Can. J. Math., Vol. XXIV, No. 5, 1972, pp. 947–956

EXTENSIONS OF HOMOMORPHISMS OF PARABOLIC SUBGROUPS

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1. Introduction and definitions. If one wishes to construct a homomorphism of a group, an obvious approach is to begin with a homomorphism of a subgroup and attempt to extend it in some way. Any such extended homomorphism is determined by its action on elements of the group which, together with the subgroup, generate the whole group. The values assigned to such generators must then satisfy various functional equations. By a *governing list* for the extension problem, we shall mean a list of functional equations whose solutions describe all possible homomorphisms extending a given subgroup homomorphism.

Let G be a group and B a subgroup with presentation $(R; X)$, i.e., R is a set of generators for B and X is a set of relations on R which define B. Let S be a set of elements in G such that $R \cup S$ is a set of generators for G. A governing list for the pair (G, B) corresponds to a set of relations Y on $R \cup S$ such that $(R \cup S; X \cup Y)$ is a presentation of G. Since the approach here is directed towards extending a homomorphism of B to a homomorphism of G, we shall usually take $(R; X)$ to be the trivial group-table presentation of B.

Our primary interest is in obtaining governing lists convenient from the point of view of representation theory. To illustrate our criteria of desirability, let G, B and S be as above, and suppose that $f : B \rightarrow H$ is a homomorphism. Select a set of elements $H_S = \{h_s : s \in S\}$. Suppose there exists a homomorphism $\tilde{f} : G \rightarrow H$ with $\tilde{f}(b) = f(b)$ for all $b \in B$ and $\tilde{f}(s) = h_s$ for all $s \in S$. For each $s \in S$, define $N_s = B \cap s^{-1}Bs$. Then for $n \in N_s$, we have the functional relation

$$\tilde{f}(sn) = h_s f(n) = f(sns^{-1})h_s.$$

In other words, h_s intertwines the homomorphisms $n \rightarrow f(n)$ and $n \rightarrow f(sns^{-1})$ of N_s. Conditions of this form in a governing list will be denoted by (I) and will be referred to as *intertwining conditions*. If $s_i \in S$ and $b_i \in B$ $(i = 1, 2, 3)$ with $s_1b_1s_2 = b_2s_3b_3$, then we must also have $h_{s_1}f(b_1)h_{s_2} = f(b_2)h_{s_3}f(b_3)$. Conditions of this form involving three letter words in the generators of G will be referred to as triple conditions and will be denoted by (T). In representation theory, *triple conditions* are often much more difficult to analyze than intertwining conditions. An attempt is made here to exhibit governing lists involving a minimal number of triple conditions.

Received September 23, 1971.

https://doi.org/10.4153/CJM-1972-095-4 Published online by Cambridge University Press

A more obvious criterion of desirability is of course the size of the set S. If S is taken to be a set of (B, B) double coset representatives of G, it is straightforward to check that the extension process is governed by an intertwining condition for each $s \in S$ and triple conditions involving a set of representatives of $(N_s, N_{s'})$ double cosets of B for each pair $(s, s') \in S \times S$. In section 2 we investigate a case when S may be taken to be a singleton set. In sections 3 and 4, we investigate a case when the double coset space $W = B\backslash G/B$ has a group structure. In particular, this situation arises when G is a semi-simple Lie group and B a parabolic subgroup. W is commonly called the Weyl group of the pair (G, B). In addition to intertwining and triple conditions, the associated governing lists contain equations arising from a presentation of W. Such conditions will be denoted by (W).

2. Governing lists for subgroups with large double coset. Let G be a group and A a subset of G. Set $A^{-1} = \{g \in G : g^{-1} \in A\}$ and for $x \in G$, $Ax = \{g \in G : gx^{-1} \in A\}$. A is said to be *large* in G if $A^{-1} \cap Ax \cap Ay \cap Az \neq \emptyset$ for all x, y, z in G. If G is a locally compact group and dx is a (right or left) Haar measure on G, then a subset is large in G if its complement has zero measure.

Let A be large in G and let f be a mapping of A into a group H. Weil has shown [**4**] that f extends to a homomorphism of G into H if and only if $f(a) = f(a_1)f(a_2)$ whenever a, a_1, and a_2 are in A and $a = a_1a_2$.

Let G be a group, B a subgroup and suppose there exists $p \in G$ such that $A = BpB$ is large in G. Let $C = B \cap p^{-1}Bp$, $\tilde{C} = pCp^{-1}$, and $B = \bigcup_{j \in J} Cv_j\tilde{C}$ a union of distinct double cosets. Let $J' = \{j \in J : Cv_j\tilde{C} \subset A\}$. For each $j \in J'$ select elements v_j' and v_j'' in B such that $pv_jp = v_j'pv_j''$.

THEOREM 1. *With the notations above, let $f : B \rightarrow H$ be a homomorphism. Then f extends to a homomorphism $\tilde{f} : G \rightarrow H$ with $\tilde{f}(p) = h$ if and only if h satisfies the following list of conditions*:

(I) *$hf(c) = f(\tilde{c})h$ for $c \in C$ and $\tilde{c} = pcp^{-1}$.*

(T) *$hf(v_j)h = f(v_j')hf(v_j'')$ for $j \in J'$.*

When h satisfies (I) *and* (T), *$\tilde{f}$ is uniquely determined by $\tilde{f}(p) = h$.*

Proof. Necessity of the conditions (I) and (T) is obvious. It is immediate from the definition of a large subset that $A^2 = G$. Hence G is generated by B and p and the uniqueness statement is trivial.

Suppose h satisfies (I) and (T). Check that $b_1pb_2 = b_1'pb_2'$ implies that $b_2(b_2')^{-1} \in C$. From (I), it follows that for $a = b_1pb_2 \in A$, $a \rightarrow \tilde{f}(a) = f(b_1)h\, f(b_2)$ is a well defined mapping of A into H. It remains to check that $\tilde{f}$ is multiplicative. But if $a_i = b_ipb_i'\,(i = 1, 2)$ and $a = a_1a_2 \in A$, then $b_1'b_2 \in Cv_j\tilde{C}$ for some $j \in J'$. It is straightforward to check using (I) and (T) that under these conditions, $\tilde{f}(a) = \tilde{f}(a_1)\tilde{f}(a_2)$. The theorem now follows from the result of Weil quoted above.

We now consider a family of examples for which the hypotheses of Theorem 1 are satisfied. Let G be a connected semi-simple Lie group and $B = MAN$, a minimal parabolic subgroup (see section 5 for definitions). If $q \in W$ is the element of the Weyl group mapping all positive roots to negative roots, then $G - BqB$ is a finite union of manifolds of dimension less than the dimension of G. Hence $L = BqB$ is large in G. It is easy to check that in this situation $C = \tilde{C} = MA$. However in general, the index set J' of Theorem 1 will be infinite and hence the governing list which results will be quite complicated. If $\tilde{B}$ is any parabolic subgroup, i.e., if $\tilde{B}$ is a subgroup containing a conjugate of B, then for some $\tilde{p} \in G$, $\tilde{L} = \tilde{B}\tilde{p}\tilde{B}$ will be large in G. We now describe several cases where the resulting triple conditions are finite in number.

Let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. For $n \geqq 1$, set

$$p_1 = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \text{ and } p_2 = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$$

where I_n is the $n \times n$ identity matrix. For any matrix g, let ${}^tg$ (respectively, $^*g$) denote the matrix transpose (respectively, matrix adjoint) of g. Now set

$$\mathrm{Sp}(n, \mathbf{F}) = \{g \in \mathrm{GL}(2n, \mathbf{F}) : {}^tgp_1g = p_1\},$$
$$\mathrm{U}(n, n) = \{g \in \mathrm{GL}(2n, \mathbf{C}) : {}^*gp_1g = p_1\},$$

and for n even,

$$\mathrm{SO}(n, n, \mathbf{F}) = \{g \in \mathrm{SL}(2n, \mathbf{F}) : {}^tgp_2 g = p_2\}.$$

In the following discussion, G will denote one of the groups $\mathrm{GL}(2n, \mathbf{F})$, $\mathrm{SL}(2n, \mathbf{F})$, $\mathrm{Sp}(n, \mathbf{F})$, $\mathrm{U}(n, n)$, $\mathrm{SU}(n, n) = \mathrm{U}(n, n) \cap \mathrm{SL}(2n, \mathbf{C})$, or $\mathrm{SO}(n, n, \mathbf{F})$. Then G is a connected reductive Lie group. Denote the elements of G in block form, e.g.,

$$g = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \text{ for } x, y, z, w \text{ in } \mathbf{F}^{n\times n}.$$

Set

$$C_0 = \{c(x, w) = \begin{bmatrix} x & 0 \\ 0 & w \end{bmatrix} : x, w \in \mathrm{GL}(n, \mathbf{F})\},$$

$$V_0 = \{v(z) = \begin{bmatrix} I_n & 0 \\ z & I_n \end{bmatrix} : z \in \mathbf{F}^{n\times n}\}$$

$$C = C_0 \cap G,$$

$$V = V_0 \cap G,$$

and

$$B = CV.$$

Then B is a semi-direct product of C and V. For $p = p_2$ in the case $G = \mathrm{SO}(n, n, \mathbf{F})$ and $p = p_1$ in the other cases, $L = BpB = BpV$ is large in G. Indeed, it is routine to check that $\dim_{\mathbf{R}} L = \dim_{\mathbf{R}} G$. If B_0 is a minimal parabolic subgroup of G contained in B and $L_0 = B_0qB_0$ is the large set described above, it follows that $L \supset L_0$ and hence that L is large in G. Trivially, $C = B \cap p^{-1}Bp = pCp^{-1}$. The decomposition of B into (C, C) double cosets is described by the orbits of the action $c(x, w) \cdot v(z) = v(w^{-1}zx)$ of C on V. By an easy matrix calculation, if $pv(z)p \in L$, then $\det z \neq 0$. A double coset $Cv(z)C$ will be called *non-singular* if $\det z \neq 0$. From linear algebra, there are only finitely many non-singular double cosets. Indeed, for $G = \mathrm{GL}(2n, \mathbf{F})$ or $\mathrm{SL}(2n, \mathbf{F})$ this is trivial. For $G = \mathrm{Sp}(n, \mathbf{R})$, $v(z) \in V$ if and only if $z = {}^tz \in \mathbf{R}^{n\times n}$ and $c(x, w) \in C$ if and only if $x \in \mathrm{GL}(n, \mathbf{R})$ and $w = {}^tx^{-1}$. The action of C on V corresponds to the change of basis of a symmetric bilinear form on $\mathbf{R}^n$. Set

$$z_j = \begin{bmatrix} -I_j & 0 \\ 0 & I_{n-j} \end{bmatrix} \text{ for } j = 0, 1, \ldots, n.$$

Then the double cosets $Cv(z_j)C$ $(j = 0, 1, \ldots, n)$ exhaust the non-singular double cosets of B. The other groups may be treated similarly.

From Theorem 1 it follows that a homomorphism f of B into H extends to a homomorphism $\tilde{f}$ of G into H with $\tilde{f}(p) = h$ if and only if h satisfies the intertwining condition (I) on C and triple conditions (T) for the representatives of the non-singular double cosets of B.

3. Structure of Tits systems with finite Weyl group. A quadruplet (G, B, M', S) is said to be a *Tits system* if G is a group, B and M' are subgroups of G, S is a subset of $M'/B \cap M'$, and the following axioms are satisfied:

(T1) G is generated by $B \cup M'$ and $M = B \cap M'$ is a normal subgroup of M';

(T2) S is a set of involutive generators of $W = M'/M$;

(T3) $sBs \neq B$ for all $s \in S$;

(T4) $sBp \subset BsB \cup BspB$ for all $(s, p) \in S \times W$.

If $X_1, X_2, \ldots, X_k$ are subsets of W and $B_1, B_2, \ldots, B_{k+1}$ are subsets of B, expressions of the form $B_1X_1B_2X_2 \ldots B_kX_kB_{k+1}$ are understood to mean the set of all quantities $b_1m_1'b_2m_2' \ldots b_km_k'b_{k+1}$ where $b_j \in B_j$ $(j = 1, 2, \ldots, k+1)$, $m_j' \in M'$ and $m_j'M \in X_j$ $(j = 1, 2, \ldots, k)$.

We now summarize some of the consequences of these axioms. For proofs of the results quoted, see Bourbaki [**1**, pp. 9–36]. For s and s' in S, let $n(s, s')$ denote the order of ss' in W (in general, $n(s, s')$ may be infinite). Then S and the set of relations $(ss')^{n(s,s')} = 1$ for $n(s, s') < \infty$ form a presentation of the group W. For $p \in W$, let $l(p)$ denote the length of p, i.e., the smallest integer k for which p may be written as a product of k elements of S. Then (T3) is equivalent to

$$\text{(T3}'\text{)}\quad BsBpB = \begin{cases} BspB \text{ if } l(sp) > l(p) \\ BpB \cup BspB \text{ if } l(sp) < l(p) \end{cases}$$

for all $(s, p) \in S \times W$.

https://doi.org/10.4153/CJM-1972-095-4 Published online by Cambridge University Press

For p and r distinct elements of W, BpB and BrB are disjoint. Moreover, $G = \bigcup_{p \in W} BpB$. Hence the Weyl group W may be identified with $B\backslash G/B$.

We shall henceforth assume that (G, B, M', S) is a Tits system and that W is finite. It then follows from an exercise in Bourbaki [**1**, p. 43] that there is a unique element $q \in W$ with maximal length. Indeed, q may be characterized as the unique element in W for which $l(qs) < l(q)$ for all $s \in S$. It is easy to see that $q^2 = 1$ and $l(pq) = l(qp) = l(q) - l(p)$ for all $p \in W$. In particular this implies that $qSq = S$. Set $\tilde{B} = qBq$.

LEMMA 1. *$(G, \tilde{B}, M', S)$ is a Tits system.*

Proof. Since $\tilde{B} \cap M' = B \cap M'$, axioms (T1) and (T2) are satisfied. Axioms (T3) and (T4) are consequences of the fact that $p \to qpq$ is an automorphism of W which leaves S invariant.

We shall often refer to $(G, \tilde{B}, M', S)$ as the *dual* system.

LEMMA 2. *If $(s, p) \in S \times W$ with $l(sp) < l(p)$, then $sBs \cap sp\,B(sp)^{-1} \subset B$.*

Proof. Suppose $l(sp) < l(p)$. Then $l(ssp) > l(sp)$ and from $(T3')$

$$\begin{aligned}(sp)B(sp)^{-1} \cap BsB &= (spB \cap BsBsp)(sp)^{-1} \\ &\subset (BspB \cap BpB)(sp)^{-1} \\ &= \emptyset\end{aligned}$$

The lemma now follows from the fact that $sBs \subset B \cup BsB$.

Let $C = B \cap \tilde{B}$. Lemma 3 shows that C is the subgroup of B normalized by M'.

LEMMA 3. $C = \bigcap_{p \in W} pBp^{-1}$.

Proof. Let $s \in S$. Then $sCs \subset B \cup BsB$. But $BsB \cap sCs \subset BsB \cap sqBqs = \emptyset$ by Lemma 2. Hence $sCs \subset B$. By applying the same argument to the dual system, it follows that $sCs \subset \tilde{B}$ and hence that $sCs \subset C$. But then $sCs = C$ and since W is generated by S, $C = pCp^{-1} \subset pBp^{-1}$ for all $p \in W$. Let $C' = \bigcap_{p \in W} pBp^{-1}$. It has just been shown that $C \subset C'$. Since the converse statement is trivial, the lemma is proved.

Now for any $p \in W$, set $N_p = B \cap p^{-1}Bp$ and $U_p = B \cap p^{-1}\tilde{B}p$.

LEMMA 4. *The following equations hold for all p and r in W.*
(1) $pN_pp^{-1} = N_{p^{-1}}$.
(2) $N_{qp} = U_p$.
(3) $p^{-1}(N_{p^{-1}} \cap N_r)p = N_p \cap N_{rp} = N_p \cap p^{-1}N_rp$.
(4) $N_p \cap U_p = C$.

Proof. Statements (1)–(3) follow immediately from the definitions. To prove (4), use Lemma 3 to get

$$N_p \cap U_p = B \cap p^{-1}(B \cap \tilde{B})p = B \cap p^{-1}Cp = B \cap C = C.$$

LEMMA 5. *Let* $(s, p) \in S \times W$ *with* $l(sp) > l(p)$. *Then*
(1) $N_{(sp)^{-1}} \subset N_s$.
(2) $N_{sp} \subset N_p$.
(3) $U_s \subset N_{p^{-1}}$.
(4) $U_p \subset U_{sp}$.

Proof. Let $l(sp) > l(p)$. From $(T3')$, it follows as in Lemma 2 that $BsB \cap pBp^{-1} = \emptyset$ and hence that $sBs \cap pBp^{-1} \subset B$. But then

$$N_{(sp)^{-1}} = B \cap (sp)B(sp)^{-1} \subset sBs \cap B = N_s$$

so (1) is proved. From Lemma 4, we get

$$N_{sp} = (sp)^{-1}N_{(sp)^{-1}}\, sp = (sp)^{-1}\,(N_{(sp)^{-1}} \cap N_s)sp = N_{sp} \cap N_p$$

which proves (2).

To prove (3), it suffices to show that $p^{-1}\,U_s\,p \subset B$ whenever $l(sp) > l(p)$. We do this by induction on the length of p. The case $p = 1$ (i.e., $l(p) = 0$) is trivial. Suppose (3) holds for $p \in W$ and let $p' = ps'$ with $l(sp') > l(p') > l(p)$. Then $l(sp) > l(p)$ and by hypothesis $p^{-1}U_s\,p \subset B$. Hence $(p')^{-1}$ $U_s\,p' \subset s'Bs'$. But by Lemma 2

$$(p')^{-1}\,U_sp' = s'Bs' \cap (p')^{-1}\,U_sp' \subset s'Bs' \cap (qsp')^{-1}Bqsp' \subset B$$

so (3) holds for p' as well. By induction (3) holds in general.

(4) follows from (2) by Lemma 4. Indeed, if $\tilde{s} = qsq$, then $l(qp) = l(\tilde{s}qsp) > l(qsp)$ and hence

$$U_p = N_{qp} = N_{\tilde{s}qsp} \subset N_{qsp} = U_{sp}.$$

THEOREM 2. (1) $B = N_pU_p$ *for all* $p \in W$.
(2) *If* $l(sp) > l(p)$ *then* $N_{sp} = N_p \cap p^{-1}N_sp$ *and* $U_{sp} = (p^{-1}U_sp)\cdot\, U_p$.

Proof. The proof will again use induction on the length of p. First consider the case $p = s \in S$. From $(T3')$ it follows that $sBsq \subset BqB$ and hence that $B \subset (sBs)\,(s\tilde{B}s)$. Thus for each element $b \in B$ there exist elements $n \in sBs$ and $u \in s\tilde{B}s$ such that $b = nu$. We claim that $n \in B$. Indeed, if $n \notin B$, then $n \in BsB$. But then we would have $b \in (BsB\,sqB)qs \subset (BqB)qs$ which is impossible since $BsqB \cap BsB = \emptyset$. Hence $n \in B \cap sBs = N_s$ and $u = bn^{-1} \in B \cap s\tilde{B}s = U_s$. Thus we have the decomposition $B = N_sU_s$ for all $s \in S$.

Now suppose the decomposition $B = N_pU_p$ holds for some $p \in W$. Let $s \in S$ with $l(sp) > l(p)$. Since $B = N_sU_s$, then in particular for $n \in N_{p^{-1}}$, there exists $n_1 \in N_s$ and $u_1 \in U_s$ such that $n = n_1u_1$. From Lemma 5, $U_s \subset N_{p^{-1}}$

https://doi.org/10.4153/CJM-1972-095-4 Published online by Cambridge University Press

so $n_1 \in N_s \cap N_{p^{-1}}$. Thus $N_{p^{-1}} = (N_{p^{-1}} \cap N_s) \cdot U_s$. From Lemma 4 it follows that

$$\begin{aligned} B &= N_p U_p \\ &= p^{-1} N_{p^{-1}} p U_p \\ &= (p^{-1}(N_{p^{-1}} \cap N_s)p)\,(p^{-1}U_s p) U_p \\ &= (N_p \cap p^{-1}N_s p)\,(p^{-1}U_s p) U_p. \end{aligned}$$

But by Lemmas 4 and 5, $N_{sp} \subset N_p$ and $N_p \cap p^{-1}N_s p = N_p \cap N_{sp} = N_{sp}$. Also from Lemma 5, $U_p \subset U_{sp}$ and $U_s \subset N_{p^{-1}}$ so

$$p^{-1}U_s p = p^{-1}(U_s \cap N_{p^{-1}})p = U_{sp} \cap N_p.$$

From the argument given above, it follows that

$$U_{sp} = (U_{sp} \cap N_p)\,(U_{sp} \cap U_p) = (p^{-1}U_s p) \cdot U_p.$$

By induction on the length of p, the decomposition (1) holds for all $p \in W$ and the formulae in (2) have been proved as well.

Statements analogous to those in Lemma 5 and part (2) of Theorem 2 hold for the cases $l(ps) < l(p)$, $l(sp) > l(p)$ and $l(sp) < l(p)$. Such statements are easily proved from Lemmas 4 and 5 by substitution of p for p^{-1} and the use of Lemma 5 in the dual system.

Borrowing terminology from the Lie group context, the subgroups $p^{-1}U_s p$ may be called *positive* (respectively, *negative*) *root subgroups* for $(s, p) \in S \times P$ and $l(sp) > l(p)$ (respectively, $l(sp) < l(p)$). It follows easily from Theorem 2 that B is generated by the positive root subgroups. Since the conjugate of a positive root subgroup under q is a negative root subgroup, it follows that G is generated by the positive root subgroups and any element $\bar{q} \in G$ with $\bar{q} \in q$. These statements are of course well known in the context of semi-simple Lie groups. The standard proofs exploit the differentiable structure of such groups. Once the decomposition theorem $B = N_p U_p$ has been proven, the key axiom (T3) for Tits system is an immediate consequence.

4. Governing lists for Tits systems. Let (G, B, M', S) be a Tits system with finite Weyl group. For each $s \in S$ pick once and for all an element $\bar{s} \in M'$ such that $s = \bar{s}M$. Let $\bar{S} = \{\bar{s} : s \in S\}$. Recall that $n(s, s')$ denotes the order of ss' in W. Hence for all $(\bar{s}, \bar{s}') \in \bar{S} \times \bar{S}$, there exists an element $m(\bar{s}, \bar{s}') \in M$ such that $(\bar{s}\,\bar{s}')^{n(s,s')} = m(\bar{s}, \bar{s}')$. Let $(L; Y)$ be any presentation of M. For $\bar{s}$ and $\bar{s}'$ in $\bar{S}$ and $m \in W$, $m(\bar{s}, \bar{s}')$ and $\bar{s}m\bar{s}^{-1}$ may be regarded as words in the elements of the generating set W. Let X_1 be the set of relations of the form $(\bar{s}\,\bar{s}')^{n(s,s')} = m(\bar{s}, \bar{s}')$ and X_2 the set of relations of the form $\bar{s}m\bar{s}^{-1}(\bar{s}m\bar{s}^{-1})^{-1} = 1$. Set $Z = Y \cup X_1 \cup X_2$. Then Z is a set of relations on the generating set $L \cup \bar{S}$ of M'.

LEMMA 7. *$(L \cup \bar{S}\,; Z)$ is a presentation of M'.*

Proof. The lemma follows easily from the fact that $(S; X)$ is a presentation of W where X is the set of relations of the form $(ss')^{n(s,s')} = 1$.

Now for $s \in S$ and U_s as in section 3, consider the double coset space $C\backslash U_s/C$. Let D_s be a set of representatives for the non-trivial double cosets. A typical element of D_s will be denoted by $\bar{u}$. From Lemma 3, $sD_s s \subset BsB$. Hence for each $\bar{u} \in D_s$, we may select elements $v_1(\bar{u}) \in B$ and $v_2(\bar{u}) \in U_s$ such that $\bar{s}\bar{u}\bar{s} = v_1(\bar{u})\,\bar{s}\,v_2(\bar{u})$. Let $(B; Z_1)$ be the group-table presentation of B. Let Z_2 be the set of relations of the form $\bar{s}n\bar{s}^{-1}\,\tilde{n} = 1$ for $\bar{s} \in \bar{S}$, $u \in N_s$ and $\tilde{n} = (\bar{s}n\bar{s}^{-1})^{-1}$ in B. Let Z_3 be the set of relations of the form

$$\bar{s}\bar{u}\bar{s}(v_1(\bar{u})\bar{s}v_2(\bar{u}))^{-1} = 1$$

for $\bar{s} \in \bar{S}$ and $\bar{u} \in D_s$. For Z as above, set $\tilde{Z} = Z \cup Z_1 \cup Z_2 \cup Z_3$. Then $\tilde{Z}$ may be regarded as a set of relations on the generating set $B \cup \bar{S}$ of G.

THEOREM 3. *Let $f\colon B \to H$ be a homomorphism. Select a set of elements $H_{\bar{s}} = \{h_{\bar{s}} : \bar{s} \in \bar{S}\}$. Then f extends to a homomorphism $\tilde{f}\colon G \to H$ with $\tilde{f}(s) = h_{\bar{s}}$ if and only if the following conditions are satisfied*:

(W) $(h_{\bar{s}}h_{\bar{s}'})^{n(s,s')} = m(\bar{s}, \bar{s}')$ for $(\bar{s}, \bar{s}) \in \bar{S} \times \bar{S}$.

(I) $h_{\bar{s}}f(n) = f(\bar{s}\,n\,\bar{s}^{-1})h_{\bar{s}}$ for $(\bar{s}, n) \in \bar{S} \times N_s$.

(T) $h_{\bar{s}}f(\bar{u})h_{\bar{s}} = f(v_1(\bar{u}))h_{\bar{s}}f(v_2(\bar{u}))$ for $(\bar{s}, \bar{u}) \in \bar{S} \times D_s$.

Proof. The statement of the theorem is equivalent to the statement that $(\bar{S} \cup B; \tilde{Z})$ is a presentation of G. Let $\mathscr{G}$ be the free group on the generating set $\bar{S} \cup B$. The set of relations $\tilde{Z}$ may be identified with a set of words in $\mathscr{G}$ which describe the identity element in G. By an elementary word operation we shall mean insertion or deletion of any element from $\tilde{Z}$. By a word operation on $\mathscr{G}$, we shall mean a composite of elementary word operations. In order to show that $(\bar{S} \cup B; \tilde{Z})$ is a presentation of G, it suffices to show that whenever w is a word in $\mathscr{G}$ which represents the identity element in G, then w may be mapped to the identity word by a word operation.

By Lemma 7, two words in the elements of $\bar{S} \cup M$ which represent the same group element in M' may be mapped into one another by a word operation. Hence every element in $\mathscr{G}$ may be regarded as a word of the form $b_1m_1b_2m_2 \ldots b_nm_n$ where $b_j \in B$ and $m_j \in M'$ for $j = 1, 2, \ldots, k$. To prove the theorem it suffices to show that every word of the form m_1bm_2 may be mapped to a word of the form $b_1m_3b_2$ by a word operation. Let $p = m_1M$ and $p' = m_2M$. By induction on the length of p, it suffices to consider the case $m_1 = \bar{s} \in \bar{S}$. Since $sBp' = N_s s U_s p'$, we may assume $b = u \in U_s - C$. There are now two cases. First assume $l(sp') > l(p') = k$. Write p' in the form $p' = s_1s_2 \ldots s_k$. Let $p_0 = 1$ and $p_j = s_1s_2 \ldots s_j$ for $j = 1, 2, \ldots, k$. Then for all j, $l(sp_j) > l(p_j)$ and by Lemma 5, $U_s \subset N_{p_j^{-1}}$. It follows that $p_{j-1}^{-1} U_s p_{j-1} \subset N_{s_j}$ for $j = 1, 2, \ldots, k$. Set $m_2 = \bar{s}_1\bar{s}_2 \ldots \bar{s}_k m_0$ for some $m_0 \in M$, $u_0 = u$, $u_j = (\bar{s}_1\bar{s}_2 \ldots \bar{s}_j)^{-1}\,u\,(\bar{s}_1\bar{s}_2 \ldots \bar{s}_j)^{-1}$ for $j = 1, 2, \ldots, k$, and $b_2 = u_k m_0$. Then $\bar{s}um_2$ may be mapped to $\bar{s}m_2b_2$ by successively replacing $u_j\bar{s}_{j+1}$ by $\bar{s}_{j+1}u_{j+1}$ for $j = 0, 1, 2, \ldots, k - 1$. Clearly such replacements may be described by elementary word operations. Now consider the case $l(sp') < l(p')$. Then $\bar{s}u$ may be mapped by a word operation to a word of the form $v_1\bar{s}v_2(\bar{s})^{-1}$ for $v_1 \in B$ and $v_2 \in U_s$. Since $l(ssp') > l(sp')$, the argument just given implies

https://doi.org/10.4153/CJM-1972-095-4 Published online by Cambridge University Press

that $v_2(\bar{s})^{-1}m_2$ may be mapped by a word operation to a word of the form $\bar{s}^{-1}m_2b_2$. This completes the proof of the theorem.

5. Examples.

Example 1. Let $G = \mathrm{GL}(n, k)$ for $n \geqq 2$ and k any field. Let B be the upper triangular subgroup of G and M' the set of elements in G having precisely one non-zero entry in each row and column. Then $M = B \cap M'$ is the set of diagonal elements in G and $W = M'/M$ may be identified with the permutation group on n letters. For $i = 1, 2, \ldots, n - 1$ let s_i be the element of W identified with the transposition $(i, i + 1)$. Then (G, B, M', S) is a Tits system [**1**, p. 24]. The element $q \in W$ of maximal length is the coset determined by any matrix in M' with non-zero elements along the secondary diagonal. For $1 \leqq i \neq j \leqq n - 1$, s_is_j is of order 2 if $|i - j| > 1$ and of order 3 if $|i - j| = 1$. $\tilde{B} = qBq$ is the lower triangular subgroup and $B \cap \tilde{B} = M$. For $i = 1, 2, \ldots, n - 1$,

$$N_{s_i} = \{b \in B : b_{i,i+1} = 0\}$$

and

$$U_{s_i} = \{b \in B\colon b_{kl} = 0 \text{ whenever } k \neq l \text{ and } (k, l) \neq (i, i + 1)\}.$$

U_{s_i} contains precisely one non-trivial (M, M) double coset. Hence in the governing list for this system given by Theorem 3, there are $n - 1$ triple conditions.

Now let $G_0 = \mathrm{SL}(n, k)$ $B_0 = G_0 \cap B$ and $M_0' = M' \cap G_0$. Then (G_0, B_0, M_0', S) is again a Tits system. For $n \geqq 3$, the resulting governing list again contains $n - 1$ triple conditions. For $n = 2$, however, there are $[k^*\colon (k^*)^2]$ such conditions. Here $k^* = \{x \in k\colon x \neq 0\}$, $(k^*)^2 = \{x^2\colon x \in k^*\}$ and $[k^*\colon (k^*)^2]$ is the cardinality of $k^*/(k^*)^2$.

Example 2. Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$ and Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. The notation follows that of Kunze and Stein [**2**, pp. 385–392]. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and Δ the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Then $\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X$ for all $H \in \mathfrak{a}\}$. Fix $H_0 \in \mathfrak{a}$ with $\alpha(H_0) \neq 0$ for all $\alpha \in \Delta$ and let Δ^+ (respectively, Δ^-) be the set of $\alpha \in \Delta$ for which $\alpha(H_0) > 0$ (respectively, $\alpha(H_0) < 0$). Set $\mathfrak{n} = \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ and $\mathfrak{v} = \sum_{\alpha\in\Delta^-}\mathfrak{g}_\alpha$. Let K, A, N, and V denote the Lie sugbroups of G with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, and $\mathfrak{v}$. Let M be the centralizer of A in K, M' the normalizer of A in K and $W = M'/M$. Via the contragredient of the adjoint representation, W may be regarded as a transformation group on Δ. Select a set $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of simple roots in Δ^+. For $\alpha_i \in \Pi$ let s_i denote the reflection of Δ in α_i. Then $S = \{s_1, s_2, \ldots, s_n\}$ is a set of involutive generators for W. Set $B = MAN$. Then (G, B, M', S) is a Tits system. The element $q \in W$ having maximal length in the elements of S is the unique element in W mapping Δ^+ onto Δ^-. It follows that $\tilde{B} = qBq = MAV$ and $B \cap \tilde{B} = MA$. Also

$$U_{s_i} = MA \exp(\mathfrak{g}_{\alpha_i} + \mathfrak{g}_{2\alpha_i}) \quad \text{and} \quad N_{s_i} = MA \exp\Big(\sum_{\alpha \in \Delta^+ - \{\alpha_i, 2\alpha_i\}} \mathfrak{g}_\alpha\Big).$$

For $i \neq j$, the order $n(s_i, s_j)$ of $s_i s_j$ is either 2, 3, 4, or 6.

Suppose, in particular, that G has a complex structure. Then U_{s_i}/MA is a one parameter subgroup for $i = 1, 2, \ldots, n$. Thus the governing list of Theorem 3 contains precisely one triple condition for each simple root.

Now return to the general situation. Let T be a strongly continuous representation of B on a Banach space $\mathfrak{B}$. Suppose that T' is an extension of T to an algebraic homomorphism of G into $\mathrm{GL}(B)$. It is easy to see that T' is strongly continuous. If $\mathfrak{B}$ is a Hilbert space, then T' is uniformly bounded if and only if T is uniformly bounded. In the complex case, B is solvable and it follows that any uniformly bounded representation of B on an infinite dimensional Hilbert space is similar to a unitary representation. Thus, up to similarity, all infinite dimensional uniformly bounded representations of G arise from unitary representations of B via the governing lists of Theorem 3.

References

1. N. Bourbaki, *Groupes et algebres de Lie* (Chapitre 4–6, Hermann, Paris, 1968).
2. R. A. Kunze and E. M. Stein, *Uniformly bounded representations III, Intertwining operators of the principal series on semisimple groups*, Amer. J. Math. *89* (1967), 385–392.
3. J. Tits, *Algebraic and abstract simple groups*, Ann. of Math. *80* (1964), 313–329.
4. A. Weil, *Sur certaines groupes d'operateurs unitaires*, Acta Math. Vol. III (1964), 143–211.

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