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A NOTE ON THE SOLUTION OF THE ONE-DIMENSIONAL UNSTEADY EQUATIONS OF ARTERIAL BLOOD FLOW BY THE METHOD OF CHARACTERISTICS

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Abstract

The "Hartree hybrid method" has recently been employed in one-dimensional non-linear aortic blood-flow models, and the results obtained appear to indicate that shock-waves could only form in distances which exceed physiologically meaningful values. However, when the same method is applied with greater numerical accuracy to these models, the existence of a shock-wave in the vicinity of the heart is predicted. This appears to be contrary to present belief.

In the past two decades, one-dimensional non-linear models of arterial blood flow have received much attention. These models predict a system of non-linear hyperbolic equations for the velocity and pressure of the blood in the aorta, which are well suited to solution by the method of characteristics. They were originally solved by Lambert [3] using the natural grid of characteristics, but the clumsiness of this method has since caused it to fall into disuse. Subsequent models [1, 2, 5, 6, 7] have featured the solution of the one-dimensional blood-flow equations by an "hybrid method" apparently due to Hartree. The Hartree hybrid method, or method of specified time intervals [4], combines the features of the method of characteristics with those of conventional finite-difference techniques, for, although the original partial differential equations are still required to be reduced to normal form, interpolative procedures are used to ensure that the solution is presented only at regularly spaced lattice-points, as it would be if a straight finite-difference solution had been attempted.

When the Hartree scheme is applied to problems in which the dependent variables are known to be continuous functions, the numerical results obtained show a strong artificial diffusion whose severity depends on the grid sizes Δx and Δt. As these grid sizes are reduced, convergence to the exact solution is observed. However, when the method is applied to problems in which the solution is known to possess discontinuities, such as gas-dynamic problems with embedded shocks, the method will again produce a numerical damping effect, but when the grid sizes are reduced, convergence to the exact answer is not observed. Instead, in the approximate vicinity of the shock, the Hartree scheme inserts large finite gradients in the dependent variables. It can be demonstrated that these gradients may be made arbitrarily large by choosing sufficiently small Δx and Δt, although, in general, the height of the jump is not the same as its correct value. On the basis of these observations, we make the following conjecture for the limit Δx, $\Delta t \to 0$; the Hartree hybrid method is incapable of correctly computing irreversible discontinuities without the explicit introduction of "Rankine–Hugoniot" jump conditions. However, in the approximate vicinity of the irreversible discontinuity, the Hartree method will place a reversible one. The presence of the reversible discontinuity in the numerical results is an indication that the correct solution to the original partial differential equations contains a shock. We now consider the possibility of shock-waves in one-dimensional non-linear models of arterial blood flow as revealed by the Hartree method.

The major success of non-linear blood-flow models is their apparent ability to predict the observed steepening of the pulse from the heart as it proceeds down the aorta. Anliker *et al.* [1] have claimed that, although the blood-flow equations predict this steepening, the original signal from the heart is neither sufficiently steep nor strong to produce a shock-wave within the physical dimensions of the body. We shall now show that results qualitatively similar to those of Anliker *et al.* [1] may be obtained when large grid sizes Δx and Δt are chosen, but by simply refining the numerical mesh, the existence is made evident of a pulse close to the heart for which the pressure and velocity gradients at the beginning of the cardiac cycle are greatly increased. As there is every indication that these gradients may be made arbitrarily large by further reducing the grid spacing, we are justified in assuming that the blood-flow equations predict the formation of a shock-wave close to the heart.

In the one-dimensional model, the flow of blood in the aorta is described by the continuity and momentum equations

$$\frac{\partial A}{\partial t}+\frac{\partial(uA)}{\partial x}+\psi=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}=f, \tag{2}$$

subject to a constitutive relation of the form

$$A = A(p, x), \tag{3}$$

where $u(x, t)$ and $p(x, t)$ are the velocity and pressure of the blood inside the vessel, ρ is the density of blood and A is the cross-sectional area of the aorta. The function $\psi(p, x)$ is designed to simulate the outflow of blood from the aorta into discrete side-branches, and $f(u, A)$ is chosen to model the effect of frictional forces. We choose to retain the equations in their dimensional form, to facilitate comparison with the results of other authors.

The characteristics of equations (1)–(3) are

$$\frac{dx}{dt} = u \pm c \tag{4}$$

along which the compatibility conditions

$$du \pm \frac{1}{\rho c} dp = \left[f \mp \frac{uc}{A} \left(\frac{\partial A}{\partial x} \right)_p \mp \frac{c}{A} \psi \right] dt \tag{5}$$

apply, where the wave-speed $c(p, x)$ is defined as

$$c(p, x) = \left[\frac{A}{\rho (\partial A / \partial p)_x} \right]^{\frac{1}{2}}. \tag{6}$$

The plus sign in equation (4) applies to the forward running RQ characteristic in Fig. 1, and the minus sign to the backward running SQ characteristic. In Fig. 1, it is assumed that the values of u and p are known at the points A, C and B, and the aim is to find their values at the point Q, thereby marching the solution forward to the next time step. It is first necessary to compute the values of u and p at the points R and S, and for the Hartree method of first-order accuracy, this may be achieved by using equations (4) and linear interpolation. To first order, the following formulae for u and p at the points R and S are valid;

$$\left. \begin{aligned} u_R &= u_c[1 - (u+c)_c \theta] + u_A \theta (u+c)_c, \\ p_R &= p_c[1 - (u+c)_c \theta] + p_A \theta (u+c)_c, \\ u_S &= u_c[1 + (u-c)_c \theta] - u_B \theta (u-c)_c, \\ p_S &= p_c[1 + (u-c)_c \theta] - p_B \theta (u-c)_c, \end{aligned} \right\} \tag{7}$$

where $\theta = \Delta t / \Delta x$.

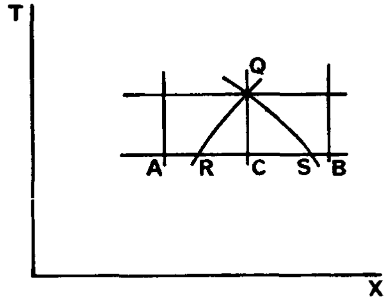

Fig. 1. Rectangular lattice and characteristics in the Hartree scheme.

These results are derived in detail elsewhere [6, 7]. Equations (5) must now be integrated along the RQ and SQ characteristics and solved for u and p at the point Q. This yields

$$\begin{aligned} u_Q &= \tfrac{1}{2}[u_R+u_S+(p_R-p_S)/\rho c_c]+f_c\,\Delta t, \\ p_Q &= \tfrac{1}{2}[p_R+p_S+(u_R-u_S)\,\rho c_c]-\rho c_c\left[\frac{uc}{A}\left(\frac{\partial A}{\partial x}\right)_p+\frac{c}{A}\psi\right]_c\Delta t. \end{aligned} \tag{8}$$

The algorithm described by equations (7) and (8) constitutes the first-order Hartree hybrid scheme, and is stable provided that the classical Courant–Friedrichs–Lewy condition is satisfied.

The functions $c(p,x)$, $A(p,x)$ and $\psi(p,x)$ used in the present problem were all taken from reference [1], and apparently represent approximations to physiological data obtained from experiments performed on dogs. The Poiseuille formula for the viscous force on steady flow in a pipe [1] was chosen as the expression for $f(u,A)$. At the proximal boundary, a periodic volume flow rate similar to that used by Anliker *et al.* [1] was specified, whilst the distal boundary condition was satisfied by utilizing the concept of peripheral resistance [1]. The length of the aorta was chosen to be 100 cm, and the quiescent initial conditions $p = 25$ mm Hg and $u = 0$ cm/sec throughout the entire aorta were used to start the computation, which was then allowed to continue until a total of three cardiac cycles had been completed. It is assumed here that the results obtained at the third cardiac cycle may be seen as representative of the steady-state behaviour of the system, and, indeed, there appears to be an approach with increasing number of cardiac cycles to a limiting behaviour similar to that at the third cardiac cycle. In view of the large amount of computing time required to run this problem with fine numerical grid spacing, however, the computation was not continued for this case beyond the third cycle. It is of some interest to speculate as to whether it is possible in general to begin with arbitrary initial conditions and to achieve a steady-state situation by computing a large number of cardiac cycles, for if, as is suggested in the present note, the equations of motion predict the existence of a shock-wave in the aorta but the numerical solution technique does not adequately account for the shock, then it is not clear that a steady-state situation will ever be reached with large time. Alternatively, if a steady-state situation is achieved, as appears to be the case in the present example, it is doubtful that this will reflect the true steady state which presumably is predicted by the differential equations of motion.

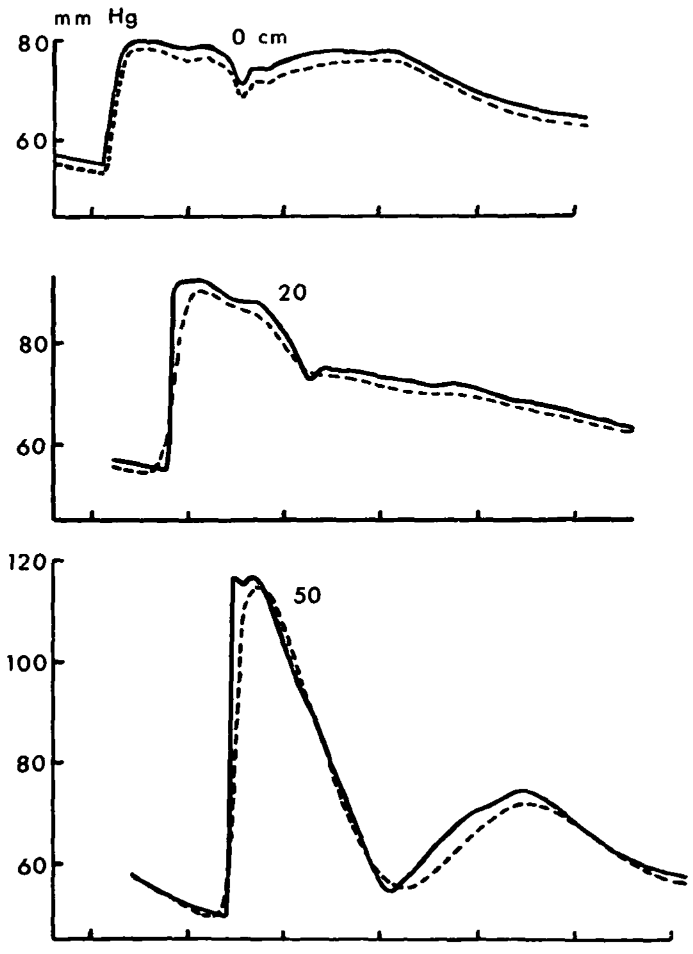

Fig. 2. Pressure–time profiles for the third cardiac cycle at the postions $x = 0, 20, 50, 80, 100$ cm along the aorta as predicted by the one-dimensional blood flow model. The broken lines are the graphs obtained with coarse grid spacing, the solid lines are those obtained with fine grid spacing.

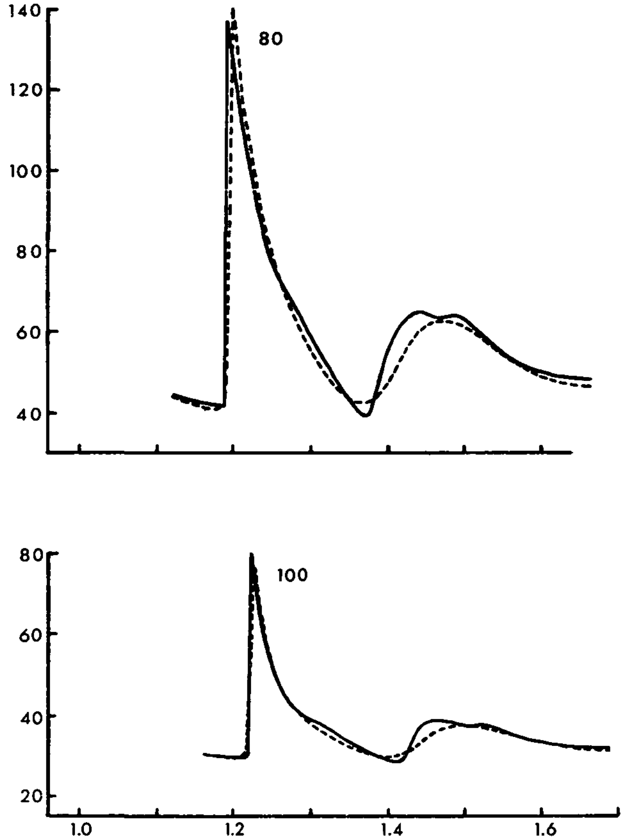

Fig. 2 (cont.)

The results for two different values of grid spacing are shown in Figs 2 and 3, for the third cardiac cycle, and for the five different positions $x = 0, 20, 50, 80, 100$ cm along the aorta. The graphs obtained with the coarse grid spacing $\Delta x = 2.0$ cm, $\Delta t = 0.001$ sec (shown as broken lines in Figs. 2 and 3) are qualitatively similar to results presented by Anliker *et al.* [1] and give the impression that, although wave-steepening is evident, the possibility of shock formation is indeed remote. However, the more accurate graphs, obtained with $\Delta x = 0.25$ cm, $\Delta t = 0.000125$ sec (shown as solid lines) indicate the existence of greatly increased pressure and velocity gradients at the beginning of the cardiac cycle, which, by virtue of the preceding conjecture, may be taken as being indicative of the presence of a shock. The difference between the the results obtained with the coarse and fine grid spacings is greatest close to the heart, at $x = 20$ cm, but the graphs become more similar as one proceeds toward the distal boundary, at $x = 100$ cm. This would appear to be due to the effects of numerical diffusion which become more significant the further a signal from the heart propagates down the aorta. The existence of a shock has been confirmed by independently calculating the physical characteristics of the present problem and observing that these coalesce in the vicinity of the heart, signifying shock formation. Details of this work will be presented in a future publication of the author. As shocks are not actually observed in the aorta under

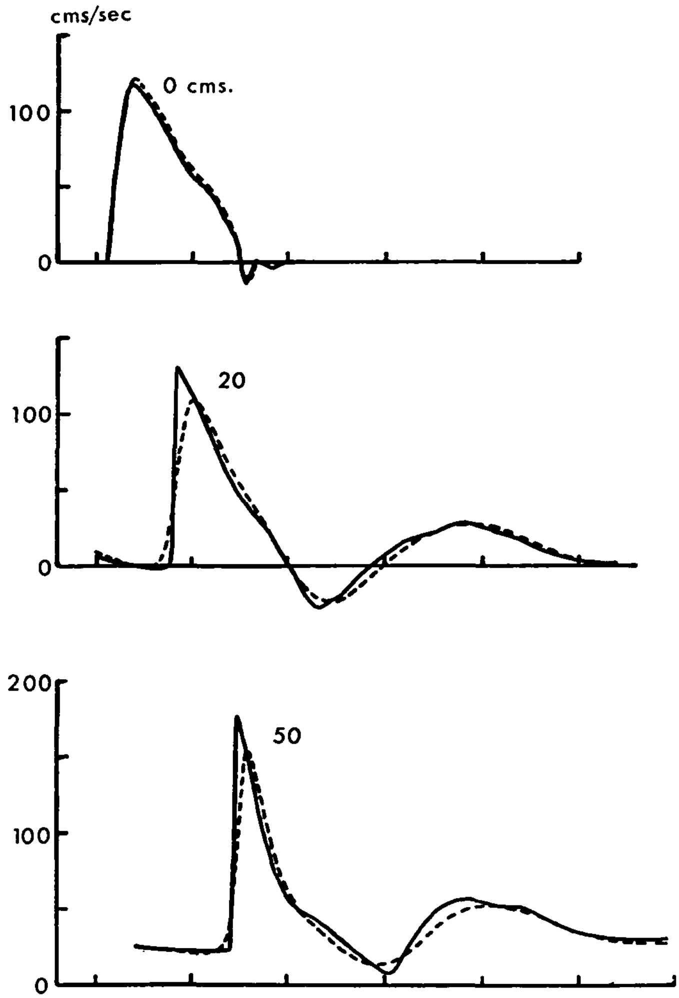

Fig. 3. Velocity–time profiles for the third cardiac cycle at the positions $x = 0, 20, 50, 80$ 100 cm along the aorta as predicted by the one-dimensional blood flow model. The broken lines are the graphs obtained with coarse grid spacing, the solid lines are those obtained with fine grid spacing.

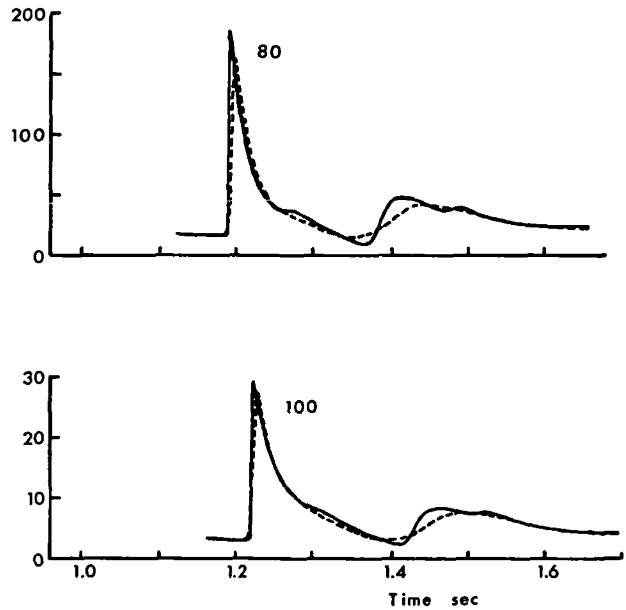

Fig. 3 (cont.)

normal circumstances, we must conclude that the original equations (1), (2) and (3), together with the formulae for c, ψ and f used in this study, do not adequately represent the situation existing in large arteries.

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