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REGIONS OF THE n-SPHERE AND RELATED INTEGRALS

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1. Introduction

In this note the volumes of certain regions in the n-sphere will be found in two ways: (a) by using a symmetry argument, (b) by expressing the volumes as repeated integrals over the $(n-1)$-cube. By considering the 4 and 5 spheres and equating the integrals obtained by method (b) to the solution obtained by method (a) we evaluate integrals of the form

$$I(a, b, c) = \int_0^a \frac{x \tan^{-1} x}{(b - x^2)\sqrt{(c - x^2)}}\, dx, \quad b > c > 0, \sqrt{c} \geqslant a > 0$$

for certain values of a, b and c; it does not appear easy (if indeed it is possible) to evaluate these integrals by direct methods.

These integrals arose in the evaluation of the distribution function of a random variable W defined by Shapiro and Wilk in (**1**). They define

$$W = \frac{n(\bar{X} - X_{(1)})^2}{(n-1)S^2}$$

where $X_1, X_2, \ldots, X_n$ are independent exponential variables, i.e.

$$f_X(x) = e^{-x}, \quad x > 0,$$

$$\bar{X} = \sum_{i=1}^{n} X_i/n,$$

$$X_{(1)} = \min\{X_1, X_2, \ldots, X_n\}$$

and

$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2.$$

The evaluation of the $I(a, b, c)$ was an unexpected bonus.

2. The Volumes of Two Regions in the n-sphere

Let the volume of the n-sphere $\{\boldsymbol{x}: \Sigma_{i=1}^n x_i^2 \leqslant 1\}$ be denoted by S_n. Then it is immediately obvious that the volume U_n of the region R_n,

$$R_n = \left\{\boldsymbol{x}: 0 \leqslant x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 \leqslant 1\right\}$$

is given by

$$U_n = S_n/(n!2^n).$$

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The second region T_n is defined by

$$T_n = \left\{ \boldsymbol{x}: 0 \leqslant \surd(1.2)x_1 \leqslant \surd(2.3)x_2 \leqslant \cdots \leqslant \surd(n(n+1))x_n \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 \leqslant 1 \right\}$$

and has volume V_n given by

$$V_n = S_n/(n+1)!$$

We prove this as follows: consider the region R in the $(n+1)$-sphere defined by

$$R = \left\{ \boldsymbol{x}: 0 \leqslant \surd(1.2)x_1 \leqslant \surd(2.3)x_2 \leqslant \cdots \leqslant \surd(n(n+1))x_n \quad \text{and} \quad \sum_{i=1}^{n+1} x_i^2 \leqslant 1 \right\}$$

and apply the Helmert transformation

$$x_i = \frac{y_1 + y_2 + \cdots + y_i - iy_{i+1}}{\surd(i(i+1))}, \quad i = 1, \ldots, n$$

$$x_{n+1} = \frac{y_1 + y_2 + \cdots + y_{n+1}}{\surd(n+1)}.$$

Then this transformation maps R into R' where

$$R' = \left\{ y: y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{n+1} \quad \text{and} \quad \sum_{i=1}^{n+1} y_i^2 \leqslant 1 \right\}.$$

By considering the $(n+1)!$ permutations of the suffices of the y's we see, by symmetry, that the volume of R' is

$$S_{n+1}/(n+1)! \tag{1}$$

Since the Helmert transformation is orthogonal the volume of R is also given by (1).

Suppose the hyperplane $\{\boldsymbol{x}: x_{n+1} = a\}$ intersects R in a surface R_a with area C_a. Then

$$\int_{-1}^{1} C_a da = S_{n+1}/(n+1)! \tag{2}$$

R_a is given by

$$R_a = \left\{ \boldsymbol{x}: 0 \leqslant \surd(1.2)x_1 \leqslant \surd(2.3)x_2 \leqslant \cdots \leqslant \surd(n(n+1))x_n, \, x_{n+1} = a, \quad \sum_{i=1}^{n} x_i^2 \leqslant 1 - a^2 \right\}.$$

and so R_0 has area V_n. Now R_a is mapped onto R_0 by the transformation

$$x_i = \surd(1-a^2)y_i, \quad i = 1, 2, \ldots, n$$

$$x_{n+1} = y_{n+1} + a.$$

The Jacobian of this transformation is $(1-a^2)^{n/2}$ and so

$$C_a = (1-a^2)^{n/2} V_n$$

Substituting in (2) gives

$$V_n \int_{-1}^{1} (1-a^2)^{n/2} da = S_{n+1}/(n+1)!$$

and the formula for V_n follows.

3. U_n and V_n as Repeated Integrals

We apply the transformation

$$x_i = \frac{y_i}{\sqrt{(i(i+1))}}, \quad i = 1, \ldots, n$$

to the region T_n. The Jacobian of this transformation is

$$1/(n!\sqrt{(n+1)})$$

and T_n transforms into

$$\left\{y: 0 \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \quad \text{and} \quad \sum_{i=1}^{n} \frac{y_i^2}{i(i+1)} \leqslant 1\right\}.$$

This region is transformed by setting

$$y_i = z_1 z_2 \ldots z_{n-i+1}, \quad i = 1, 2, \ldots, n \tag{3}$$

The Jacobian of this transformation is $z_1^{n-1} z_2^{n-2} \ldots z_{n-1}$, and the region is mapped into

$$\left\{z: 0 \leqslant z_1, 0 \leqslant z_i \leqslant 1, i = 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} \frac{z_1^2 z_2^2 \ldots z_i^2}{(n-i+1)(n-i+2)} \leqslant 1\right\}$$

and so we have

$$V_n = \frac{1}{n!\sqrt{(n+1)}} \int_0^1 \cdots \int_0^1 \int_0^{\theta(z)} z_1^{n-1} z_2^{n-2} \ldots z_{n-1} dz$$

where

$$1/\theta(z) = \left\{\frac{1}{n(n+1)} + \sum_{i=2}^{n} \frac{z_2^2 z_3^2 \ldots z_i^2}{(n-i+1)(n-i+2)}\right\}^{1/2}.$$

Carrying out the integration with respect to z_1 and using the value of V_n we find

$$\int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_2^{n-2} x_3^{n-3} \ldots x_{n-1}}{\left\{\frac{1}{n(n+1)} + \sum_{i=2}^{n} \frac{x_2^2 x_3^2 \ldots x_i^2}{(n-i+1)(n-i+2)}\right\}^{n/2}} dx_2 dx_3 \ldots dx_n$$

$$= \frac{n}{\sqrt{(n+1)}} \cdot \frac{\pi^{n/2}}{\Gamma(n/2+1)}, \quad n \geqslant 2. \tag{4}$$

In order to express U_n as a repeated integral we use only the transformation (3) and obtain in the same way

$$\int_0^1 \int_0^1 \cdots \int_0^1 \frac{x_2^{n-2} x_3^{n-3} \ldots x_{n-1}}{\left(1 + \sum_{i=2}^{n} x_2^2 x_3^2 \ldots x_i^2\right)^{n/2}} dx_2 dx_3 \ldots dx_n = \frac{\pi^{n/2}}{(n-1)!2^n \Gamma(n/2+1)}, \; n \geqslant 2. \tag{5}$$

4. Applications

The cases $n = 4$ and 5 in expressions (4) and (5) are interesting because if one tries to evaluate the integrals one is led to integrals of the form $I(a, b, c)$ as defined in Section 1. For example, if $n = 4$ in (5), and if we integrate out the variables in the

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order x_3 followed by x_2, we find that we must evaluate $I(1/\sqrt{3}, 3, 1)$. The details are elementary and rather tedious but the end result is that

$$I(1/\sqrt{3}, 3, 1) = \frac{\sqrt{2}}{576}\pi^2.$$

By considering the evaluation of (4) and (5) for $n = 4$ and 5, with different orders of integration we can obtain the following results:

Source	a	b	c	$I(a, b, c)$	Order of integration
(5) $n = 4$	$\frac{1}{\sqrt{3}}$	3	1	$\frac{\sqrt{2}}{576}\pi^2$	x_3, x_2
(5) $n = 4$	$\frac{1}{\sqrt{3}}$	1	$\frac{1}{2}$	$\frac{\sqrt{2}}{96}\pi^2$	x_3, x_4
(4) $n = 4$	$\sqrt{\frac{5}{3}}$	3	2	$\frac{\pi^2}{30}$	x_3, x_4
(4) $n = 4$	$\frac{1}{\sqrt{3}}$	3	$\frac{1}{2}$	$\sqrt{(10)}\left\{\frac{\pi^2}{25} - \frac{2\pi}{15}\tan^{-1}\left(\sqrt{\frac{5}{3}}\right)\right\}$	x_3, x_2
(5) $n = 5$	$\frac{1}{\sqrt{3}}$	3	2	$\frac{\pi^2}{20} - \frac{\pi}{6}\tan^{-1}\left(\sqrt{\frac{5}{3}}\right)$	x_4, x_2, x_3
(4) $n = 5$	$\sqrt{2}$	3	2	$\frac{\pi^2}{12}$	x_4, x_2, x_3
-	1	3	2	$\frac{\pi^2}{96}$	—

It is easy to show, by using straight forward methods, that

$$I(\sqrt{2}, 3, 2) = \frac{\pi^2}{16} + 2I(1, 3, 2).$$

This gives the final entry in the table.

Note: Because of the amount of algebra needed to obtain the above results, all the results were checked by numerical integration.

5. Comments

Three obvious questions can be asked:

(i) Can the integrals obtained in 4 be evaluated directly?
(ii) For what other values of a, b and c do nice results like those in 4 hold?
(iii) If the answer to (i) is "no", can other regions in the n-sphere be defined which will lead to evaluation of integrals of the form $I(a, b, c)$?

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