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PRODUCT-FORM DISTRIBUTIONS FOR AN $M/G/k$ GROUP-ARRIVAL GROUP-DEPARTURE LOSS SYSTEM

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Abstract

In this work we investigate under what circumstances the equilibrium distribution of the numbers of groups of various sizes in a certain $M/G/k$ group-arrival group-departure loss system can be obtained in a closed product form.

1. Introduction

The following service system is considered. Groups of customers arrive at a service station in accordance with a Poisson process of rate λ. The sizes of successive arriving groups are independent identically distributed random variables and independent of the arrival times. Let (g_j) $j = 1, 2, \cdots$ be the group size probability distribution and $q_j = g_j + g_{j+1} + \cdots$ $(j = 1, 2, \cdots)$ be the corresponding tail probabilities.

The service is rendered by a number k of identical servers, each capable of serving one customer at a time, and the system operates as follows. Whenever a group of size j $(j = 1, 2, \cdots)$ finds on its arrival n $(n = 0, 1, \cdots, k)$ servers busy, then a number $\min(j, k - n)$ of its members, chosen at random, occupy an equal number of idle servers and start being served. The remaining customers if any, are rejected by the system (are blocked) and do not return later. Hence no queue is allowed to form and the blocked customers are considered as lost to the system.

We assume that the accepted customers of each group have equal service times with probability 1 and therefore depart together from the system. Moreover this common service time may depend upon their number (effective group size), but not upon the service times of other accepted customers. Hence let $B_j(x)$ be the service time distribution function for groups of effective size j $(j = 1, 2, \cdots, k)$ and b_j be the corresponding mean value which is supposed to be finite.

The service system described above will be referred to as an $M/G/k$ group-arrival group-departure loss system; when $g_1 = 1$, i.e. when customers arrive singly, it will be called an $M/G/k$ loss system. This system, apart from its theoretical interest, is also of practical value since it can represent real-life service systems in which customers may arrive in groups, such as first-aid stations, hotels, cafeterias, etc..

The $M/G/k$ loss system as well as several variants of it have been extensively studied; see for example Franken et al. (1981). For this system it is known that the equilibrium (limiting) probability distribution (p_n) $n = 0, 1,, \cdots, k$ of the number of busy servers is insensitive to the form of the service time distribution function $B(x)$, depending only on its mean value b through the traffic intensity $\rho = \lambda b$. Specifically this is a Poisson distribution truncated at k with parameter ρ, i.e.

$$(1) \qquad p_n = \frac{\rho^n}{n!} \sum_{v=0}^{k} \frac{\rho^v}{v!} \qquad (n = 0, 1, \cdots, k)$$

which is the Erlang B formula. Systems having this insensitivity property are discussed for example, by Kelly (1979), Schassberger (1986) and Whittle (1986).

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The alternative case of the $M/G/k$ group-arrival group-departure loss system, where the whole group is blocked whenever its size is greater than the number of idle servers, has been studied by Fakinos (1982). Among other results it has been shown that the equilibrium distribution $p(n_1, n_2, \cdots, n_k)$ $(n_j \in \mathbb{N}_0\ j = 1, 2, \cdots, k)$ of the numbers of groups of various sizes in the system is given by

$$(2)\qquad p(n_1, n_2, \cdots, n_k) = C\delta(n_1, n_2 + \cdots + kn_k) \prod_{j=1}^{k} \frac{(g_j\rho_j)^{n_j}}{n_j!} \qquad (n_j \in \mathbb{N}_0; j = 1, 2, \cdots, k)$$

where C is the appropriate normalizing constant, $\delta(n)$ is a function assuming the values 1 and 0 if $n \leqq k$ or $n > k$ respectively and $\rho_j = \lambda b_j$ $(j = 1, 2, \cdots, k)$. Formula (2), which is a generalization of the Erlang B formula, shows that the equilibrium distribution $p(n_1, n_2, \cdots, n_k)$ has the above intrinsic product form for any group-size distribution. In the present work we investigate the circumstances under which a similar product-form equilibrium distribution is obtained for the previously described alternative model.

2. The results

The equilibrium distribution $p(n_1, n_2, \cdots, n_k)$ of the numbers of groups of various effective sizes in the system is not affected if we modify the arrival process so that the group size cannot be greater than k, and is k with probability $g_k + g_{k+1} + \cdots$. Therefore for given (g_j) $j = 1, 2, \cdots,$ we can assume without loss of generality that the group size distribution is (g_j') $j = 1, 2, \cdots, k$ where

$$(3)\qquad g_j' = g_j \qquad (j = 1, 2, \cdots, k-1); \quad g_k' = q_k.$$

But then consider a network of k infinite-server queues where the jth queue accepts exclusively groups of size j $(j = 1, 2, \cdots, k)$. Since the arrival process is a compound Poisson process and customers who belong to the same group have equal service times it follows that the jth queue in isolation behaves as an $M/G/\infty$ single arrival queue with arrival rate $\lambda g_j'$ and mean service time b_j, and moreover its state is independent of the states of the other queues. Hence denoting by $\pi(n_1, n_2, \cdots, n_k)$ the equilibrium distribution of the numbers of groups of various sizes in the network and letting $\rho_j = \lambda b_j$ $(j = 1, 2, \cdots, k)$ we have that

$$(4)\qquad \pi(n_1, n_2, \cdots, n_k) = \prod_{j=1}^{k} \exp(-g_j'\rho_j) \frac{(g_j'\rho_j)^{n_j}}{n_j!} \qquad (n_j \in \mathbb{N}_0; j = 1, 2, \cdots, k).$$

Now the $M/G/k$ group-arrival group-departure loss system can be viewed as such a network of queues but with state-dependent arrival rates $\lambda_j(n_1, n_2, \cdots, n_k)$ for groups of size j, where

$$(5)\qquad \lambda_j(n_1, n_2, \cdots, n_k) = \begin{cases} \lambda g_j' & n^* + j < k \\ \lambda g_j' \quad \text{if} & n^* + j = k \\ 0 & n^* + j > k \end{cases} \qquad (j = 1, 2, \cdots, k)$$

and $n^* = n_1 + 2n_2 + \cdots + kn_k$. Of course $q_j' = q_j (j = 1, 2, \cdots, k-1); q_k' = g_k'$.

The above network of queues with constant arrival rates $\lambda g_j'$ $(j = 1, 2, \cdots, k)$ is quasi-reversible. Therefore based on Kelly (1979), § 3.5, if we can define a function $\psi(n_1, n_2, \cdots, n_k)$ such that

$$(6)\qquad \lambda_j(n_1, n_2, \cdots, n_k) = \frac{\psi(n_1, n_2, \cdots, n_j + 1, \cdots, n_k)}{\psi(n_1, n_2, \cdots, n_j, \cdots, n_k)} \lambda g_j' \qquad (j = 1, 2, \cdots, k; n_j \in \mathbb{N}_0)$$

then

$$(7)\qquad p(n_1, n_2, \cdots, n_k) = B\psi(n_1, n_2, \cdots, n_k)\pi(n_1, n_2, \cdots, n_k)$$

where $\pi(n_1, n_2, \cdots, n_k)$ is given by (4) and B is the appropriate normalizing constant making the above probabilities sum to unity.

Because of (5) and (6), $\psi(.)$ must necessarily be of the form

(8) $$\psi(n_1, n_2, \cdots, n_k) = \begin{cases} 1 & n^* < k \\ \sigma(n_1, n_2, \cdots, n_k) \quad \text{if} & n^* = k \\ 0 & n^* > k \end{cases}$$

where the function $\sigma(.)$, defined only for $n_1, n_2, \cdots, n_k \in \mathbb{N}_0 : n^* = k$, must satisfy the relations

(9) $$\sigma(n_1, n_2, \cdots, n_k)\lambda g'_j = \lambda q'_j \qquad (j = 1, 2, \cdots, k: n_j > 0).$$

Let d' be the greatest common divisor of all j such that $g'_j > 0$. When $d' > 1$ and $k = sd'$ for some $s \in \mathbb{N}$ then, considering each d customers of the same group as a single customer and each d specific servers as a single server, the system is reduced to one with s servers and group-size distribution $\bar{g}_j = g'_{jd'}$ $(j = 1, 2, \cdots, s)$ for which $\bar{d} = 1$. On the other hand, when $d' > 1$ and $k = sd' + t$ for some $s \in \mathbb{N}$ and $t = 1, 2, \cdots, d' - 1$ then it is possible to have $n^* = k$ with $n_t > 0$. But then no function $\sigma(.)$ satisfies (9) since $g'_t = 0$ and $q'_t = 1$. Hence it remains to consider the case $d' = 1$. Since $q'_1 = 1 > 0$ we have from (9) that certainly $g'_1 > 0$. Also, ignoring the trivial case $g_1 = 1$, we have that $l = \max\{j : g'_j > 0\} > 1$. But then it is possible to have $n^* = k$ with $n_{l-i} > 0$ for any $i = 1, 2, \cdots, l - 1$ and since $q'_{l-i} > 0$ from (9) it again follows that necessarily $g'_{l-i} > 0$. Therefore if (9) is to hold for some function $\sigma(.)$, there must necessarily exist an l $(l = 1, 2, \cdots, k)$: $g'_j > 0$ $(j = 1, 2, \cdots, l)$, $g'_j = 0$ $(j = l + 1, \cdots, k)$, and then we obtain the equivalent form

(10) $$\sigma(n_1, n_2, \cdots, n_l, 0, \cdots, 0) = 1 + \frac{q'_{j+1}}{g'_j} \qquad (j = 1, 2, \cdots, l: n_j > 0).$$

From (10) it follows that the only possibility for $\sigma(.)$ is

(11) $$\sigma(n_1, n_2, \cdots, n_l, 0, \cdots, 0) = \begin{cases} 1 & \text{if} \quad n_l > 0 \\ \sigma & \phantom{\text{if}} \quad n_l = 0 \end{cases} \qquad (n_j \in \mathbb{N}_0; j = 1, 2, \cdots, l: n^* = k)$$

which can happen if and only if the ratio q'_{j+1}/g'_j $(j = 1, 2, \cdots, l - 1)$ is constant equal to ξ, say, or equivalently

(12) $$g'_j = \left(\frac{1}{1+\xi}\right)\left(\frac{\xi}{1+\xi}\right)^{j-1} \qquad (j = 1, 2, \cdots, l - 1); \quad g'_l = \left(\frac{\xi}{1+\xi}\right)^{l-1}$$

in which case $\sigma = 1 + \xi$. We claim now that $l = k$ must necessarily hold. In fact if $1 < l < k$, there always exists at least one $j < l$ such that $n^* + j = k$ with $n_l > 0$. But then

(13) $$\begin{aligned} \lambda q'_j &= \lambda_j(n_1, \cdots, n_j, \cdots, n_l, 0, \cdots, 0) \\ &= \frac{\psi(n_1, \cdots, n_j + 1, \cdots, n_l, 0, \cdots, 0)}{\psi(n_1, \cdots, n_j, \cdots, n_l, 0, \cdots, 0)} \lambda g'_j \\ &= \sigma(n_{1,}, \cdots, n_j + 1, \cdots, n_l, 0, \cdots, 0)\lambda g'_j \end{aligned}$$

and because of the form of (11) for $n_l > 0$, it follows that $q'_j = g'_j$ which is impossible. Thus when $d' = 1$ (and $(g'_1 \neq 1)$) a function $\psi(n_1, n_2), \cdots, n_k)$ of the form (6) can be defined if and only if

(14) $$g'_j = (1 - r)r^{j-1} \; (j = 1, 2, \cdots, k - 1); g'_k = r^{k-1} \qquad (0 < r < 1)$$

and then

$$\psi(n_1, n_2, \cdots, n_k) = \begin{cases} 1 & n^* < k \quad \text{or} \quad n_k = 1 \\ \dfrac{1}{1-r} & n^* = k, \quad n_k = 0 \\ 0 & n^* > k. \end{cases} \tag{15}$$

In this case relation (7) holds and taking into account (14) we obtain the formula

$$p(n_1, n_2, \cdots, n_k) = C\psi(n_1, n_2, \cdots, n_k)(1-r)^{\sum_{i=1}^{k-1} n_i} r^{\sum_{i=2}^{k} (i-1)n_i} \prod_{j=1}^{k} \frac{\rho_j^{n_j}}{n_j!} \tag{16}$$

$$(n_j \in \mathbb{N}_0; j = 1, 2, \cdots, k)$$

where $C = p(0, 0, \cdots, 0)$ is chosen so that the sum of the above probabilities is unity.

Summarizing, we have the following interesting result. The equilibrium distribution $p(n_1, n_2, \cdots, n_k)$ is insensitive to the form of the service time distributions and has a product form if and only if the group size distribution (g_j) $j = 1, 2, \cdots$ is such that the modified distribution (g'_j) $j = 1, 2, \cdots, k$ defined by (3) is either a censored geometric distribution of the form (14) or a lattice distribution whose positive probabilities $\bar{g}_j = g'_{jd'}$ $(j = 1, 2, \cdots, s)$ have the form (14) with k replaced by s, and $k = sd'$.

Formula (16) can be used in telephone networks. Specifically, consider a network of k telephone lines where calls arrive in accordance with a Poisson process of rate λ while their durations are independent and identically distributed random variables. As is well known, whenever a call is connected, there always exists a possibility of 'double coverage', that is with a very small but positive probability r each such call forces one more idle line to become engaged and to remain at that state up to the end of the call. By making the obvious distinction between 'single', 'double', etc. calls it is clear that the telephone network can be represented (at least to a first approximation) as an $M/G/k$ group-arrival, group-departure loss system of the above form where of course $\rho_j = \rho$ $(j = 1, 2, \cdots, k)$.

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