A CLASS OF GROUPS RICH IN FINITE QUOTIENTS

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Introduction. If **X** is a class of groups, the class of *counter-***X** *groups* is defined to consist of all groups having no non-trivial **X**-quotients. The counter-abelian groups are the perfect groups and the counter-counter-abelian groups are the imperfect groups studied by Berrick and Robinson [**2**]. This paper is concerned with the class of counter-counter-finite groups. It turns out that these are the groups in which any non-trivial quotient has a non-trivial representation over any finitely generated domain (Theorem 1.1), so we shall call these groups *highly representable* or **HR**-groups.

A group G is an **HR**-group if and only if it has no non-trivial counter-finite quotients, or equivalently, if any non-trivial quotient G/N has a non-trivial finite quotient G/M where $N \leq M$. Examples of infinite **HR**-groups include $\mathbb{Z}$, all abelian groups of finite exponent, as well as the polycyclic groups. Although the class of **HR**-groups is quite similar to the class of residually finite groups in that both classes consist of groups which are rich in finite quotients, the classes are incomparable. It is possible, however, to characterize **HR**-groups in terms of their finite residual (Proposition 1.4).

We begin our study by establishing the fundamental closure properties of **HR**-groups, and by characterizing them in terms of quasicentral chief factors, that is, chief factors in which conjugation by any group element is an inner automorphism (Proposition 1.3).

In Section 2 we characterize the abelian **HR**-groups (Theorem 2.1) as extensions of a finitely generated subgroup, called a *fundamental subgroup*, by a direct product of bounded p-groups. In addition, an arbitrary abelian **HR**-group can be written as a product of a collection of certain subgroups indexed by primes (Theorem 2.3). In the torsion-free case, these subgroups are free abelian groups of a fixed finite rank n, as is the fundamental subgroup. Conversely, it is possible to construct a torsion-free abelian **HR**-group from a collection of free abelian groups of rank n indexed by primes. In fact, any torsion-free abelian **HR**-group is isomorphic to a group constructed in this manner.

The theory of nilpotent **HR**-groups, which is developed in Section 3, is similar to that of abelian **HR**-groups. We show that any nilpotent **HR**-group has a finitely generated subnormal subgroup such that the intermediate subnormal factors are torsion abelian **HR**-groups (Theorem 3.5). However, torsion-free nilpotent **HR**-groups can have non-isomorphic fundamental subgroups. It is shown that a nilpotent **HR**-group can be written as a product of subgroups indexed by primes; again these subgroups are finitely generated in the torsion-free case (Theorem 3.6).

The complexity of the solvable **HR**-groups is exposed in Section 4; little can be said about these groups in general. However, a subclass of the solvable **HR**-groups yields some interesting results, including a generalization of Gruenberg's Theorem on the residual finiteness of finitely generated torsion-free nilpotent groups (Theorem 4.5). This subclass, which includes the abelian and nilpotent **HR**-groups, consists of all solvable groups having no quasicyclic sections.

In the final section we discuss **HR**-subgroups and quotients of arbitrary groups. We obtain characterizations of the **HR**-radical and residual in groups with finite composition

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length. The subnormal structure of **HR**-groups is shown to be arbitrary in the sense that any group may be two-step subnormally embedded in an **HR**-group (Corollary 5.9). However, the normal structure of **HR**-groups is restricted. Necessary and sufficient conditions for a group to be normally embeddable in an **HR**-group are given in Theorem 5.10.

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1. Basic results. We now examine some properties of counter-counter-finite groups. These groups can be characterized in terms of their linear representations. If R is a finitely generated integral domain, then for every set $y_1, \ldots, y_r$ of distinct elements of $GL(n, R)$ there exists a finite field K and a homomorphism θ of $GL(n, R)$ into $GL(n, K)$ such that $\theta(y_1), \ldots, \theta(y_r)$ are distinct ([**13**], p. 51).

THEOREM 1.1. *Let G be a group. Then G is counter-counter-finite if and only if every non-trivial quotient of G has a non-trivial representation over any finitely generated domain.*

Proof. If G is counter-counter-finite and G/M is a non-trivial quotient, then there exists a non-trivial finite quotient G/N where $M \leq N$. The permutation representation of G/M on the cosets of N gives rise to a representation of G/M by permutation matrices with entries from any domain. The converse follows easily from the above result in [**13**]. □

It is easy to see that the class of counter-finite groups and the class of **HR**-groups are both Q-closed classes. Both classes are also P-closed. We give the result in the **HR** case; the proof in the counter-finite case is similar.

LEMMA 1.2. *If $N \lhd G$ and N, G/N are both* **HR**-*groups, then G is also* **HR**.

Proof. Assume that G/L is a counter-finite quotient. Then $G = LN$ and $N/(L \cap N)$ is counter-finite. Thus $N \leq L$ and $G = L$. □

A normal factor H/K of a group G is said to be *quasicentral* in G if each element of G induces by conjugation an inner automorphism of H/K. It follows from the definition that a normal factor H/K is quasicentral if and only if G/K is the direct product of H/K and $C_{G/K}(H/K)$ with $Z(H/K)$ amalgamated ([**2**], p. 6). Thus a quasicentral factor is central precisely when it is abelian. Quasicentrality will be important when we consider normal embeddings in Section 5.

PROPOSITION 1.3. (i) *Let H/K be an infinite chief factor of an* **HR**-*group G. If H/K is quasicentral, then it is central.*

(ii) *Let G be a group satisfying max-n, the maximal condition on normal subgroups. If every quasicentral infinite chief factor of G is central, then G is an* **HR**-*group.*

Proof. (i) If H/K is a non-central quasicentral infinite chief factor of G, then $Z(H/K)$ must be trivial. Therefore, G/K is the direct product of H/K and $C_{G/K}(H/K)$, which implies that G has an infinite simple quotient. This is impossible.

(ii) If $G \notin$ **HR**, then by max-n there exists an infinite simple quotient G/M. Then G/M is a non-central quasicentral infinite chief factor of G. □

The proof of Theorem 2.1 will show that a free abelian group of infinite rank is not an **HR**-group. Conversely, a countable extra-special p-group is **HR** but not residually finite. Therefore the class of **HR**-groups and the class of residually finite groups are incomparable.

PROPOSITION 1.4. *Let G be a group with finite residual R. Then G is an* **HR**-*group if and only if G/R is an* **HR**-*group and R has no proper normal supplements in G.*

Proof. If $G \in$ **HR**, then $G/R \in$ **HR**. If N is a proper normal supplement of R, then G/N is a non-trivial **HR**-group, so it has a non-trivial finite quotient G/M. Therefore $R \leq M$, which contradicts the assumption that $G = RN$. Conversely, if the conditions are satisfied and G/L is counter-finite, then $G = RL$. Thus $L = G$, so $G \in$ **HR**. □

If $G = \mathbb{Z} \oplus \mathbb{Q}$, then $\mathbb{Q}$ is the finite residual R. Thus even though G/R is an **HR**-group, G is not. Therefore the condition that R have no proper normal supplements cannot be removed in general.

2. Abelian HR-groups. In this section we shall study abelian **HR**-groups with the aim of classifying these groups as well as giving methods for constructing them. All groups will be written additively in this section.

An abelian group is counter-finite if and only if it is divisible. Therefore an abelian group is an **HR**-group if and only if it has no non-trivial divisible quotients if and only if it has no quasicyclic sections. Clearly $\mathbb{Z}_{p^\infty}$ and $\mathbb{Q}$ will be our basic examples of abelian counter-finite groups; examples of abelian **HR**-groups include finitely generated abelian groups and bounded abelian groups.

We now show that abelian **HR**-groups have the structure described in the introduction.

THEOREM 2.1. *An abelian group G is an* **HR**-*group if and only if it has a finitely generated subgroup X such that G/X is a direct sum of bounded p-groups for distinct primes p.*

Proof. If G is a p-group, then $G \in$ **HR** if and only if it is equal to a basic subgroup B. However, B has no non-trivial divisible quotients if and only if it is bounded.

In the torsion case G is a direct sum of its p-primary components G_p, as is any quotient of G. Therefore G has no non-trivial divisible quotients if and only if each G_p is bounded.

Finally assume that G is an abelian **HR**-group. If X is a subgroup of G generated by a maximal linearly independent set of elements of infinite order, then X must be finitely generated since G has no $\mathbb{Z}_{p^\infty}$ sections. Since G/X is a torsion abelian **HR**-group, G has the desired form. The converse is clear. □

COROLLARY 2.2. *A subgroup of an abelian* **HR**-*group is an* **HR**-*group.*

If G is a group, H a subgroup of G, and π a collection of primes, then we define

$$G_\pi(H) = \{g \in G : l(g)g \in H \text{ for some positive } \pi\text{-number } l(g)\}.$$

If G is an abelian group, then $G_\pi(H)$ is a subgroup of G containing H.

THEOREM 2.3. *Let G be an abelian* **HR**-*group and let $\{x_1, \ldots, x_n\}$ be a maximal linearly independent set of torsion-free elements of G. If $X = \langle x_1, \ldots, x_n \rangle$, then*:

(i) $G = \langle G_p(X) : p \in \mathcal{P} \rangle$;

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(ii) $G_p(X) \cap \langle G_q(X): q \in \mathcal{P}, q \neq p\rangle = X$;

(iii) *if* G *is torsion-free, then the* $G_p(X)$ *are free abelian groups of rank* n *and* $|G_p(X):X| = p^{e(p)}$ *where* $e(p)$ *is a non-negative integer.*

Proof. $G_p(X)/X$ is the p-primary component of G/X, so (i) and (ii) follow from the Primary Decomposition Theorem. Since $G_p(X)/X \in \mathbf{HR}$ it must be bounded with exponent $p^{e(p)}$. If G is torsion-free, the map $\tau: G_p(X) \to X$ defined by $\tau(g) = p^{e(p)}g$ is an injective homomorphism, which gives (iii). □

COROLLARY 2.4. *Let* G *be a torsion-free abelian* **HR**-*group of rank* 1 *with* $\vec{t}(G) = (h_1, h_2, \ldots)$. *Then* G *is an* **HR**-*group if and only if every* h_i *is finite.*

When G is a torsion-free abelian **HR**-group we shall call the subgroup X defined in Theorem 2.3 a *fundamental subgroup* of G. Any two fundamental subgroups of a torsion-free abelian **HR**-group are free abelian groups of the same rank.

Now we want to consider torsion-free abelian **HR**-groups of rank $n > 1$. If G is such a group, Theorem 2.3(iii) shows that G can be written as a product of free abelian groups $G_p(X)$ of rank n. Conversely, given a collection of free abelian groups $\{X_p : p \in \mathcal{P}\}$, of rank n, we shall give a method to construct a torsion-free abelian **HR**-group. It can be shown that any torsion-free abelian **HR**-group is isomorphic to a group constructed in this manner. The structure of these groups can be much more complicated than for torsion-free abelian **HR**-groups of rank 1.

Let X_p be a free abelian group of rank $n > 0$ and let $e(p)$ be a non-negative integer for each prime p. Then $p^{e(p)}X_p$ is a free abelian subgroup of rank n, as is any choice of Y_p such that $p^{e(p)}X_p \leq Y_p \leq X_p$. Equivalently, we can choose $Y_p \leq X_p$ such that X_p/Y_p is a finite p-group. All such Y_p are isomorphic. Fix a collection of subgroups $\{Y_p : p \in \mathcal{P}\}$ and isomorphisms $\pi_{p,q}: Y_p \to Y_q$ subject to the conditions

$$\pi_{p,r} = \pi_{q,r}\pi_{p,q} \quad \text{and} \quad \pi_{p,p} = \text{the identity map}.$$

Identify X_p with a subgroup of $\bigoplus_{p \in \mathcal{P}} X_p$ and consider the subgroup $S = \langle -y_p + \pi_{p,q}(y_p): y_p \in Y_p;\, p, q \in \mathcal{P}\rangle$ of $\bigoplus_{p \in \mathcal{P}} X_p$. It can be shown that

$$G(X_p, Y_p, \pi_{p,q}) = \left(\bigoplus_{p \in \mathcal{P}} X_p\right) \Big/ S$$

is a torsion-free abelian **HR**-group and that any torsion-free abelian **HR**-group is isomorphic to a group of this form.

Now we would like to give examples to show that torsion-free abelian **HR**-groups can have a complex structure.

PROPOSITION 2.5. *Let* $V = \mathbb{Q}a \oplus \mathbb{Q}b$ *be a two-dimensional vector space. Assume that* $\{p_1, p_2, \ldots\}$, $\{q_1, q_2, \ldots\}$, $\{r\}$ *are sets of distinct primes. If we define* $E_0 = \left\langle \frac{a}{p_i} : i = 1, 2, \ldots \right\rangle$, $E_1 = \left\langle \frac{b}{q_j} : j = 1, 2, \ldots \right\rangle$ *and* $G = \left\langle E_0, E_1, \frac{a+b}{r} \right\rangle$, *then* G *is an indecomposable torsion-free abelian* **HR**-*group of rank* 2.

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Proof. G is clearly a torsion-free abelian group of rank 2 and $G \in \mathbf{HR}$ by Theorem 2.1. The set $\{E_0, E_1\}$ is a rigid system of groups, so G is indecomposable by [**3**, vol. 2, p. 124]. □

Before proceeding, we note that there are continuously many non-isomorphic groups G of the above form. To illustrate the construction mentioned above we show how to construct a group of the form $G(X_p, Y_p, \pi_{p,q})$ isomorphic to the group G in Proposition 2.5. Let $X_{p_i} = \langle a_{p_i}, b_{p_i}\rangle$, $X_{q_j} = \langle a_{q_j}, b_{q_j}\rangle$, $X_r = \langle a_r, b_r\rangle$ be free abelian groups of rank 2 with subgroups $Y_{p_i} = \langle p_i a_{p_i}, b_{p_i}\rangle$, $Y_{q_j} = \langle a_{q_j}, q_j b_{q_j}\rangle$, $Y_r = \langle r(a_r + b_r) - b_r, b_r\rangle$. If the isomorphisms $\pi_{p,q}$ are defined by

$$\pi_{p_i,q_j}: \begin{cases} p_i a_{p_i} \mapsto a_{q_j} \\ b_{p_i} \mapsto q_j b_{q_j} \end{cases},$$

$$\pi_{q_j,r}: \begin{cases} a_{q_j} \mapsto r(a_r + b_r) - b_r \\ q_j b_{q_j} \mapsto b_r \end{cases},$$

then $G \simeq G(X_p, Y_p, \pi_{p,q})$.

3. Nilpotent HR-groups. In this section we study the structure of nilpotent **HR**-groups, obtaining characterizations that bear a strong resemblance to the results for abelian **HR**-groups in Section 2. Despite this similarity, the theory of nilpotent **HR**-groups is inherently more complex, as is shown by the existence of non-isomorphic fundamental subgroups in torsion-free nilpotent **HR**-groups.

Although the first two lemmas deal with solvable and abelian groups respectively, they will have applications here.

LEMMA 3.1. *Let G be a solvable group. Then G is an* **HR**-*group if and only if G_{ab} is an* **HR**-*group.*

Proof. The necessity is clear. Conversely, the solvability of G implies that any non-trivial counter-finite quotient G/M has a non-trivial abelian quotient G/N. Therefore $G_{ab} \notin \mathbf{HR}$. □

LEMMA 3.2. *If G_1 and G_2 are abelian* **HR**-*groups, then $G_1 \otimes G_2$ is an abelian* **HR**-*group.*

Proof. From Theorem 2.1 we know that G_i has a finitely generated subgroup N_i such that G_i/N_i is a direct sum of bounded abelian p-groups. Then $G_1 \otimes G_2$ has a subgroup $R = \langle x \otimes y; x \in N_1, \text{ or } y \in N_2\rangle$, and G/R is a direct sum of bounded abelian p-groups. Since R is a sum of a finite number of **HR**-groups, it follows that $G_1 \otimes G_2 \in \mathbf{HR}$. □

LEMMA 3.3. *Let G be a nilpotent group. Then the following conditions are equivalent*:

(i) $G \in \mathbf{HR}$;

(ii) $\gamma_i(G)/\gamma_{i+1}(G) \in \mathbf{HR}$ *for all i*;

(iii) *G has no quasicyclic sections.*

Proof. (i) ⇒ (ii) Clearly $G_{ab} \in \mathbf{HR}$. Since G is nilpotent, there exists an epimorphism from $\underbrace{G_{ab} \otimes \ldots \otimes G_{ab}}_{i}$ to $\gamma_i(G)/\gamma_{i+1}(G)$ by [**9**, p. 127].

(ii) $\Rightarrow$ (i) This follows from the P-closure of **HR**.
(ii) $\Leftrightarrow$ (iii) This is clear. □

COROLLARY 3.4.(i) *A subgroup of a nilpotent* **HR**-*group is* **HR**.
(ii) *Let* $G = \mathrm{Dr}_{p \in \mathcal{P}} G_p$ *be a nilpotent torsion group. Then G is an* **HR**-*group if and only if the p-component* G_p *has finite exponent for each prime p.*

THEOREM 3.5. *Let G be a nilpotent group. Then G is an* **HR**-*group if and only if it has a finitely generated subgroup X and a subnormal series* $X = H_0 \lhd \ldots \lhd H_n = G$ *such that* H_{i+1}/H_i *is a torsion abelian* **HR**-*group for* $i = 0, \ldots, n-1$.

Proof. If G has such a series, then $G \in$ **HR**. Conversely, assume that G is a nilpotent **HR**-group. Lemma 3.3 implies that each factor $\gamma_i(G)/\gamma_{i+1}(G)$ is an abelian **HR**-group. By Theorem 2.1 $\gamma_i(G)/\gamma_{i+1}(G)$ has a finitely generated subgroup $X_i/\gamma_{i+1}(G) = \langle g_{i_1}\gamma_{i+1}(G), \ldots, g_{i_{l_i}}\gamma_{i+1}(G)\rangle$ such that $\gamma_i(G)/X_i$ is a torsion abelian **HR**-group. Let X be generated by all the g_{i_j}. Then $X = X\gamma_n(G) \leq \ldots \leqq X\gamma_1(G) = G$ is a series with the desired properties where $n+1$ is the nilpotent class of G. □

If G is a nilpotent **HR**-group, we shall call a subnormal series of the form described in Theorem 3.5 a *nilpotent* **HR**-*series for G*.

The fundamental subgroups of a torsion-free abelian **HR**-group are all free abelian groups of a fixed finite rank. It is important to note, however, that the fundamental subgroups of a torsion-free nilpotent **HR**-group need not be isomorphic. If $G = \langle a, b, c;$ $[a, c] = [b, c] = 1,\ [a, b] = c\rangle$, then G is a torsion-free nilpotent **HR**-group. Since G is finitely generated, it is the fundamental subgroup of itself. However, $X = \langle a^2, b^2, c\rangle$ is also a fundamental subgroup of G, and $G \not\cong X$.

Although the fundamental subgroups of a torsion-free nilpotent **HR**-group need not be isomorphic, they are quite similar. If X and Y are fundamental subgroups of a torsion-free nilpotent **HR**-subgroup G, then they both have the same Hirsch number. Also, $X \cap Y$ is a common subgroup of finite index, so they have the same nilpotent class.

In Theorem 2.3 we saw that an abelian **HR**-group could be written as the product of certain subgroups $G_p(X)$ for $p \in \mathcal{P}$. If G was torsion-free, we saw that the $G_p(X)$ were finitely generated. We shall establish similar results for nilpotent **HR**-groups after some preliminary results.

Recall that if H is a subgroup of a group G and π is a collection of primes, then $G_\pi(H) = \{g \in G;\ g^{l(g)} \in H$ for some positive π-number $l(g)\}$. It follows from [**12**, p. 14] that if G is nilpotent, $G_\pi(H) \leq G$. If G is a nilpotent **HR**-group, then the nilpotent **HR**-series shows that for any prime p there exists a positive integer m such that $g^{p^m} \in X$ for all $g \in G_p(X)$.

THEOREM 3.6. *Let G be a nilpotent* **HR**-*group with fundamental subgroup X. Then*
(i) $G = \langle G_p(X) : p \in \mathcal{P}\rangle$;
(ii) $G_p(X) \cap \langle G_q(X) : p \in \mathcal{P}, q \neq p\rangle = X$;
(iii) $G_p(X)G_q(X) = G_q(X)G_p(X)$ *for all* $p \in \mathcal{P}$;
(iv) *If G is torsion-free, then* $G_p(X)$ *is finitely generated for all* $p \in \mathcal{P}$.

Proof. (i) Given $g \in G$, there is a positive integer $m(g)$ such that $g^{m(g)} \in X$. If $m(g) = p_1^{a_1} \dots p_n^{a_n}$ and $l_i = \dfrac{m(g)}{p_i^{a_i}}$, then $g^{l_i} \in G_{p_i}(X)$. Since $g = g^{b_1 l_1 + \dots + b_n l_n}$ we see that $G = \langle G_p(X) \rangle_{p \in \mathcal{P}}$.

(ii) Certainly $X \leq G_p(X) \cap \langle G_q(X) : q \in \mathcal{P}, q \neq p \rangle$. If $y \in G(X) \cap \langle G_q(X) : q \in \mathcal{P}, q \neq p \rangle$, then there exist a p-number k and a p'-number l such that $y^k \in X$ and $y^l \in X$. Then $y = y^{ak+bl} \in X$.

(iii) Let $\pi = \{q, p\}$. Then $G_p(X)G_q(X) \subseteq G_\pi(X)$. Given $x \in G_\pi(X)$ we know that $x^{p^r q^s} \in X$ for some r, $s > 0$. Now $x = x^{bq^s} x^{ap^r}$. But $x^{ap^r} \in G_q(X)$ and $x^{bq^s} \in G_p(X)$, so $G_\pi(X) \subseteq G_p(X)G_q(X)$, which implies the result.

(iv) We shall induct on the nilpotent class of G. If G has nilpotent class ≤ 1, the result follows from Theorem 2.3(iii). Now assume that the result holds for all groups of nilpotent class $c \geq 1$ and let G have class $c + 1$.

Let $X = H_0 \lhd \dots \lhd H_n = G$ be a nilpotent **HR**-series for G. if $\bar{G} = G/Z(G)$, then $\bar{G}$ is torsion-free and

$$\bar{X} = XZ(G)/Z(G) \lhd \dots \lhd H_n Z(G)/Z(G) = \bar{G}$$

is a nilpotent **HR**-series for $\bar{G}$. We know that $\bar{G}_p(\bar{X})$ is finitely generated by induction. But $G_p(X)Z(G)/Z(G) \leq \bar{G}_p(\bar{X})$, so it is also finitely generated. Thus we need only show that $G_p(X) \cap Z(G)$ is finitely generated. There exists an integer m such that $g^{p^m} \in X$ for all $g \in G_p(X)$. Since $G_p(X) \cap Z(G)$ is torsion-free, the map $\theta : G_p(X) \cap Z(G) \to X \cap Z(G)$ defined by $\theta(g) = g^{p^m}$ is an injective homomorphism. Thus $G_p(X) \cap Z(G)$ is finitely generated. □

4. Solvable HR-groups. In Sections 2 and 3 we saw that it was possible to characterize abelian and nilpotent **HR**-groups in terms of a series involving a finitely generated subnormal subgroup and intermediate torsion abelian **HR**-factors. From these results we saw that the classes of abelian and nilpotent **HR**-groups were closed under the formation of subgroups. We also noted that torsion-free abelian and torsion-free nilpotent **HR**-groups could be built up from collections of finitely generated subgroups. Unfortunately, the structure of solvable **HR**-groups can be quite complicated, so that no such results are possible. A subclass of the solvable **HR**-groups will, however, yield some interesting results.

We begin by pointing out that the center of a solvable **HR**-group can be any countable abelian group. By a result of Philip Hall [**9**, p. 402], we know that if A is any non-trivial countable abelian group, there exist $2^{\aleph_0}$ non-isomorphic 2-generator groups G such that $G'' = Z(G) \simeq A$, and $G/Z(G)$ has trivial center. Lemma 3.1 implies that any finitely generated solvable group is an **HR**-group, so we see that it is possible to find solvable **HR**-groups with divisible centers. Therefore we cannot say much about solvable **HR**-groups in general. For this reason we shall consider a subclass of solvable **HR**-groups.

Let G be a group. If all the normal subgroups of G are **HR**-groups, we shall call G an $\overline{\mathbf{HR}}$-group. Clearly any nilpotent **HR**-group is an $\overline{\mathbf{HR}}$-group, and polycyclic groups are also examples of solvable $\overline{\mathbf{HR}}$-groups.

LEMMA 4.1. *Any subgroup of a solvable* $\overline{\mathbf{HR}}$*-group is an* **HR***-group.*

Proof. Let G be a solvable $\overline{\mathbf{HR}}$-group and choose $H \leq G$. We know that each factor $G^{(i)}/G^{(i+1)}$ is an abelian $\mathbf{HR}$-group, so $(G^{(i)} \cap H)/(G^{(i+1)} \cap H)$ is an abelian $\mathbf{HR}$-group. Therefore $H \in \mathbf{HR}$. □

The next result characterizes solvable $\overline{\mathbf{HR}}$-groups by their sections.

THEOREM 4.2. *Let G be a solvable group. Then the following conditions are equivalent*:

(i) *G is an $\overline{\mathbf{HR}}$-group*;

(ii) *G is poly-(abelian $\mathbf{HR}$)*;

(iii) *G has no non-trivial quasicyclic sections.*

Proof. (i) ⇒ (ii) is clear.

(ii) ⇒ (iii) Since G is a poly-(abelian $\mathbf{HR}$) group, any subgroup, and hence any section, of G must be $\mathbf{HR}$. Therefore G has no non-trivial quasicyclic sections.

(iii) ⇒ (i) If $H \leq G$, then $H_{ab} \in \mathbf{HR}$. Therefore Lemma 3.1 implies that $H \in \mathbf{HR}$. □

PROPOSITION 4.3. *The class of solvable $\overline{\mathbf{HR}}$-groups is the smallest* S, P, *and* Q*-closed class containing $\mathbb{Z}$ and all direct products of bounded abelian p-groups.*

Proof. Clearly the class of solvable $\overline{\mathbf{HR}}$-groups contains the smallest class with these proper. Conversely, the smallest class with these properties contains all poly-(abelian $\mathbf{HR}$) groups, so it contains all solvable $\overline{\mathbf{HR}}$-groups. □

From these simple results we can already see that solvable $\overline{\mathbf{HR}}$-groups form an interesting class of infinite solvable groups with finite torsion-free rank; it does not appear to have been previously studied.

We shall now investigate the structure of solvable $\overline{\mathbf{HR}}$-groups. If G is a solvable $\overline{\mathbf{HR}}$-group, the abelian sections of G have finite torsion-free rank. By [**8**, vol. 2, p. 131] we see that $G/T \in \mathbf{S}_1$, where T is the maximum normal torsion subgroup of G. The following theorem collects some other properties of G/T.

THEOREM 4.4. *Let G be a solvable $\overline{\mathbf{HR}}$-group with maximum normal torsion subgroup T. Then*

(i) *G/T is an extension of a torsion-free nilpotent $\overline{\mathbf{HR}}$-group by a finitely generated abelian-by-finite group*;

(ii) *G/T is locally polycyclic*;

(iii) *there is a finite set of primes π such that G/T is a residually finite π-group.*

Proof. (i) Since $G/T \in \mathbf{S}_1$ its Fitting subgroup F/T is a torsion-free nilpotent group and G/F is polycyclic and abelian-by-finite by [**8**, vol. 2, p. 169].

(ii) If X/T is a finitely generated subgroup of G/T, then $X/T \in \overline{\mathbf{HR}}$ and $X/T \in \mathbf{S}_1$. Thus X/T is a solvable minimax group by [**8**, vol. 2, p. 176], and hence polycyclic since it contains no quasicyclic sections.

(iii) Since $G/T \in \mathbf{S}_1$ and the center of the Baer radical is reduced the result follows from [**8**, vol. 2, p. 138]. □

Before proceeding we make a few remarks about Theorem 4.4. If the Sylow p-subgroups of T have bounded exponent and G/T satisfies the conditions in (i), then G is a solvable $\overline{\mathbf{HR}}$-group. The result in (ii) cannot be extended to G since the standard wreath product $\mathbb{Z}_p wr \mathbb{Z}$ is a solvable $\overline{\mathbf{HR}}$-group that is not locally polycyclic. A countable extra-special p-group is a solvable $\overline{\mathbf{HR}}$-group which is not residually finite, so (iii) also cannot be extended to G.

We conclude this section by establishing a type of residual property of torsion-free nilpotent **HR**-groups.

THEOREM 4.5. *If G is a torsion-free nilpotent* **HR**-*group, then G is residually a finite p-group for all primes p.*

Proof. By Corollary 3.4(i) it follows that $G \in \overline{\mathbf{HR}}$. We therefore know that G has finite abelian section rank and $Z(G)$ is $\mathcal{P}$-reduced since it is an **HR**-group. Thus the result follows from [**8**, vol. 2, p. 135]. □

Theorem 4.5 gives a generalization of the well-known theorem of Gruenberg: If *G is a finitely generated torsion-free nilpotent group, then G is residually a finite p-group for all primes p.*

5. Highly representable subgroups and quotients. We now wish to characterize the **HR**-radical and residual in groups with finite composition length. The **HR**-radical of a group G, which we denote by $\mathbf{HR}(G)$, is locally **HR** but need not be **HR** since a free abelian group of infinite rank is not **HR**. The following property will be useful for characterizing the **HR**-radical.

We say that a group G is *purely infinite* if each subnormal composition factor of G is infinite. It follows from the definition that a group is purely infinite if and only if each of its subnormal subgroups is counter-finite. The connection between $\mathbf{HR}(G)$ and purely infinite quotients of G is shown in the next result.

THEOREM 5.1. *Let G be a group satisfying min-sn, the minimal condition on subnormal subgroups. Then $G/\mathbf{HR}(G)$ is the largest purely infinite quotient of G.*

Proof. In proving that $G/\mathbf{HR}(G)$ is purely infinite we may assume that $\mathbf{HR}(G) = 1$. If G is not purely infinite, then there is a subnormal subgroup H that is minimal with respect to being non-counter-finite. Let $H = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$. Suppose that $\mathbf{HR}(H_i) = 1$ and let L be a normal **HR**-subgroup of H_{i-1}. Then L is subnormal in G, so it has only finitely many conjugates in G since G satisfies min-*sn* [**8**, p. 385]. Therefore $\langle L^{H_i} \rangle$ is **HR** and $L \leq \mathbf{HR}(H_i) = 1$. Since $\mathbf{HR}(G) = 1$, induction on $n - i$ implies that $\mathbf{HR}(H) = 1$, so $H \notin \mathbf{HR}$. Thus H has a non-trivial counter-finite quotient H/N. The minimality of H implies that N is counter-finite, whence so is H, which contradicts our choice of H. It follows that G is purely infinite.

Now assume that G/J is purely infinite and that M is a normal **HR**-subgroup of G. then $M/(M \cap J)$ is **HR** and counter-finite, so $M \subseteq J$. Therefore $\mathbf{HR}(G) \leq J$. □

COROLLARY 5.2. *Let G be a group with finite composition length. Then $\mathbf{HR}(G)$ is simultaneously the maximum normal* **HR**-*subgroup of G and the minimum normal subgroup of G with purely infinite quotient.*

It is important to note that $\mathbf{HR}(G)$ need not be an **HR**-group even when G satisfies min-*sn*. Section 5 of [**2**] provides an example of such a group.

We now wish to examine the **HR**-residual a group G which we denote by $\mathbf{HR}^*(G)$. Before proceeding we give an example to show that **HR** is not an R_0-closed class.

Let H be an **HR**-group such that $Z(H)$ is non-trivial and divisible; such groups have been constructed by Philip Hall [**9**, p. 402]. Define $\bar{G} = H \times H$ and let $H_1 = \{(h,1): h \in H\}$, $H_2 = \{(1,h): h \in H\}$, $D = \{(h,h): h \in H\}$ be subgroups of $\bar{G}$. If $G = DZ(H_1) = DZ(H_2)$, then $Z(H_1)$ and $Z(H_2)$ our normal in G. Clearly $G/Z(H_1)$ and $G/Z(H_2)$ are **HR** and $Z(H_1) \cap Z(H_2) = 1$, but G is not **HR**.

However the following lemma shows that it is useful to discuss the **HR**-residual in groups satisfying max-n.

LEMMA 5.3. *Let G be a group satisfying the maximal condition on normal subgroups. If M and N are normal subgroups of G such that G/M and G/N are* **HR**-*groups, then $G/(M \cap N)$ is an* **HR**-*group.*

Proof. Without loss we can assume that $M \cap N = 1$. If G/L is a non-trivial counter-finite quotient, then $G = LM = LN$. We can further assume that $L \cap M = 1$. Then $[L, M] = [N, M] = 1$, so $M \le Z(G)$. Since $G/L \simeq M$ is counter-finite, M is a divisible central subgroup of G. But G has max-n, so M has max, and therefore is finitely generated. Thus M must be trivial, which contradicts our choice of L. □

COROLLARY 5.4. *If G is a group with finite composition length, then $G/$***HR***$^*(G)$ is an* **HR**-*group.*

Our immediate goal is to provide a characterization of the **HR**-residual analogous to that for the **HR**-radical. First we need to define a technical property of normal subgroups which is similar to one used in [**2**].

Let $\mathscr{A}$ denote the following property of normal subgroups N of a group G: all non-trivial G-quotients of N are infinite and each G-simple quotient of N is quasicentral in G.

PROPOSITION 5.5. *In any group G there exists a unique largest normal subgroup with property $\mathscr{A}$.*

Proof. Let $\{N_i\}_{i \in I}$ be a chain of normal subgroups of G having property $\mathscr{A}$ and let U be the union of the chain. If U/V is a finite G-quotient, then $U \le N_i V$ for some i. Thus $U/V \overset{G}{\simeq} N_i/(N_i \cap V)$, which shows that $U = V$. If U/V is a G-simple quotient, the same argument shows that U/V must be quasicentral in G. Thus U has $\mathscr{A}$ and Zorn's Lemma implies there is a normal subgroup M that is maximal with respect to having $\mathscr{A}$.

If $N \lhd G$ has $\mathscr{A}$, then we claim that MN has $\mathscr{A}$. This will imply that $N \le M$ and M will have the desired properties. If MN/L is a finite G-quotient, then $M/(M \cap L)$ must be trivial. Hence $MN = L$. A G-simple quotient of MN is isomorphic with such a quotient of M or N. The result follows. □

Let $\rho(G)$ denote the normal subgroup of Proposition 5.5. We now use this subgroup to characterize the **HR**-residual in groups with finite chief length.

THEOREM 5.6. *If G is a group with finite chief length, then* **HR***$^*(G) = \rho(G)$.*

Proof. First we shall show that **HR**$^*(G)$ has $\mathscr{A}$, which will imply that **HR**$^*(G) \le \rho(G)$. Let $R =$ **HR**$^*(G)$. Since G has finite chief length, $G/R \in$ **HR**. If R/N is a finite G-quotient, then $G/N \in$ **HR**, so $R = N$. Now suppose that R/M is an infinite G-simple quotient. Then $G/M \notin$ **HR** by the minimality of R, so there is a non-trivial counter-finite

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quotient G/L where $M \leq L$. Since R/M is G-simple and $R \not\leq L$, we conclude that $R \cap L = M$. Since $G = LR$, we have $G/M = L/M \times R/M$, which shows that R/M is quasicentral in G. Thus R has $\mathscr{A}$.

Now assume that $R \neq J = \rho(G)$. Since G satisfies max-n, there is an infinite G-simple quotient J/K where $R \leq K$. By definition J has $\mathscr{A}$, so J/K is quasicentral in G, and hence is central by Proposition 1.3 since $G/R \in \mathbf{HR}$. Therefore J/K is an infinite abelian simple group, which is impossible. □

COROLLARY 5.7. *Let G be a group with finite chief length. Then $\mathbf{HR}^*(G)$ is simultaneously the maximum normal subgroup with property $\mathscr{A}$ and the minimum normal subgroup whose quotient in G is an* **HR**-*group.*

The following result shows that the subnormal subgroups of an **HR**-group can be arbitrary.

THEOREM 5.8. *If H is an arbitrary group and K is an* **HR**-*group containing an element of infinite order, then the complete wreath product $H\overline{\mathrm{wr}}K$ is an* **HR**-*group.*

Proof. Let G/N be a non-trivial counter-finite quotient. Then $G = BN$ where B is the base group. Let $x \in K$ have infinite order, and write $x = bn$ $(b \in B, n \in N)$. Then $n = b^{-1}x$ and N contains $[B, n]$. We claim that $[B, n] = B$. To show this it suffices to prove that given $d \in B$, there exists $c \in B$ such that $d = [c, n] = [c, b^{-1}x] = c^{-1}(c^{b^{-1}})^x$. Take k-components $(k \in K)$ to get

$$d_k = c_k^{-1}(c_{kx^{-1}})^{b_{kx}^{-1}-1}. \qquad (*)$$

This is an infinite linear system for c_k over H. To solve it choose a left transversal T to $\langle x \rangle$ in K. For each $t \in T$ assign c_t arbitrarily in H, then use $(*)$ to solve back and find c_{tx^i}. It follows that $B \leq N$ and $G/N \in \mathbf{HR}$, so $G = N$. □

COROLLARY 5.9. *An arbitrary group H is isomorphic to a 2-step subnormal subgroup of an* **HR**-*group H^*. If H is solvable of derived length d, then H^* is solvable and has derived length $d+1$.*

It turns out that there are restrictions on the normal subgroups of **HR**-groups. We shall say that a group H satisfies *condition $\mathscr{B}$* if there exists an **HR**-subgroup X of $\mathrm{Aut}(H)$ such that for any non-trivial counter-finite X-quotient H/M, some element of X induces an outer automorphism of H/M.

THEOREM 5.10. (i) *If H is a normal subgroup of an* **HR**-*group and H_{ab} is an* **HR**-*group, then H satisfies condition $\mathscr{B}$.*

(ii) *If H is a group satisfying condition $\mathscr{B}$, then H is isomorphic with a normal subgroup of an* **HR**-*group.*

Proof. (i) Let $H \lhd G$, where $G \in \mathbf{HR}$, and put $X = G/C_G(H)$ considered as a subgroup of $\mathrm{Aut}(H)$. Assume that H/M is a non-trivial counter-finite quotient with $M \lhd G$. If H/M is quasicentral in G, then Proposition 1.3(i) implies that H/M is central. Hence $H' \leq M$. But $H_{ab} \in \mathbf{HR}$, so $H = M$, contradicting our choice of H/M.

(ii) Assume that H satisfies $\mathscr{B}$. If $X \leq \mathrm{Aut}(H)$ has the required property, define $G = H \rtimes X$. If G/L is a non-trivial counter-finite quotient, then $G = HL$. Therefore $G/L \simeq H/(H \cap L)$, which is a non-trivial quotient of H, and $H \cap L$ is X-admissible because $L \lhd G$. Since $G = HL$ and $[H, L] \leq H \cap L$, every element of G, and thus of X, induces an inner automorphism of $H/(H \cap L)$, contradicting the assumption that H satisfies condition $\mathcal{B}$. Hence $G \in \mathbf{HR}$, and of course $H \lhd G$. □

COROLLARY 5.11. (i) *If H is a finitely generated group, then H is isomorphic with a normal subgroup of an* **HR**-*group if and only if H satisfies $\mathcal{B}$.*

(ii) *If H is an infinite simple group, then H embeds normally in an* **HR**-*group if and only if H is not complete.*

In light of Corollary 5.11(ii) it is important to note that there are many infinite simple groups that are not complete (see [**6**]).

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