Canad. Math. Bull. Vol. **29** (4), 1986

HAUSDORFF DISTANCE AND A COMPACTNESS CRITERION FOR CONTINUOUS FUNCTIONS

BY
GERALD BEER

ABSTRACT. Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be metric spaces and let h_ρ denote Hausdorff distance in $X \times Y$ induced by the metric ρ on $X \times Y$ given by $\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$. Using the fact that $h\rho$ when restricted to the uniformly continuous functions from X to Y induces the topology of uniform convergence, we exhibit a natural compactness criterion for $C(X, Y)$ when X is compact and Y is complete.

1. **Introduction**. Let X be a compact metric space and let $C(X,R)$ denote the continuous real functions on X, made a metric space in the usual way by defining the distance d_1 between continuous functions f and g to be

$$d_1(f,g) = \sup\{|f(x)-g(x)| : x \in X\}$$

No student of mathematics can avoid exposure to the following compactness criterion for subspaces Ω of $C(X,R)$.

THE ARZELA-ASCOLI THEOREM. *Let X be a compact metric space and let Ω be a closed subset of $C(X,R)$. Then Ω is compact if and only if (i) Ω is equicontinuous (ii) For each x in X, $\Omega_x = \{f(x): f \in \Omega\}$ is bounded.*

More generally, we can replace R by any complete metric space Y, provided that we insist that each set Ω_x has compact closure [10], i.e., that Ω_x be totally bounded. It is the purpose of this note to set forth a new compactness criterion for subsets of $C(X, Y)$ with X compact and Y complete. To do this, we view elements of $C(X, Y)$ as subsets of $X \times Y$, not as transformations. Our characterization will also involve two conditions, one weaker than equicontinuity, the other stronger than pointwise total boundedness. Our approach will be to remetrize the topology of uniform convergence on $C(X, Y)$ induced by the metric $d_1(f,g) = \sup\{d_Y(f(x), g(x)): x \in X\}$. Using this different metric, conditions equivalent to total boundedness and completeness (which together characterize compactness in arbitrary metric spaces) will become apparent. But first, we need to recall some basic metric space topology.

Received by the editors 19 February 1985 and, in revised form, 28 August 1985.
AMS Subject Classification Primary: 40A30, 54B20, Secondary: 54C35, 54E50.
Key Words: Hausdorff metric, Arzela-Ascoli Theorem, complete metric space, functions with closed graph.

DEFINITION. *A subset K of a pseudometric space is* totally bounded *if for each $\epsilon > 0$, K is contained in a finite union of ϵ-balls.*

In a complete metric space, a set has compact closure if and only if it is totally bounded because (1) closed subsets of a complete metric space are complete; (2) the closure of a totally bounded subset of any metric space is totally bounded; (3) the compact subsets of a metric space are precisely those subsets that are complete and totally bounded; (4) subsets of totally bounded sets are totally bounded. Now let $\langle W, d\rangle$ be a metric space. If K is a subset of W and ϵ is positve, let $S_\epsilon[K]$ denote the union of all open ϵ-balls whose centers run over K. If K_1 and K_2 are nonempty subsets of W and for some $\epsilon > 0$ both $S_\epsilon[K_1] \supset K_2$ and $S_\epsilon[K_2] \supset K_1$, we define the *Hausdorff distance* h_d between them to be

$$h_d(K_1, K_2) = \inf\{\epsilon : S_\epsilon[K_1] \supset K_2 \text{ and } S_\epsilon[K_2] \supset K_1\}.$$

Otherwise, we write $h_d(K_1, K_2) = \infty$. It is easy to check that h_d defines an infinite valued pseudometric on the nonempty subsets of W, and that $h_d(K_1, K_2) = 0$ if and only if K_1 and K_2 have the same closure. Thus, if we restrict h_d to the closed (resp. compact) subsets of W, then h_d defines an infinite (resp. finite) valued metric on such sets. Most importantly, if $\{K_n\}$ is a sequence of nonempty subsets of W h_d-convergent to a nonempty set K, then $\{K_n\}$ also converges to $\bar{K}$, and $\bar{K} = Li\ K_n = Ls\ K_n$, where $Li\ K_n$ (resp. $Ls\ K_n$) is the set of all points w each neighborhood of which meets all but finitely many (resp. infinitely many) sets K_n [6].

In the sequel, we shall denote the closed (resp. compact) nonempty subsets of a metric space W by $CL(W)$ (resp. $K(W)$). We need two results proved in Chapter II of [5] that we state in a single lemma.

LEMMA 1. *Let h_d be Hausdorff distance for a complete metric space $\langle W, d\rangle$. Then:* (1) *$\langle CL(W), h_d\rangle$ is complete* (2) *$\langle K(W), h_d\rangle$ is a closed subspace of $\langle CL(W), h_d\rangle$.*

Now consider the product of a compact metric space $\langle X, d_X\rangle$ with an arbitrary metric space $\langle Y, d_Y\rangle$, metrized in the following way:

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)\}$$

If we identify members of $C(X, Y)$ with their graphs, then h_ρ defines a metric on $C(X, Y)$, which we denote by d_2. It is easy to see that $d_2(f, g) \leq d_1(f, g)$. In fact, by Theorem 4.7 of [7], the metrics d_1 and d_2 are equivalent, and as a result, Hausdorff metric convergence of graphs has drawn some attention from researchers in constructive approximation theory (see, e.g., [8] and [9]). We note that the class of spaces X for which d_1 and d_2 are equivalent on $C(X, Y)$ for each target space Y turns out to properly include the compact spaces [4]. This class, first thoroughly studied by Atsuji [1], can be characterized in any of the following ways: (1) each pair of disjoint closed sets in X lie a positive distance apart; (2) the set of limit points X' of X is compact, and for each positive ϵ, the set $(S_\epsilon[X'])^c$ is uniformly discrete; (3) each open cover of X has a Lebesgue number; (4) each continuous function with domain X is uniformly continuous. The relationship between d_1-convergence, d_2-convergence, the relation $Li\, f_n = Ls\, f_n = f$, and uniform convergence on compacta has also been studied by this author in [2] and [3]. One can show that for compact X any two metrics on $X \times Y$ that generate the product topology will yield the same Hausdorff metric topology for $C(X, Y)$ because the elements of $C(X, Y)$ are all compact sets. For noncompact X this is not in general true.

2. **Results**. If X is compact and Y is complete, then $\langle C(X, Y), d_1\rangle$ is of course complete. However, $\langle C(X, Y), d_2\rangle$ need not be.

EXAMPLE 1. For each $n \in Z^+$ let $f_n \in C([0, 1], R)$ be the function whose graph consists of the line segment joining $(0, 1)$ to $(1/n, 0)$ plus the one joining $(1/n, 0)$ to $(1, 0)$. Clearly, $\{f_n\}$ h_ρ-converges to the set consisting of the line segment from $(1, 0)$ to $(0, 0)$ plus the one from $(0, 0)$ to $(0, 1)$. Since this is not the graph of a continuous function, $\{f_n\}$ is d_2-Cauchy but is not d_2-convergent.

THEOREM 1. *Let $\langle X, d_X\rangle$ be a compact metric space and let $\langle Y, d_Y\rangle$ be a complete metric space. Suppose $\Omega \subset C(X, Y)$ is d_2-closed. The following are equivalent:* (1) *Ω is d_2-complete* (2) *Each d_2-Cauchy sequence in Ω is d_1-Cauchy* (3) *Whenever $\{f_n\}$ is a sequence in Ω h_ρ-convergent to a closed set E, then E is the graph of a function from X to Y.*

PROOF. (1) $\rightarrow$ (2) Suppose Ω is d_2-complete. Let $\{f_n\}$ be a d_2-Cauchy sequence in Ω. Then $\{f_n\}$ is d_2-convergent, and since d_1 and d_2 induce the same topology, $\{f_n\}$ is d_1- convergent. Thus $\{f_n\}$ is d_1-Cauchy. (2) $\rightarrow$ (3) Suppose $\{f_n\}$ in Ω converges in the Hausdorff metric to a closed set $E \subset X \times Y$. Then $\{f_n\}$ is d_2-Cauchy and is therefore by assumption d_1-Cauchy. Since $\langle C(X, Y), d_1\rangle$ is complete, $\{f_n\}$ d_1-converges to a continuous function f. Since it now follows that $\lim_{n\to\infty} h_\rho(f_n, f) = 0$ and the graph of f is a closed set, we conclude that $E = f$. (3) $\rightarrow$ (1) Let $\{f_n\}$ be a d_2-Cauchy sequence in Ω. Since $\langle X \times Y, \rho\rangle$ is a complete metric space, Lemma 1 implies that $\langle K(X \times Y), h_\rho\rangle$ is a complete metric space. Now each f_n, viewed as a subset of $X \times Y$, is compact; so, $\{f_n\}$ h_ρ-converges to a compact set E. By assumption E is the graph of a function f, and the compactness of its graph implies $f \in C(X, Y)$. Since Ω is closed, Ω is thus complete.

It is important to note that without completeness of Y, a function with closed graph that is the h_ρ-limit of a sequence in $C(X, Y)$ need not be continuous.

EXAMPLE 2. Let $X = \{1/n : n \in Z^+\} \cup \{0\}$ and let $Y = [0, 1)$, both viewed as subspaces of the line. Define for each $n \in Z^+$ a function $f_n \in C(X, Y)$ by

$$f_n(x) = \begin{cases} 1 - \dfrac{1}{k} \text{ if } x = \dfrac{1}{k} \text{ for some } k \leq n \\ 0 \quad \text{otherwise} \end{cases}$$

Then $\{f_n\}$ converges in the Hausdorff metric to the following discontinuous function:

https://doi.org/10.4153/CMB-1986-073-5 Published online by Cambridge University Press

$$f(x) = \begin{cases} 1 - \dfrac{1}{k} & \text{if } x = \dfrac{1}{k} \text{ for some } k \\ 0 & \text{if } x = 0 \end{cases}$$

To characterize the d_2-totally bounded subsets of $C(X, Y)$ we prove a more general lemma that extends results in [5] and [6].

LEMMA 2. *Let $\langle W, d\rangle$ be a metric space and let Ω be a collection of totally bounded subsets of W. The following are equivalent:* (1) $\cup\,\Omega$ *is a totally bounded subset of W* (2) Ω *is h_d-totally bounded.*

PROOF. Suppose $\cup\,\Omega$ is totally bounded. Let $\epsilon > 0$. We can find a finite subset F of $\cup\,\Omega$ such that $\cup\,\Omega \subset S_\epsilon[F]$. Let Σ denote the set of finite subsets of F. We claim that if $C \in \Omega$ there exists $K \in \Sigma$ for which $h_d(K, C) \leq \epsilon$. To see this since $C \subset S_\epsilon[F]$, there exists a minimal subset K *of* F (with respect to inclusion) such that $C \subset S_\epsilon[K]$. The minimality of K implies that $K \subset S_\epsilon[C]$ whence $h_d(C, K) \leq \epsilon$. Since Σ is finite it is clear that there is now a corresponding finite subset Σ^* of Ω such that each member of Ω has Hausdorff distance at most 2ϵ from some member of Σ^*. We conclude that Ω is h_d-totally bounded. Conversely, suppose Ω is h_d-totally bounded. Let $\epsilon > 0$ and choose a finite subset Σ of Ω such that for each C in Ω there exists K in Σ such that $h_d(C, K) < \epsilon/2$. For each K in Σ choose a finite subset A_k for which $K \subset S_{\epsilon/2}[A_k]$. Clearly $\cup\,\Omega \subset S_\epsilon[\cup\{A_K : K \in \Sigma\}]$, and we conclude that $\cup\,\Omega$ is a totally bounded subset of W.

As an immediate corollary of Lemma 2 we have

THEOREM 2. *Let $\langle X, d_X\rangle$ be a compact metric space and let $\langle Y, d_Y\rangle$ be an arbitrary metric space. Then $\Omega \subset C(X, Y)$* is d_2*-totally bounded if and only if $\{(x, f(x)) : x \in X$ and $f \in \Omega\}$ is a totally bounded subset of $X \times Y$.*

Theorems 1 and 2 together yield our variant of the Arzela-Ascoli Theorem.

THEOREM 3. *Let $\langle X, d_X\rangle$ be a compact metric space and let $\langle Y, d_Y\rangle$ be a complete metric space. Let Ω be a subset of $C(X, Y)$. Then Ω is d_1-compact if and only if* (1) *Ω is d_1-closed and whenever $\{f_n\}$ is a sequence in Ω convergent in the Hausdorff metric to a closed subset E of $X \times Y$, then E is the graph of a function* (2) *$\{(x, f(x)) : x \in X$ and $f \in \Omega\}$ is a ρ-totally bounded subset of $X \times Y$.*

PROOF. The collection Ω is d_1-compact if and only if Ω is d_2-compact, and Ω is d_2-compact if and only if Ω is d_2-complete and d_2-totally bounded.

It is illuminating to derive the sufficiency of equicontinuity and pointwise total boundedness for the compactness of closed subsets of $C(X, Y)$ from Theorem 3. Equicontinuity alone implies that the identity map from $\langle \Omega, d_2\rangle$ to $\langle \Omega, d_1\rangle$ is uniformly continuous, a stronger property than condition (2) of Theorem 1, equivalent to condition (1) of Theorem 3. To see this let $\epsilon > 0$ and choose $\delta \in (0, \epsilon/2]$ such that for each $f \in \Omega$ whenever $d_X(x, w) < \delta$ then $d_Y(f(x), f(w)) < \epsilon/2$. Suppose now that $\{f, g\} \subset \Omega$ and $d_2(f, g) < \delta$, i.e., both $g \subset S_\delta[f]$ and $f \subset S_\delta[g]$. Fix x in X. Since $(x, g(x)) \in S_\delta[f]$ there exists $z \in X$ for which $\rho[(x, g(x)), (z, f(z))] < \delta$. Since $d_X(x, z) < \delta$, we have $d_Y(f(x), f(z)) < \epsilon/2$ and it follows that $d_Y(f(x), g(x)) < \epsilon$. Thus in Ω $d_2(f, g) < \delta$ implies $d_1(f, g) < \epsilon$. To show that the union of the graphs of the functions in Ω is ρ-totally bounded requires both conditions. We shall apply Lemma 2 in an unexpected way. For each x in X let $\Omega_x = \{f(x) : f \in \Omega\}$. Since each set Ω_x is totally bounded, so is $B_x = \{x\} \times \Omega_x$ as a subset of $X \times Y$; indeed $\langle \Omega_x, d_Y \rangle$ and $\langle B_x, \rho \rangle$ are isometric. Thus if we can show that $\{B_x : x \in X\}$ is h_ρ-totally bounded, then by Lemma 2 $\cup \{B_x : x \in X\} = \{(x, f(x)) : x \in X \text{ and } f \in \Omega\}$ will be totally bounded. To this end let $\epsilon > 0$ and choose $\delta < \epsilon$ such that for each f in Ω whenever $d_X(x, w) < \delta$ then $d_Y(f(x), f(w)) < \epsilon$. Since X is compact there is a finite subset F of X such that $X \subset S_\delta[F]$. We claim that for each x in X there exists w in F for which $h_\rho(B_x, B_w) \leq \epsilon$: simply choose w such that $d_X(x, w) < \delta$. If $(x, y) \in B_x$ there exists $f \in \Omega$ such that $y = f(x)$; evidently $(w, f(w)) \in B_w$ and $\rho[(x, y), (w, f(w))] < \epsilon$. Thus $B_x \subset S_\epsilon[B_w]$, and in the same way $B_w \subset S_\epsilon[B_x]$.

Pointwise total boundedness of a class of continuous functions Ω is a weaker condition than total boundedness of Ω relative to the Hausdorff metric, whereas the d_2-completeness of Ω is a weaker condition than equicontinuity. The two weaker conditions combined do not ensure compactness of Ω.

EXAMPLE 3. For each $n \in Z^+$ let $f_n \in C([0, 1], R)$ be the function whose graph consists of the line segment joining $(0, 0)$ to $(1/2n, n)$, the one joining $(1/2n, n)$ to $(1/n, 0)$, and the one joining $(1/n, 0)$ to $(1, 0)$. Then $\Omega = \{f_n : n \in Z^+\}$ is pointwise totally bounded and is d_2-complete (since it admits no Cauchy sequences other than the eventually constant ones), but Ω is noncompact.

REFERENCES

1. M. Atsuji, *Uniform continuity of continuous functions of metric spaces,* Pacific J. Math. **8** (1958), 11–16.

2. G. Beer, *On uniform convergence of continuous functions and topological convergence of sets,* Canad. Math. Bull. **26** (1983), 418–24.

3. ———, *More on convergence of continuous functions and topological convergence of sets,* Canad. Math. Bull. **28** (1985), 52–59.

4. ———, *Metric spaces on which continuous functions are uniformly continuous and Hausdorff distance,* Proc. Amer. Math Soc. **95** (1985), 653–58.

5. C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions,* Springer-Verlag, Berlin, 1977.

6. K. Kuratowski, *Topology,* Vol. 1., Academic Press, New York, 1966.

7. S. Naimpally, *Graph topology for function spaces,* Trans. Amer. Math. Soc. **123** (1966), 267–72.

8. B. Penkov and Bl. Sendov, *Hausdorffsche metrik und approximationen*, Num. Math **9** (1966), 214–26.

9. Bl. Sendov, *Hausdorff approximations*, Bolgar. Acad. Nauk., Sofia, 1979 (Russian).

10. K. Stromberg, *An introduction to classical real analysis*, Wadsworth, Belmont, CA, 1981.

DEPARTMENT OF MATHEMATICS
CALIFORNIA STATE UNIVERSITY, LOS ANGELES
LOS ANGELES, CALIFORNIA 90032

https://doi.org/10.4153/CMB-1986-073-5 Published online by Cambridge University Press