. By Lemma 4.4, it is enough to prove that the inequality

$$\prod_{i=1}^{m}\Big(\int_{B_iE} f_i\,\mathrm{d}x\Big)^{c_i} \times \prod_{i=1}^{m}\Big(\int_{(B_iE)^\perp} g_i\,\mathrm{d}x\Big)^{c_i} \le C_r\Big(\int_E f\,\mathrm{d}x\Big)\Big(\int_{E^\perp} g\,\mathrm{d}x\Big)$$

holds for every set of compactly supported upper semi-continuous functions $(f_i)_{i\le m}$ and $(g_i)_{i\le m}$, where f and g are defined by

$$f\colon x\in E \mapsto \sup\Big\{\prod_{i=1}^{m} f_i(x_i)^{c_i},\ \sum_{i=1}^{m} c_i (B_{i,E})^* x_i = x\Big\},$$

$$g\colon y\in E^\perp \mapsto \sup\Big\{\prod_{i=1}^{m} g_i(y_i)^{c_i},\ \sum_{i=1}^{m} c_i (B_{i,E^\perp})^* y_i = y\Big\}.$$

Let $\epsilon > 0$. For $i \le m$, define a function h_i on $\mathbb{R}^{n_i}$ by

$$h_i(x+y) = f_i(x/\epsilon) g_i(y), \quad x \in B_i E,\ y \in (B_i E)^\perp,$$

and let

$$h\colon z \in \mathbb{R}^n \mapsto \sup\Big\{ \prod_{i=1}^m h_i(z_i)^{c_i},\ \sum_{i=1}^m c_i B_i^* z_i = z \Big\}.$$

By definition of the reversed Brascamp–Lieb constant C_r,

$$\prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} h_i \,\mathrm{d}x \Big)^{c_i} \le C_r \int_{\mathbb{R}^n} h \,\mathrm{d}x. \tag{4.4}$$

Using the equality $\sum_{i=1}^m c_i \dim(B_i E) = \dim(E)$, we get

$$\epsilon^{-\dim(E)} \prod_{i=1}^m \Big(\int_{\mathbb{R}^{n_i}} h_i \,\mathrm{d}x \Big)^{c_i} = \prod_{i=1}^m \Big(\int_{B_i E} f_i \,\mathrm{d}x \Big)^{c_i} \times \prod_{i=1}^m \Big(\int_{(B_i E)^\perp} g_i \,\mathrm{d}x \Big)^{c_i}.$$

On the other hand, we let the reader check that for every $x \in E$, $y \in E^\perp$,

$$h(\epsilon x + y) \le f(x) g_\epsilon(y),$$

where

$$g_\epsilon(y) = \sup\big\{ g(y'),\ |y - y'| \le K\epsilon \big\}$$

and K is a constant depending on the diameters of the supports of the functions f_i. Therefore

$$\epsilon^{-\dim E} \int_{\mathbb{R}^n} h \,\mathrm{d}x = \int_{E \times E^\perp} h(\epsilon x + y) \,\mathrm{d}x\mathrm{d}y \le \Big(\int_E f \,\mathrm{d}x \Big) \Big(\int_{E^\perp} g_\epsilon \,\mathrm{d}x \Big).$$

Inequality (4.4) becomes

$$\prod_{i=1}^m \Big(\int_{B_i E} f_i \,\mathrm{d}x \Big)^{c_i} \times \prod_{i=1}^m \Big(\int_{(B_i E)^\perp} g_i \,\mathrm{d}x \Big)^{c_i} \le C_r \Big(\int_E f \,\mathrm{d}x \Big) \Big(\int_{E^\perp} g_\epsilon \,\mathrm{d}x \Big).$$

By Lemma 4.5, the function g has compact support and is upper semi-continuous. This implies easily that

$$\lim_{\epsilon \to 0} \int_{E^\perp} g_\epsilon \,\mathrm{d}x = \int_{E^\perp} g \,\mathrm{d}x,$$

which concludes the proof.

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