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On infinite tensor products of factors of type I_2

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Abstract. It is proved, using Krieger's theorem, that ITPFI's of bounded type are $ITPFI_2$. This answers a question asked by E. J. Woods.

0. *Introduction*

The main result of this paper asserts that every infinite tensor product of factors of type I (ITPFI) of bounded type is an $ITPFI_2$ factor (theorem 2.1). Its proof is based on Krieger's theorem ([**12**]).

The same ideas yield a sufficient condition for two $ITPFI_2$ factors to be isomorphic (theorem 3.1). This comes from Kakutani's criterion ([**11**]) for the equivalence of infinite product measures, applied in the context of the Connes-Takesaki flow of weights.

Theorem 3.1, in turn, allows us to sharpen theorem 2.1: namely, given an ITPFI factor of bounded type, we realize it explictly as an $ITPFI_2$ in terms of eigenvalues and multiplicities (corollary 3.6). Together with a partial converse (proposition 4.4), theorem 3.1 gives also a rather surprising example (4.5).

In the proof of theorem 2.1, when Connes' invariant T is $\{0\}$, we use the rather easy proposition 1.1. In the appendix we indicate the construction of the flow of weights used in this text. Theorem 2.1 solves question 4 of [**16**] and theorem 3.1 may be used to give an answer to question 7 of [**16**] (corollary 3.7). Note, finally, that theorem 2.1 admits a more elementary, but technical proof (remark 2.3).

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All the definitions and the notation can easily be found in the literature (for instance in [**15**]). However, we shall recall those definitions which are frequently used.

Definitions (1) A factor M is called ITPFI if it is of the form

$$M=\bigotimes_{k=1}^{\infty}(M_{n_k}(C),\phi_k)$$

acting on the Hilbert space $\bigotimes_{k=1}^{\infty}(H_k,\xi_k)$ where $M_{n_k}(C)$ denotes the algebra of $n_k\times n_k$ matrices acting on the Hilbert space $H_k(n_k\geq 2)$ and $\phi_k(x)=\langle x\xi_k,\xi_k\rangle$.

(2) If all the n_k are bounded by a number n, M is said to be an ITPFI of bounded type.

(3) If all the n_k are equal to 2 (or to m) M is said to be an ITPFI$_2$ (or ITPFI$_m$).

Finally, if M is a factor, $T(M)$ and $S(M)$ denote Connes' invariants T and S ([**2**, §§ 1, 3]).

1. *A technical result*

Let M be an ITPFI factor of bounded type. In order to prove that M is an ITPFI$_2$ factor, we use, if $T(M)=\{0\}$, the following:

(1.1) Proposition. *Let $M=\bigotimes_{k\geq 1}(M_{m+1}(C),\phi_k)$ be an* ITPFI$_{m+1}$, *with $S(M)\subseteq\{0,1\}$. Then, for every positive number a, there exist sequences $(\psi_k)_{k\geq 1}$ of states on $M_{m+1}(C)$ and $(\lambda_p)_{p\geq 0}$ of real numbers such that*:

(i) *the states $\bigotimes_{k\geq 1}\phi_k$ and $\bigotimes_{k\geq 1}\psi_k$ are weakly equivalent*;

(ii) *the eigenvalues of ψ_k are $(1/\Lambda'_k,\eta_{k,1}/\Lambda'_k,\dots,\eta_{k,m}/\Lambda'_k$ where $\eta_{k,j}\in\{\lambda_p;\,p\geq 0\}$, $(\Lambda'_k=1+\sum_{j=1}^{m}\eta_{k,j})$; and*

(iii) *$\lambda_0=1$ and $e^{-(p+1)a}<\lambda_p\leq e^{-pa}$*, for $p\geq 1$.

We write the eigenvalues of the state ϕ_k in the form $(\xi_{k,0}/\Lambda_k,\xi_{k,1}/\Lambda_k,\dots,\xi_{k,m}/\Lambda_k)$, where $1=\xi_{k,0}\geq\xi_{k,1}\geq\cdots\geq\xi_{k,m}>0$ and $\Lambda_k=\sum_{i=0}^{m}\xi_{k,i}$.

We need the following application of lemma 8.6 of [**1**].

(1.2) Lemma. *Let a be a positive real number. If there exist disjoint subsets Y, Z of $\mathbb{N}\times\{0,\dots,m\}$ and a bijection $\alpha\colon Y\to Z$ such that $\sum_{(k,i)\in Y}|\xi_{k,i}^{\frac12}-\xi_{\alpha(k,i)}^{\frac12}|^2=\infty$ and $|\log(\xi_{\alpha(k,i)}/\xi_{k,i})|\leq a$, for all $(k,i)\in Y$, then $S(M)\cap[e^{-a},1)\neq\varnothing$.*

Proof. Write $\alpha(k,i)=(l_{k,i},j_{k,i})$. Replacing if necessary the pair $\{(k,i),\alpha(k,i)\}$ by $\{\alpha(k,i),(k,i)\}$, we may assume that for all $(k,i)\in Y$, $l_{k,i}\geq k$. Put $\beta(k,i)=|\xi_{k,i}^{\frac12}-\xi_{\alpha(k,i)}^{\frac12}|^2$, $(k,i)\in Y$.

If $\sum_{(k,i)\in Y,l_{k,i}=k}\beta(k,i)=\infty$, then one may apply lemma 8.6 of [**1**] and get the conclusion (putting $K_k^1=\{\xi_{k,i}/\Lambda_k|(k,i)\in Y$ and $l_{k,i}=k\}$;$\phi_k(\xi_{k,i}/\Lambda_k)=\xi_{\alpha(k,i)}/\Lambda_k$ if $\xi_{k,i}/\Lambda_k\in K_k^1$; $K_k^2=\phi_k(K_k^1)$; note that $\Lambda_k\leq m+1$).

So, replacing Y by $\{(k,i)\in Y|l_{k,i}>k\}$, we may furthermore assume that $l_{k,i}>k$, for all $(k,i)\in Y$.

Let $p\colon\mathbb{N}\times\{0,\dots,m\}\to\mathbb{N}$ be the projection, $(p(k,i)=k)$. For $k\in p(Y)$, let i_k be an element maximizing $\{\beta(k,i)|(k,i)\in Y\}$. We have

$$\sum_{k\in p(Y)}\beta(k,i_k)\geq\frac{1}{m+1}\sum_{(k,i)\in Y}\beta(k,i)=+\infty.$$

Replacing Y by $\{(k, i_k)|k \in p(Y)\}$, we may now assume that the map $p: Y \to \mathbb{N}$ is one-to-one. Maximizing, for a given k, $\beta(\alpha^{-1}(k, j))$, we may also suppose that the map $p: Z \to \mathbb{N}$ is one-to-one.

Let $l: p(Y) \to p(Z)$ be given by $l(k) = l_{k, i_k}$. For $k \in p(Y)$, put

$$r(k) = \begin{cases} 0 & \text{if } k \notin p(Z), \\ \max\{i | k \in l^i(p(Y))\} & \text{if } k \in p(Z). \end{cases}$$

As $l(k) > k$, for all $k \in p(Y)$, $r(k) \le k$. As $\sum_{k \in p(Y)} \beta(k, i_k) = \infty$, either $\sum_{r(k)\,\text{even}} \beta(k, i_k) = \infty$ or $\sum_{r(k)\,\text{odd}} \beta(k, i_k) = \infty$.

Assume, for instance, that $\sum_{k \in p(Y), r(k)\,\text{even}} \beta(k, i_k) = \infty$. Then, replacing Y by $\{(k, i_k)|r(k) \text{ even}\}$, the map $p: Y \cup Z \to \mathbb{N}$ is one-to-one. Write

$$M = \left(\bigotimes_{k \notin p(Y \cup Z)} (M_{m+1}, \phi_k) \right) \otimes \left(\bigotimes_{(k,i) \in Y} (M_{m+1} \otimes M_{m+1}, \phi_k \otimes \phi_{l(k)}) \right)$$

and apply lemma 8.6 of [1], (putting

$$K^1_{k,i} = \left\{ \frac{\xi_{k,i}\xi_{l(k),0}}{\Lambda_k \Lambda_{l(k)}} \right\}, \qquad K^2_{k,i} = \left\{ \frac{\xi_{k,0}\xi_{\alpha(k,i)}}{\Lambda_k \Lambda_{l(k)}} \right\}$$

and recalling that $\xi_{k,0} = 1$ for all k and that $\Lambda_k \le m+1$). □

End of the proof of proposition 1.1. For $p \ge 0$, let

$$E_p = \{(k, i) \in \mathbb{N} \times \{1, \ldots, m\} | \xi_{k,i} \in (e^{-(p-1)a}, e^{-pa}]\}.$$

We have, by lemma 1.2, $\sum_{(k,i) \in E_p} |\xi^{\frac{1}{2}}_{k,i} - 1|^2 < \infty$. Hence, for $p \ne 0$, $\# E_p$ is finite, where $\# E_p$ denotes the cardinality of E_p.

Choose $\lambda_p \in (e^{-(p+1)a}, e^{-pa}]$ such that

$$\#\{(k, i) \in E_p | \xi_{k,i} < \lambda_p\} \le \#\{(k, i) \in E_p | \xi_{k,i} \ge \lambda_p\}$$

and

$$\#\{(k, i) \in E_p | \xi_{k,i} > \lambda_p\} \le \#\{(k, i) \in E_p | \xi_{k,i} \le \lambda_p\}.$$

Let F_p and G_p be disjoint sets with the same cardinality, such that

$$F_p \supseteq \{(k, i) \in E_p | \xi_{k,i} < \lambda_p\}$$

and

$$G_p \supseteq \{(k, i) \in E_p | \xi_{k,i} > \lambda_p\}$$

and let $\alpha_p: F_p \to G_p$ be a bijection.

If $(k, i) \in F_p$, we have

$$\begin{aligned} |\xi^{\frac{1}{2}}_{k,i} - \xi^{\frac{1}{2}}_{\alpha_p(k,i)}|^2 &= |\lambda^{\frac{1}{2}}_p - \xi^{\frac{1}{2}}_{k,i} + \xi^{\frac{1}{2}}_{\alpha_p(k,i)} - \lambda^{\frac{1}{2}}_p|^2 \\ &\ge |\lambda^{\frac{1}{2}}_p - \xi^{\frac{1}{2}}_{k,i}|^2 + |\xi^{\frac{1}{2}}_{\alpha_p(k,i)} - \lambda^{\frac{1}{2}}_p|^2. \end{aligned}$$

Therefore

$$\sum_{(k,i) \in F_p} |\xi^{\frac{1}{2}}_{k,i} - \xi^{\frac{1}{2}}_{\alpha_p(k,i)}|^2 \ge \sum_{(k,i) \in E_p} |\xi^{\frac{1}{2}}_{k,i} - \lambda^{\frac{1}{2}}_p|^2.$$

We get, using lemma 1.2,

$$\sum_{p=1}^{\infty} \sum_{(k,i)\in E_p} |\xi_{k,i}^{\frac{1}{2}} - \lambda_p^{\frac{1}{2}}|^2 \le \sum_{p=1}^{\infty} \sum_{(k,i)\in F_p} |\xi_{k,i}^{\frac{1}{2}} - \xi_{\alpha_p(k,i)}^{\frac{1}{2}}|^2 < \infty.$$

Put $\lambda_0 = 1$. If $(k, i) \in E_p$, $p \in \mathbb{N}$, put $\eta_{k,i} = \lambda_p$. As $\sum_{(k,i)\in E_0} |\xi_{k,i}^{\frac{1}{2}} - 1|^2 < \infty$, we get $\sum_{(k,i)\in E_0} |\xi_{k,i}^{\frac{1}{2}} - \eta_{k,i}^{\frac{1}{2}}|^2 < \infty$. Put $\Lambda'_k = \sum_{i=0}^m \eta_{k,i}$, $(\eta_{k,0} = 1)$.

Note that if x and y are non-zero vectors of a normed vector space, then we have

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \le \frac{1}{\|x\|} \left(\|x - y\| + \left\| y \left(1 - \frac{\|x\|}{\|y\|} \right) \right\| \right)$$
$$\le \frac{1}{\|x\|} (\|x - y\| + | \, \|y\| - \|x\| \, |)$$
$$\le \frac{2}{\|x\|} \|x - y\|.$$

Therefore,

$$\sum_{i=0}^{m} \left| \left(\frac{\xi_{k,i}}{\Lambda_k} \right)^{\frac{1}{2}} - \left(\frac{\eta_{k,i}}{\Lambda'_k} \right)^{\frac{1}{2}} \right|^2 \le \frac{4}{\Lambda_k} \sum_{i=0}^{m} |\xi_{k,i}^{\frac{1}{2}} - \eta_{k,i}^{\frac{1}{2}}|^2$$
$$\le 4 \sum_{i=0}^{m} |\xi_{k,i}^{\frac{1}{2}} - \eta_{k,i}^{\frac{1}{2}}|^2.$$

Hence

$$\sum_{(k,i)} \left| \left(\frac{\xi_{k,i}}{\Lambda_k} \right)^{\frac{1}{2}} - \left(\frac{\eta_{k,i}}{\Lambda'_k} \right)^{\frac{1}{2}} \right|^2 < \infty.$$

This inequality gives the weak equivalence of $\bigotimes_{k\ge1} \phi_k$ and $\bigotimes_{k\ge1} \psi_k$. □

2. *Infinite tensor products of bounded type*

In this section, we prove

(2.1) THEOREM. *Every* ITPFI *factor of bounded type is (isomorphic to) an* ITPFI_2 *factor.*

This answers a question of E. J. Woods [**16**, 4 § 6]).

Up to isomorphism, there exists a unique ITPFI factor, in each of the cases I, II_1, II_∞ and III_λ, $\lambda \in (0, 1]$ and it can be realized as an ITPFI_2 ([**1**]). Therefore, the only interesting case of theorem 2.1 is the type III_0 case.

An ITPFI factor of bounded type is a finite tensor product of homogeneous ITPFI factors.

We proceed by induction and show that every ITPFI_{m+2} factor M $(m \ge 1)$ is a tensor product of an ITPFI_{m+1} factor by an ITPFI_2 factor.

Therefore, let $M = \bigotimes_{k\ge1} (M_{m+2}(C), \phi_k)$ be an ITPFI_{m+2} factor of type III_0. We write the eigenvalues of the state ϕ_k in the form $(\xi_{k,0}/\Lambda_k, \xi_{k,1}/\Lambda_k, \ldots, \xi_{k,m}/\Lambda_k, \xi_{k,m+1}/\Lambda_k)$ where $1 = \xi_{k,0} \ge \xi_{k,1} \ge \cdots \ge \xi_{k,m} \ge \xi_{k,m+1} > 0$ and $\Lambda_k = \sum_{i=0}^{m+1} \xi_{k,i}$.

If $T(M) \ne \{0\}$, we might assume that all the $\xi_{k,i}$ are of the form λ^p, where $\lambda \in (0, 1)$ is such that $2\pi/\log \lambda \in T(M)$. In order to treat the case $T(M) = \{0\}$, we use proposition 1.1: we get a sequence $(\lambda_p)_{p\ge0}$ of real numbers, with $\lambda_0 = 1$ and $e^{-(p+1)} < \lambda_p \le e^{-p}$, for all p; we suppose that the $\xi_{k,i}$'s are chosen in the sequence $(\lambda_p)_{p\ge0}$.

Gathering the terms which have the same eigenvalue list, we may write:

$$M=\bigotimes_{p,q}((M_{m+2}(C),\phi_{p,q})^{\otimes L_{p,q}}),$$

where $p=(p_1,\ldots,p_m)$; (p,q) runs over all the $(m+1)$-tuples $0\le p_1\le\cdots\le p_m\le q$ and $\phi_{p,q}$ is a state with eigenvalues $(1/\Lambda_{p,q},\lambda_{p_1}/\Lambda_{p,q},\ldots,\lambda_{p_m}/\Lambda_{p,q},\lambda_q/\Lambda_{p,q})$, $\Lambda_{p,q}=\Lambda_p+\lambda_q$, where $\Lambda_p=1+\sum_{i=1}^m\lambda_{p_i}$.

As M is of type III_0, $L_{p,q}<\infty$ for $(p,q)\neq(0,0)$. Since M is isomorphic to $M\otimes R$ (R is the hyperfinite factor of type II_1), we may assume that $L_{0,0}=0$.

For all (p,q) define integers $K_{p,q}$ and $J_{p,q}$ by

$$K_{p,q}=L_{p,q}-\left[\frac{\lambda_qL_{p,q}}{\Lambda_{p,q}}\right]\quad\text{and}\quad J_{p,q}=\left[\frac{L_{p,q}(1+\lambda_q)}{\Lambda_{p,q}}\right],$$

where if $x\in\mathbb{R}$, $[x]$ denotes its integral part.

Let ϕ'_p (resp. ϕ''_q) be a state on $M_{m+1}(C)$ (resp. $M_2(C)$) with eigenvalues $(1/\Lambda_p,\lambda_{p_1}/\Lambda_p,\ldots,\lambda_{p_m}/\Lambda_p)$, (resp. $(1/1+\lambda_q),\lambda_q/(1+\lambda_q))$. Set

$$N=\bigotimes_{p,q}((M_{m+1}(C),\phi'_p)^{\otimes K_{p,q}}\otimes(M_2(C),\phi''_q)^{\otimes J_{p,q}}).$$

We want to prove that M is isomorphic to N.

For all (p,q), consider the (probability) measures $\mu_{p,q}$ and $\mu'_{p,q}$ on $\mathbb{N}^m\times\mathbb{N}$ given by

$$\mu_{p,q}(k,j)=\frac{L_{p,q}!}{k!j!(L_{p,q}-|k|-j)!}\cdot\frac{\lambda_p^k\lambda_q^j}{\Lambda_{p,q}^{L_{p,q}}},$$

where $\mu_{p,q}(k,j)=0$ if $|k|+j>L_{p,q}$, and $k!=\prod_{i=1}^m k_i!$; $|k|=\sum_{i=1}^m k_i$; $\lambda_p^k=\prod_{i=1}^m\lambda_{p_i}^{k_i}$;

$$\mu'_{p,q}(k,j)=\frac{K_{p,q}!}{k!(K_{p,q}-|k|)!}\cdot\frac{J_{p,q}!}{j!(J_{p,q}-j)!}\cdot\frac{\lambda_p^k\lambda_q^j}{\Lambda_p^{K_{p,q}}(1+\lambda_q)^{J_{p,q}}},$$

with $\mu'_{p,q}(k,j)=0$ if $|k|>K_{p,q}$ or $j>J_{p,q}$.

Let (Ω,ν), (resp. (Ω,ν')) be the product measure space $(\Omega,\nu)=\prod_{(p,q)}(\mathbb{N}^m\times\mathbb{N},\mu_{p,q})$ (resp. $(\Omega,\nu')=\prod_{(p,q)}(\mathbb{N}^m\times\mathbb{N},\mu'_{p,q})$). Let $B\subseteq\Omega$, with $\nu(B)\neq0$ and $\nu'(B)\neq0$.

Let $\mathcal{R}$ be the equivalence relation on $B\times\mathbb{R}$ given by: $(x,t)\mathcal{R}(y,s)$ iff there exists a finite subset E of indices (p,q) such that $x_{p,q}=y_{p,q}$, for all $(p,q)\notin E$ and

$$\sum_{(p,q)\in E}T_{p,q}(x_{p,q})+t=\sum_{(p,q)\in E}T_{p,q}(y_{p,q})+s$$

where $T_{p,q}(k,j)=-\sum_{i=1}^m k_i\log\lambda_{p_i}-j\log\lambda_q$, for $(k,j)=(k_1,k_2,\ldots,k_m,j)\in\mathbb{N}^m\times\mathbb{N}$.

The flow of weights of M (resp. N) is given by the action of $\mathbb{R}$ by translation on $(B\times\mathbb{R},\nu\times dx)/\mathcal{R}$ (resp. $(B\times\mathbb{R},\nu'\times dx)/\mathcal{R}$) (see appendix).

Using Krieger's theorem ([**12**]) to prove the isomorphism between M and N, it is enough to show that the measures ν and ν' are not (mutually) singular (i.e. equivalent on a subset $B\subseteq\Omega$). This is done using Kakutani's criterion on infinite product measures ([**11**, p. 453]) and the following lemma.

(2.2) LEMMA. *Let $1\ge\xi_1\ge\xi_2\ge\cdots\ge\xi_m\ge\lambda>0$ be real numbers and let L be a positive integer. Let μ and μ' be the (probability) measures on $\mathbb{N}^m\times\mathbb{N}$ given by*

$$\mu(k,j)=\frac{L!}{k!j!(L-|k|-j)!}\frac{\xi^k\lambda^j}{\Lambda^L},$$

$(\mu(k,j)=0$ if $|k|+j>L)$;

$$\mu'(k,j)=\frac{K!}{k!(K-|k|)!}\cdot\frac{J!}{j!(J-j)!}\cdot\frac{\xi^k\lambda^j}{(\Lambda-\lambda)^K(1+\lambda)^J},$$

($\mu'(k,j)=0$ if $|k|>K$ or $j>J$); *where* $\Lambda=1+\sum_{i=1}^m \xi_i+\lambda$, $K=L-[L\lambda/\Lambda]$, $J=[L(1+\lambda)/\Lambda]$, ($\xi^k=\prod_{i=1}^m \xi_i^{k_i}$). *Then*

$$\rho(\mu,\mu')=\sum_{(k,j)}\mu(k,j)^{\frac12}\mu'(k,j)^{\frac12}\geq 1-2\lambda.$$

Proof. Let α and α' be the probability distributions of the random variable j, with respect to μ and μ'. They are measures on $\mathbb{N}$, given by

$$\alpha(j)=\sum_k \mu(k,j)=\frac{L!}{j!(L-j)!}\frac{\lambda^j(\Lambda-\lambda)^{L-j}}{\Lambda^L}$$

and

$$\alpha'(j)=\sum_k \mu'(k,j)=\frac{J!}{j!(J-j)!}\frac{\lambda^j}{(1+\lambda)^J}.$$

Let β_j and β' be the conditional probabilities of the random variable k, given j. (β' does not depend on j, as the random variables j and k are independent for μ'.) They are measures on $\mathbb{N}^m$ given by

$$\beta_j(k)=\frac{\mu(k,j)}{\alpha(j)}=\frac{(L-j)!}{k!(L-j-|k|)!}\frac{\xi^k}{(\Lambda-\lambda)^{L-j}},$$

$$\beta'(k)=\frac{\mu'(k,j)}{\alpha'(j)}=\frac{K!}{k!(K-|k|)!}\frac{\xi^k}{(\Lambda-\lambda)^K}.$$

We have:

$$\begin{aligned}\rho(\mu,\mu')&=\sum_{k,j}\alpha(j)^{\frac12}\beta_j(k)^{\frac12}\alpha'(j)^{\frac12}\beta'(k)^{\frac12}\\&=\sum_j \alpha(j)^{\frac12}\alpha'(j)^{\frac12}\sum_k \beta_j(k)^{\frac12}\beta'(k)^{\frac12}\\&=\sum_j \alpha(j)^{\frac12}\alpha'(j)^{\frac12}\rho(\beta_j,\beta')\\&=\rho(\alpha,\alpha')-\sum_j \alpha(j)^{\frac12}\alpha'(j)^{\frac12}(1-\rho(\beta_j,\beta')).\end{aligned}$$

As $1-\rho(\beta_j,\beta')\geq 0$ and $\frac12(\alpha(j)+\alpha'(j))\geq\alpha(j)^{\frac12}\alpha'(j)^{\frac12}$,

$$\rho(\mu,\mu')\geq\rho(\alpha,\alpha')-\tfrac12\sum_j(\alpha(j)+\alpha'(j))(1-\rho(\beta_j,\beta')). \tag{1}$$

Therefore, we need only to estimate $\rho(\alpha,\alpha')$ and the expectations with respect to α and α' of $1-\rho(\beta_j,\beta')$.

(a) *Computation of* $\rho(\beta_j,\beta')$. Let $K_1=\inf(K,L-j)$ and $K_2=\sup(K,L-j)$. If $K\geq L-j$, put $\gamma_1=\beta_j$ and $\gamma_2=\beta'$; if $K<L-j$, put $\gamma_1=\beta'$ and $\gamma_2=\beta_j$. We get:

$$\frac{\gamma_2(k)}{\gamma_1(k)}=\frac{K_2!(K_1-|k|)!}{K_1!(K_2-|k|)!}\cdot\frac{1}{(\Lambda-\lambda)^{K_2-K_1}}=\prod_{i=1}^{K_2-K_1}\frac{K_1+i}{(K_1-|k|+i)(\Lambda-\lambda)}.$$

Using the inequality $\log(a/b)\geq(a-b)/a$, we get

$$\log\frac{\gamma_2(k)}{\gamma_1(k)}\geq\sum_{i=1}^{K_2-K_1}\frac{|k|(\Lambda-\lambda)-(\Lambda-\lambda-1)(K_1+i)}{K_1+i}.$$

Hence,

$$\left(\frac{\gamma_2(k)}{\gamma_1(k)}\right)^{\frac{1}{2}} \geq 1+\frac{1}{2}\sum_{i=1}^{K_2-K_1}\frac{|k|(\Lambda-\lambda)-(\Lambda-\lambda-1)(K_1+i)}{K_1+i}.$$

Now, $\rho(\gamma_1, \gamma_2)=\sum_k \gamma_1(k)(\gamma_2(k)/\gamma_1(k))^{\frac{1}{2}}$. As $\sum_k |k|\gamma_1(k)=K_1(\Lambda-\lambda-1)/(\Lambda-\lambda)$, we get

$$\rho(\gamma_1, \gamma_2)\geq 1-\frac{1}{2}\sum_{i=1}^{K_2-K_1}\frac{(\Lambda-\lambda-1)i}{K_1+i}.$$

As for $i\leq K_2-K_1$, $i/(K_1+i)\leq(K_2-K_1)/K_2$, we derive

$$\rho(\gamma_1, \gamma_2)\geq 1-\frac{(K_2-K_1)^2}{2K_2}(\Lambda-\lambda-1).$$

Hence, $\rho(\beta_j, \beta')\geq 1-\dfrac{(K-L+j)^2}{2K}(\Lambda-\lambda-1)$. Therefore

$$\sum_j \alpha(j)(1-\rho(\beta_j, \beta'))\leq\frac{\Lambda-\lambda-1}{2K}\sum_j \alpha(j)\left(\left(j-\frac{\lambda L}{\Lambda}\right)+\left(\frac{\lambda L}{\Lambda}+K-L\right)\right)^2.$$

As the expectation $E_\alpha(j)$ of j, with respect to α, is $L\lambda/\Lambda$ and its variance $\sigma^2_\alpha(j)$ is $L\lambda(\Lambda-\lambda)/\Lambda^2$, we get:

$$\sum_j \alpha(j)(1-\rho(\beta_j, \beta'))\leq\frac{\Lambda-\lambda-1}{2K}\left(\frac{L\lambda(\Lambda-\lambda)}{\Lambda^2}+\left(\frac{\lambda L}{\Lambda}+K-L\right)^2\right).$$

As $E_{\alpha'}(j)=\lambda J/1+\lambda$ and $\sigma^2_{\alpha'}(j)=\lambda J/(1+\lambda)^2$, we get in the same way:

$$\sum_j \alpha'(j)(1-\rho(\beta_j, \beta'))\leq\frac{\Lambda-\lambda-1}{2K}\left(\frac{\lambda J}{(1+\lambda)^2}+\left(\frac{\lambda J}{1+\lambda}+K-L\right)^2\right).$$

As by the definition of K, $L(\Lambda-\lambda)/K\Lambda\leq 1$ and, by the definition of J, $J/(1+\lambda)\leq L/\Lambda\leq K/(\Lambda-\lambda)$, we get:

$$\left(\frac{\Lambda-\lambda-1}{2K}\right)\frac{L\lambda(\Lambda-\lambda)}{\Lambda^2}\leq\frac{\lambda}{2}\frac{\Lambda-\lambda-1}{\Lambda}\leq\frac{\lambda}{2}$$

and

$$\left(\frac{\Lambda-\lambda-1}{2K}\right)\frac{\lambda J}{(1+\lambda)^2}\leq\frac{\lambda}{2}\frac{\Lambda-\lambda-1}{(\Lambda-\lambda)(1+\lambda)}\leq\frac{\lambda}{2}.$$

If $L\lambda/\Lambda<1$, then $K=L$ and $(\lambda J)^2/(1+\lambda)^2\leq(\lambda L/\Lambda)^2\leq\lambda L/\Lambda$. If $L\lambda/\Lambda\geq 1$, then $0\leq(\lambda L/\Lambda)+K-L=K-((L-(\lambda L/\Lambda))<1$, by the definition of K. Also, by the definition of J, $\lambda J/(1+\lambda)\leq\lambda L/\Lambda(1+\lambda)\leq\lambda L/\Lambda$; thus,

$$-1\leq\frac{-\lambda}{1+\lambda}<\frac{\lambda J}{1+\lambda}+K-L\leq\frac{\lambda L}{\Lambda}+K-L<1.$$

In all cases,

$$\left(\frac{\lambda L}{\Lambda}+K-L\right)^2\leq\frac{\lambda L}{\Lambda}\quad\text{and}\quad\left(\frac{\lambda J}{1+\lambda}+K-L\right)^2\leq\frac{\lambda L}{\Lambda}.$$

Hence, we get:

$$\sum_j(\alpha(j)+\alpha'(j))(1-\rho(\beta_j, \beta'))\leq\frac{\lambda}{2}+\frac{\Lambda-\lambda-1}{2K}\cdot\frac{L\lambda}{\Lambda}+\frac{\lambda}{2}+\frac{\Lambda-\lambda-1}{2K}\cdot\frac{L\lambda}{\Lambda}\leq 2\lambda. \quad (2)$$

(b) *Computation of* $\rho(\alpha, \alpha')$.

$$\frac{\alpha(j)}{\alpha'(j)}=\frac{L!(J-j)!(1+\lambda)^J}{J!(L-j)!\Lambda^L}\cdot(\Lambda-\lambda)^{L-j}=\left(\prod_{i=1}^{L-J}\frac{J+i}{J-j+i}\right)\frac{(1+\lambda)^J(\Lambda-\lambda)^{L-j}}{\Lambda^L}.$$

Let $e=E_{\alpha'}(j)=J\lambda/(1+\lambda)$. Put

$$V_e=\left(\prod_{i=1}^{L-J}\frac{J+i}{J-e+i}\right)\frac{(1+\lambda)^J(\Lambda-\lambda)^{L-e}}{\Lambda^L}.$$

We have

$$\frac{\alpha(j)}{\alpha'(j)}=\left(\prod_{i=1}^{L-J}\frac{J-e+i}{J-j+i}\right)(\Lambda-\lambda)^{e-j}V_e.$$

As $\log(a/b)\geq(a-b)/a$, we get

$$\log\frac{\alpha(j)}{\alpha'(j)}\geq\log V_e+(e-j)\log(\Lambda-\lambda)+\sum_{i=1}^{L-J}\frac{j-e}{J-e+i}.$$

Hence,

$$\left(\frac{\alpha(j)}{\alpha'(j)}\right)^{\frac{1}{2}}\geq V_e^{\frac{1}{2}}\left[1+\tfrac{1}{2}(e-j)\left(\log(\Lambda-\lambda)-\sum_{i=1}^{L-J}\frac{1}{J-e+i}\right)\right].$$

Therefore,

$$\rho(\alpha,\alpha')=\sum_j\left(\frac{\alpha(j)}{\alpha'(j)}\right)^{\frac{1}{2}}\alpha'(j)\geq V_e^{\frac{1}{2}},$$

as $\sum_j(e-j)\alpha'(j)=0$.

Now,

$$V_e=\frac{L}{L-e}\frac{J-e}{J}\left(\prod_{i=0}^{L-J-1}\frac{J+i}{J-e+i}\right)\frac{(1+\lambda)^J(\Lambda-\lambda)^{L-e}}{\Lambda^L}.$$

As $\log(J+t)/(J-e+t)$ is a decreasing function of $t\geq0$, we get

$$\begin{aligned}\sum_{i=0}^{L-J-1}\log\frac{J+i}{J-e+i}&\geq\int_0^{L-J}[\log(J+t)-\log(J-e+t)]\,dt\\&\geq L\log L-J\log J-(L-e)\log(L-e)+(J-e)\log(J-e).\end{aligned}$$

Hence,

$$V_e\geq\frac{J-e}{J}\cdot\frac{L^L(J-e)^{(J-e)}}{(L-e)^{L-e}J^J}\cdot\frac{(1+\lambda)^J(\Lambda-\lambda)^{L-e}}{\Lambda^L}=\frac{J-e}{J}\cdot\frac{\left(\frac{L}{\Lambda}\right)^L(J-e)^{J-e}}{\left(\frac{L-e}{\Lambda-\lambda}\right)^{(L-e)}\left(\frac{J}{1+\lambda}\right)^J}$$

$$\geq\frac{J-e}{J}\cdot\frac{\left(\frac{L}{\Lambda(J-e)}\right)^L}{\left(\frac{L-e}{(\Lambda-\lambda)(J-e)}\right)^{L-e}}\cdot\frac{1}{\left(\frac{J}{(J-e)(1+\lambda)}\right)^J}.$$

Now, by definition of e, $J=(J-e)(1+\lambda)$. Put $\varepsilon=(L(1+\lambda)/\Lambda)-J$; (we have $0\leq\varepsilon<1$). We get

$$V_e\geq\frac{1}{1+\lambda}\cdot\frac{\left(1+\frac{\varepsilon}{J}\right)^L}{\left(1+\frac{\varepsilon\Lambda}{(\Lambda-\lambda)J}\right)^{L-e}}\geq\frac{1}{1+\lambda}\left(\frac{\left(1+\frac{\varepsilon}{J}\right)}{1+\frac{\varepsilon\Lambda}{(\Lambda-\lambda)J}}\right)^{L-e}.$$

But,

$$(L-e)\log\left(\frac{1+\frac{\varepsilon}{J}}{1+\frac{\varepsilon\Lambda}{(\Lambda-\lambda)J}}\right)\geq(L-e)\frac{\frac{\varepsilon}{J}\left(1-\frac{\Lambda}{\Lambda-\lambda}\right)}{1+\frac{\varepsilon}{J}}=-\frac{(L-e)\varepsilon\lambda}{(\varepsilon+J)(\Lambda-\lambda)}$$

$$\geq-\frac{\lambda\Lambda(L-e)\varepsilon}{L(1+\lambda)(\Lambda-\lambda)}\geq-\lambda,$$

as $L-e\leq L$, $\varepsilon<1$ and $\Lambda\leq(1+\lambda)(\Lambda-\lambda)$. Hence,

$$V_e\geq\frac{1}{1+\lambda}\exp(-\lambda)>(1-\lambda)^2.$$

Thus,

$$\rho(\alpha,\alpha')\geq V_e^{\frac{1}{2}}>1-\lambda. \tag{3}$$

By (1), (2) and (3), we get the result. □

End of the proof of theorem 2.1. For a given q, there are $(q+m)!/q!m!\leq(q+1)^m$ possible choices of (p,q). As $\sum_{p,q}\lambda_q\leq\sum_q(q+1)^m\lambda_q\leq\sum_q(q+1)^m e^{-q}<\infty$, we get $\prod_{(p,q)}\rho(\mu_{p,q},\mu'_{p,q})\neq0$. By Kakutani's criterion, ([**11**, pp. 453-455]), the product measures ν and ν' are not singular. The proof is now complete.

N.B. We actually used a straightforward consequence of Kakutani's criterion: though neither $\mu_{p,q}$, nor $\mu'_{p,q}$ is absolutely continuous with respect to the other, as $\prod_{(p,q)}\rho(\mu_{p,q},\mu'_{p,q})\neq0$, the product measures ν and ν' are not singular. (cf. [**11**, remark 22.37, p. 455]. □

(2.3) *Remark.* We can prove theorem 2.1 in a more computational way but without using Krieger's theorem. We need (keeping the above notation) the following:

PROPOSITION. *There exist:*

(i) *projections $P_{p,q}$ in $A_{p,q}=(M_{m+2}(C))^{\otimes L_{p,q}}$ and $Q_{p,q}$ in $B_{p,q}=(M_{m+1}(C)^{\otimes K_{p,q}}\otimes(M_2(C))^{\otimes J_{p,q}}$;*

(ii) *states $\chi_{p,q}$ on matrix algebras $C_{p,q}$ and integers $r_{p,q}$, $r'_{p,q}$;*

such that

(a) $\prod_{p,q}\Phi_{p,q}(P_{p,q})>0$, $\prod_{p,q}\Psi_{p,q}(Q_{p,q})>0$, *where* $\Phi_{p,q}=\phi_{p,q}^{\otimes L_{p,q}}$ *and* $\Psi_{p,q}=(\phi_p'^{\otimes K_{p,q}})\otimes(\phi_q''^{\otimes J_{p,q}})$;

(b) $(C_{p,q},\chi_{p,q})\otimes(M_{r_{p,q}}(C),\mathrm{Tr})$ *is isomorphic to* $(A_{p,q},\Phi_{p,q})$ *reduced by* $P_{p,q}$, *and* $(C_{p,q},\chi_{p,q})\otimes(M_{r'_{p,q}}(C),\mathrm{Tr})$ *is isomorphic to* $(B_{p,q},\Psi_{p,q})$, *reduced by* $Q_{p,q}$.

Then, the projections $P=\bigotimes_{p,q}P_{p,q}\in M$ and $Q=\bigotimes_{p,q}Q_{p,q}\in N$ are non-zero, and $M_p\otimes R\cong N_Q\otimes R$, where R denotes the hyperfinite factor of type II$_1$.

As M is of type III, M and N are isomorphic. ($M\cong M_P\cong M_P\otimes R\cong N_Q\otimes R\cong N$). Here Tr denotes the normalized trace on $M_n(C)$ ($\mathrm{Tr}(1)=1$).

Let us now sketch the proof of the proposition. It is enough to show that $P_{p,q}$ and $Q_{p,q}$ can be chosen, satisfying (b) and such that $\Phi_{p,q}(P_{p,q})\geq1-9\lambda_q^{\frac{1}{3}}$ and $\Psi_{p,q}(Q_{p,q})\geq1-9\lambda_q^{\frac{1}{3}}$; if $\lambda_q^{\frac{1}{3}}\geq\frac{1}{9}$ then take $P_{p,q}$ and $Q_{p,q}$ to be any rank one projections.

We can now drop the indices (p, q). We keep the notation of lemma 2.2. Let $E=\{(k,j);\ |\log(\mu(k,j)/\mu'(k,j))|\le 2\lambda^{\frac13}\}$. As

$$\sum_{(k,j)}\left(1-\left(\frac{\mu(k,j)}{\mu'(k,j)}\right)^{\frac12}\right)^2\mu'(k,j)=2(1-\rho(\mu,\mu'))\le 4\lambda$$

and as for $(k,j)\notin E$,

$$\left|\left(\frac{\mu(k,j)}{\mu'(k,j)}\right)^{\frac12}-1\right|\ge 1-\exp^{-\lambda^{\frac13}},$$

we get that

$$\mu'(E)\ge 1-\frac{4\lambda}{(1-\exp(-\lambda^{\frac13}))^2}\ge 1-5\lambda^{\frac13}.$$

In the same way

$$\mu(E)\ge 1-\frac{4\lambda}{(1-\exp(-\lambda^{\frac13}))^2}\ge 1-5\lambda^{\frac13}.$$

Put

$$m_{k,j}=\frac{L!}{(L-|k|-j)!k!j!}\quad\text{and}\quad m'_{k,j}=\frac{K!}{k!(K-|k|)!}\cdot\frac{J!}{j!(J-j)!}.$$

We have

$$\frac{m_{k,j}}{m'_{k,j}}=\frac{\mu(k,j)}{\mu'(k,j)}\cdot\frac{\mu'(0,0)}{\mu(0,0)}.$$

We distinguish two cases:

(1) $(0,0)\in E$. We have $|\log(m_{k,j}/m'_{k,j}|\le 4\lambda^{\frac13}$, for $(k,j)\in E$. Let $q_{k,j}=\min(m_{k,j}, m'_{k,j})$ and let $P_{k,j}$ (resp. $Q_{k,j}$) be a subprojection of dimension $q_{k,j}$ of the projection on the eigenspace of Φ (resp. Ψ), relative to $\xi^k\lambda^j/\Lambda^L$ (resp. $\xi^k\lambda^j/(\Lambda-\lambda)^K(1+\lambda)^J$). Set $P=\sum_{(k,j)\in E}P_{k,j}$ and $Q=\sum_{(k,j)\in E}Q_{k,j}$. We have $(A_P,\Phi_P)\cong(B_Q,\Psi_Q)$ and

$$\Phi(P)\ge\exp(-4\lambda^{\frac13})\cdot\mu(E)\ge\exp(-4\lambda^{\frac13})(1-5\lambda^{\frac13})\ge 1-9\lambda^{\frac13}.$$

Also, $\Psi(Q)\ge 1-9\lambda^{\frac13}$.

In this case, put $r=r'=1$ and $(C,\chi)=(A_P,\Phi_P)$.

(2) $(0,0)\notin E$. Note that if $\lambda L/\Lambda<1$, then $K=L$ and

$$\log\frac{\mu(0,0)}{\mu'(0,0)}=\log\left(\frac{(1+\lambda)^J(\Lambda-\lambda)^K}{\Lambda^L}\right)=J\log(1+\lambda)+L\log\left(1-\frac{\lambda}{\Lambda}\right).$$

As $0\le\log(1+\lambda)-(\lambda/(1+\lambda))\le\lambda^2/(1+\lambda)$ and $-\lambda^2/\Lambda(\Lambda-\lambda)\le\log(1-(\lambda/\Lambda))+\lambda/\Lambda\le 0$,

$$-\frac{\lambda}{\Lambda-\lambda}\le-\frac{L\lambda^2}{\Lambda(\Lambda-\lambda)}\le\log\frac{\mu(0,0)}{\mu'(0,0)}-\frac{J\lambda}{1+\lambda}+\frac{L\lambda}{\Lambda}$$
$$\le\frac{J\lambda^2}{1+\lambda}\le\frac{L\lambda}{\Lambda}\lambda<\lambda.$$

As $0\le(L\lambda/\Lambda)-(J\lambda/(1+\lambda))=(\lambda/(1+\lambda))[(L(1+\lambda)/\Lambda)-J]\le\lambda/(1+\lambda)$, we get

$$\left|\log\left(\frac{\mu(0,0)}{\mu'(0,0)}\right)\right|\le 2\lambda\quad\text{and}\quad(0,0)\in E.$$

Hence, in our case, $\lambda L/\Lambda \geq 1$ and $J = [L(1+\lambda)/\Lambda] \geq L/\Lambda \geq \lambda^{-1}$. Moreover if $m'_{k,j} = 1$ and $(k, j) \neq (0, 0)$, then either K is equal to one of the k_i's or $J = j$. The μ and μ'-measure of these points is very small. Therefore, we will neglect them. (One can also show that these points do not belong to E).

Now, if $m'_{k,j} > 1$, then $m'_{k,j} \geq J$. Let r, $r' \in \mathbb{N}$ be such that r' is smaller than $J^{\frac{1}{2}}$ and

$$\left|\frac{r}{r'} - \frac{\mu'(0,0)}{\mu(0,0)}\right| \leq J^{-\frac{1}{2}}.$$

Let $q_{k,j} = \min\,([m_{k,j}/r], [m'_{k,j}/r'])$ and $P_{k,j}$ (resp. $Q_{k,j}$ and $R_{k,j}$) be a subprojection of dimension $rq_{k,j}$ (resp. $r'q_{k,j}$ and $q_{k,j}$) of the projection on the eigenspace of Φ (resp. Ψ), relative to $\xi^k\lambda^j/\Lambda^L$ (resp. $\xi^k\lambda^j/(\Lambda-\lambda)^K(1+\lambda)^J$). Put $P = \sum_{(k,j)\in E} P_{k,j}$, $Q = \sum_{(k,j)\in E} Q_{k,j}$, $R = \sum_{(k,j)\in E} R_{k,j}$ and $(C, \chi) = (B_R, \Psi_R)$. We have

$$(A_P, \Phi_P) \cong (C, \chi) \otimes (M_r(C), \mathrm{Tr})$$

and

$$(B_Q, \Psi_Q) \cong (C, \chi) \otimes (M_{r'}(C), \mathrm{Tr}).$$

It is now not difficult to estimate $\Phi(P)$ and $\Psi(Q)$. □

3. *A sufficient condition for isomorphism of* ITPFI$_2$ factors

The main result of this section is theorem 3.1, which gives a sufficient condition for two ITPFI$_2$ factors to be isomorphic. By theorem 2.1, any ITPFI factor M of bounded type is isomorphic to some ITPFI$_2$ factor N. In corollary 3.6, we give such an N explicitly (in terms of the eigenvalues and multiplicities for M).

(3.1) THEOREM. *Let $(\phi_n)_{n\geq 1}$ be a sequence of states on $M_2(C)$ with eigenvalues $\{1/(1+\lambda_n),\ \lambda_n/(1+\lambda_n)\}$, $0<\lambda_n<1$ and such that $\sum_{n\geq 1}\lambda_n < \infty$. Let L_n, L'_n be two sequences of integers. Let M and N be the* ITPFI$_2$ factors:

$$M = \bigotimes_{n\geq 1} (M_2(C), \phi_n)^{\otimes L_n} \quad \text{and} \quad N = \bigotimes_{n\geq 1} (M_2(C), \phi_n)^{\otimes L'_n}$$

Let $e_n = L_n\lambda_n/(1+\lambda_n)$, $e'_n = L'_n\lambda_n/(1+\lambda_n)$, and d_n be the closest integer to $e'_n - e_n$; if $e'_n - e_n = b + \frac{1}{2}$, $d_n = b$ or $d_n = b+1$ does not affect the convergence of the series (cf. remark 3.4). *If*

$$\sum_{n\geq 1} \frac{(e'_n - e_n - d_n)^2}{e_n + e'_n} + \left(\frac{d_n}{e_n + e'_n}\right)^2 < \infty$$

and M is purely infinite, then M and N are isomorphic.

To simplify the coming notation, it will be convenient to assume that $L'_n \geq L_n$, for all n. (If not, put $L''_n = \sup\,(L_n, L'_n)$. Let P be the corresponding ITPFI$_2$ factor and compare M and N with P.)

We will prove that M and N have the same flow of weights. Let $M_0 = \bigotimes_{n\geq 1}(M_2(C), \phi_n)^{\otimes K_n}$. Let $(f_n)_{n\geq 1}$ be a sequence of integers and let μ_n be the probability measure on $\mathbb{Z}$, supported by $\{-f_n, 1-f_n, \ldots, K_n - f_n\}$ and given by

$$\mu_n(k - f_n) = \frac{K_n!}{k!(K_n-k)!}\cdot\frac{\lambda^k}{(1+\lambda)^{K_n}} \qquad \text{for } 0 \leq k \leq K_n.$$

(The f_n's will be used in lemma 3.3). Let (Ω, μ) be the product measure space $\prod_{n\geq 1}(\mathbb{Z}, \mu_n)$ and let $B\subset\Omega$ be a non-null set.

Let $\mathcal{R}$ be the equivalence relation on $B\times\mathbb{R}$ given by $(x, t)\mathcal{R}(y, s)$ iff there exists $q\in\mathbb{N}$ such that $x_n = y_n$, for all $n>q$, and

$$-\sum_{n=1}^{q} x_n \log \lambda_n + t = -\sum_{n=1}^{q} y_n \log \lambda_n + s.$$

The flow of weights of M_0 is given by the action of $\mathbb{R}$ by translation on $B\times\mathbb{R}/\mathcal{R}$ (see appendix).

Using Kakutani's criterion and lemma 3.3 below, we are going to show that M and N have the same flow of weights. We first need the following:

(3.2) LEMMA. *Let $0<\lambda<1$ be a real number and $L\leq L'$ be positive integers. Let $e = L\lambda/(1+\lambda)$, $e' = L'\lambda/(1+\lambda)$, d be the closest integer to $e'-e$ and $\delta = e'-e-d$. Set*

$$V_e = \frac{\lambda^d}{(1+\lambda)^{L'-L}}\cdot\prod_{i=1}^{d}\frac{1}{e+i}\prod_{i=1}^{L'-L-d}\frac{1}{L-e+i}\prod_{i=1}^{L'-L}(L+i).$$

Then,

$$V_e \geq \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}}\exp\left(-\frac{(1+\lambda)\delta^2}{e'}\right).$$

Proof. For $t\geq 0$, put $g(t) = \log(L'-d+t)/(e+t)$ and $h(t) = \log(L+t)/(L-e+t)$

$$\begin{aligned}\log V_e &= \log\left(\frac{\lambda^d}{(1+\lambda)^{L'-L}}\right)+\sum_{i=1}^{d} g(i)+\sum_{i=1}^{L'-L-d} h(i)\\ &= \tfrac{1}{2}(g(d)-g(0)+h(L'-L-d)-h(0))+\sum_{i=1}^{d}\frac{g(i-1)+g(i)}{2}\\ &\quad+\sum_{i=1}^{L'-L-d}\frac{h(i-1)+h(i)}{2}+\log\left(\frac{\lambda^d}{(1+\lambda)^{L'-L}}\right).\end{aligned}$$

One has

$$\begin{aligned}(g(d)-g(0)+h(L'-L-d)-h(0)) &= \log\left(\frac{L'}{e+d}\cdot\frac{e}{L'-d}\cdot\frac{L'-d}{L'-e-d}\cdot\frac{L-e}{L}\right)\\ &= \log\left(\frac{e}{e+d}\cdot\frac{L'}{L'-e-d}\cdot\frac{L-e}{L}\right)\\ &\geq \log\left(\frac{e}{e+d}\right)-\log(1+\lambda).\end{aligned}$$

Now, the functions g and h are convex, so that $[g(i-1)+g(i)]/2\geq\int_{i-1}^{i} g(t)\,dt$. Therefore,

$$\begin{aligned}&\sum_{i=1}^{d}\frac{g(i-1)+g(i)}{2}+\sum_{i=1}^{L'-L-d}\frac{h(i-1)+h(i)}{2}\\ &\geq \int_0^d g(t)\,dt+\int_0^{L'-L-d} h(t)\,dt\\ &= L'\log L'-(e+d)\log(e+d)-(L'-d)\log(L'-d)+e\log e\\ &\quad+(L'-d)\log(L'-d)-(L'-e-d)\log(L'-e-d)-L\log L\\ &\quad+(L-e)\log(L-e).\end{aligned}$$

Hence

$$V_e \geq \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}} \frac{L'^{L'} e^e (L-e)^{L-e}}{L^L (e+d)^{e+d} (L'-e-d)^{L'-e-d}} \frac{\lambda^d}{(1+\lambda)^{L'-L}}$$

$$= \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}} \frac{\left(\frac{L'}{1+\lambda}\right)^{L'} \left(\frac{e}{\lambda}\right)^e (L-e)^{L-e}}{\left(\frac{L}{1+\lambda}\right)^L \left(\frac{e+d}{\lambda}\right)^{e+d} (L'-e-d)^{L'-e-d}}.$$

Noting that $L/(1+\lambda) = e/\lambda = L-e$, we get

$$V_e \geq \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}} \frac{\left(\frac{L'}{1+\lambda}\right)^{L'}}{\left(\frac{e+d}{\lambda}\right)^{e+d} (L'-e-d)^{L'-e-d}}$$

$$= \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}} \left(\frac{\lambda L'}{(1+\lambda)(e+d)}\right)^{e+d} \left(\frac{L'}{(1+\lambda)(L'-e-d)}\right)^{L'-e-d}.$$

But $\lambda L'/(1+\lambda) = e'$, $L'/(1+\lambda) = L'-e'$. Using the inequality, $\log(a/b) \geq (a-b)/a$ we get:

$$(e+d)\log\frac{e'}{e+d} + (L'-e-d)\log\left(\frac{L'-e'}{L'-e-d}\right) \geq \frac{\delta(e+d)}{e'} - \frac{\delta(L'-e-d)}{L'-e'}$$

$$= \frac{\delta(e'-\delta)}{e'} - \frac{\delta(L'-e'+\delta)}{L'-e'}$$

$$= -\frac{\delta^2}{e'} - \frac{\delta^2}{L'-e'} = -\frac{(1+\lambda)\delta^2}{e'}.$$

Therefore,

$$V_e \geq \left(\frac{e}{(1+\lambda)(e+d)}\right)^{\frac{1}{2}} \exp\left(-\frac{(1+\lambda)\delta^2}{e'}\right). \qquad \square$$

(3.3) LEMMA. *Let μ and μ' be the (probability) measures on $\mathbb{Z}$ given by*

$$\mu(k) = \frac{L!}{k!(L-k)!} \frac{\lambda^k}{(1+\lambda)^L}$$

($\mu(k)=0$ for $k<0$ or $k>L$) and

$$\mu'(k) = \frac{L'!}{(k+d)!(L'-k-d)!} \frac{\lambda^{k+d}}{(1+\lambda)^{L'}}$$

($\mu'(k)=0$ for $k<-d$ or $k>L'-d$). Then we have

$$\rho(\mu,\mu') = \sum_{k=0}^{L} \mu(k)^{\frac{1}{2}} \mu'(k)^{\frac{1}{2}} \geq 1 - \frac{\lambda}{2} - \frac{\delta^2}{e'} - \frac{d^2}{2e(e+d)} - \frac{d}{6e^2}.$$

Proof. For all k, $0 \leq k \leq L$, we have

$$\frac{\mu'(k)}{\mu(k)} = V_e \prod_{i=1}^{d} \frac{e+i}{k+i} \prod_{i=1}^{L'-L-d} \frac{L-e+i}{L-k+i}.$$

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Hence,

$$\log\frac{\mu'(k)}{\mu(k)}=\log(V_e)-\sum_{i=1}^{d}\log\left(1+\frac{k-e}{e+i}\right)-\sum_{i=1}^{L'-L-d}\log\left(1+\frac{e-k}{L-e+i}\right).$$

Using the inequalities $\log(1+x)\leq x-x^2/2+x^3/3$ and $\log(1+x)\leq x$ we get:

$$\log\frac{\mu'(k)}{\mu(k)}\geq\log V_e+\sum_{i=1}^{d}\left(\frac{e-k}{e+i}+\frac{(e-k)^2}{2(e+i)^2}+\frac{(e-k)^3}{3(e+i)^3}\right)+\sum_{i=1}^{L'-L-d}\frac{k-e}{L-e+i}$$

Now $(\mu'(k)/\mu(k))^{\frac{1}{2}}\geq 1+\frac{1}{2}\log(\mu'(k)/\mu(k))$. As $\sum_{k=0}^{L}(e-k)\mu(k)=0$,

$$\sum_{k=0}^{L}(e-k)^2\mu(k)=\frac{L\lambda}{(1+\lambda)^2}=\frac{e}{1+\lambda}$$

and

$$\sum_{k=0}^{L}(e-k)^3\mu(k)=-\frac{L\lambda(1-\lambda)}{(1+\lambda)^3}=\frac{-e(1-\lambda)}{(1+\lambda)^2}$$

(cf. [7]). We deduce

$$\begin{aligned}\rho(\mu,\mu')&\geq 1+\tfrac{1}{2}\log V_e+\tfrac{1}{2}\sum_{i=1}^{d}\frac{e}{2(1+\lambda)(e+i)^2}-\frac{e(1-\lambda)}{3(1+\lambda)^2(e+i)^3}\\
&\geq 1-\tfrac{1}{4}\log(1+\lambda)-\tfrac{1}{4}\log\frac{e+d}{e}-\frac{(1+\lambda)\delta^2}{2e'}+\frac{1}{4}\frac{de}{(1+\lambda)(e+d)^2}-\frac{ed}{6e^3}\\
&\geq 1-\frac{\lambda}{4}-\frac{d}{4e}-\frac{\delta^2}{e'}+\frac{1}{4}\frac{de}{(e+d)^2}-\frac{1}{4}\frac{\lambda de}{(1+\lambda)(e+d)^2}-\frac{d}{6e^2}\\
&\geq 1-\frac{\lambda}{4}-\frac{\delta^2}{e'}-\frac{d((e+d)^2-e^2)}{4e(e+d)^2}-\frac{\lambda}{4}-\frac{d}{6e^2}\\
&\geq 1-\frac{\lambda}{2}-\frac{\delta^2}{e'}-\frac{d^2(2e+d)}{4e(e+d)^2}-\frac{d}{6e^2}\\
&\geq 1-\frac{\lambda}{2}-\frac{\delta^2}{e'}-\frac{d^2}{2e(e+d)}-\frac{d}{6e^2}.\end{aligned}$$

□

End of the proof of theorem 3.1. Let L_n, L'_n, λ_n, e_n, e'_n, d_n be as in theorem 3.1 with $L'_n\geq L_n$. Put $\delta_n=e'_n-e_n-d_n$. Let μ_n and μ'_n be the probability measures corresponding to L_n, L'_n, λ_n and d_n as in lemma 3.3. Let ν and ν' be the infinite product measures on $\prod_{n\geq 1}\mathbb{Z}$: $\nu=\bigotimes_{n\geq 1}\mu_n$, $\nu'=\bigotimes_{n\geq 1}\mu'_n$. For all n, μ_n is absolutely continuous with respect to μ'_n. Let $A=\{n\in\mathbb{N}\,|\,d_n\neq 0\}$. If $n\in A$, $e'_n-e_n\leq\frac{3}{2}d_n$. Therefore

$$\sum_{n\in A}\left(\frac{e'_n-e_n}{e'_n+e_n}\right)^2<\infty\quad\text{and}\quad\lim_{\substack{n\to\infty\\ n\in A}}\frac{e_n}{e'_n}=1.$$

Changing finitely many L_n's, we may assume that for all $n\in A$, $e_n\geq e'_n/2$. By lemma 3.3, we get

$$\begin{aligned}\sum_{n\geq 1}(1-\rho(\mu_n,\mu'_n))&\leq\sum_{n\geq 1}\frac{\lambda_n}{2}+\sum_{n\geq 1}\frac{\delta_n^2}{e'_n}+\sum_{n\in A}\frac{d_n^2}{2e_n(e_n+d_n)}+\sum_{n\in A}\frac{d_n}{6e_n^2}\\
&\leq\sum_{n\geq 1}\frac{\lambda_n}{2}+\sum_{n\geq 1}\frac{\delta_n^2}{e'_n}+\sum_{n\in A}\frac{2d_n^2}{e_n'^2}+\sum_{n\in A}\frac{2d_n^2}{3e_n'^2}<+\infty.\end{aligned}$$

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By Kakutani's criterion ([**11**, p. 453]) ν is absolutely continuous with respect to ν'. By the above description of the flow of weights we deduce (using Krieger's theorem [**12**]) that M and N are isomorphic. □

(3.4) *Remark.* Notice that the condition

$$\sum_{n\geq 1} \frac{(e'_n - e_n - d_n)^2}{e'_n + e_n} + \frac{d_n^2}{(e'_n + e_n)^2} < \infty$$

can be replaced by the equivalent ones:

(i) There exists a sequence of integers $(d'_n)_{n\geq 1}$ such that

$$\sum_{n\geq 1} \frac{(e'_n - e_n - d'_n)^2}{e'_n + e_n} + \frac{d'^2_n}{(e'_n + e_n)^2} < \infty.$$

(ii)

$$\sum_{n\geq 1} \frac{(e'_n - e_n - d_n)^2}{e'_n + e_n} + \frac{(e'_n - e_n)^2}{(e_n + e'_n + 1)^2} < \infty.$$

From theorem 3.1, we immediately get:

(3.5) Corollary. *With the notation of theorem* 3.1, *if* $\sum_{n\geq 1} |e'_n - e_n|/(e_n + 1) < \infty$, *and M is purely infinite, then M and N are isomorphic.*

Proof. As $e'_n - e_n - d_n \leq e'_n + e_n$,

$$\frac{(e'_n - e_n - d_n)^2}{e'_n + e_n} \leq |e'_n - e_n - d_n|.$$

As $|e'_n - e_n - d_n| \leq \frac{1}{2}$,

$$\frac{(e'_n - e_n - d_n)^2}{e'_n + e_n} \leq \inf\left\{|e'_n - e_n - d_n|, \frac{|e'_n - e_n - d_n|}{2(e'_n + e_n)}\right\}$$

$$\leq \frac{|e'_n - e_n|}{\sup(1, e_n)} \leq \frac{2|e'_n - e_n|}{e_n + 1}.$$

Moreover,

$$\frac{(e'_n - e_n)^2}{(e_n + e'_n + 1)^2} \leq \frac{|e'_n - e_n|}{e_n + 1}.$$

We conclude the proof using remark 3.4 (ii). □

Let a be a positive real number and λ_q be a sequence of real numbers with $e^{-(q+1)a} < \lambda_q \leq e^{-qa}$. For an integer k, let ψ_k be a state on $M_2(C)$ with eigenvalues $(1/(1+\lambda_k), \lambda_k/(1+\lambda_k))$. Let $m \geq 1$ be an integer and, for each multi-index $p = (p_1, \ldots, p_m)$, $0 \leq p_1 \leq p_2 \leq \cdots \leq p_m$, let ϕ_p be a state with eigenvalues $(1/\Lambda_p, \lambda_{p_1}/\Lambda_p, \cdots, \lambda_{p_m}/\Lambda_p)$ $(\Lambda_p = 1 + \sum_{i=1}^m \lambda_{p_i})$.

(3.6) Corollary. *Let L_p be a sequence of integers indexed by the m-tuples $p = (p_1, \ldots, p_m)$ with $0 \leq p_1 \leq \cdots \leq p_m$. For $k \in \mathbb{N}$, let*

$$R_k = \left[\sum_{i=1}^m \sum_{\{p|p_i = k\}} \frac{L_p(1+\lambda_k)}{\Lambda_p}\right].$$

If $M = \bigotimes_p (M_{m+1}(C), \phi_p)^{\otimes L_p}$ is of type III_0, *then it is isomorphic to $N = \bigotimes_k (M_2(C), \psi_k)^{\otimes R_k}$.*

Proof. We prove corollary 3.6 by induction on m. If $m=1$ there is nothing to prove.

Let $M=\bigotimes_{(p,q)}(M_{m+2}(C),\phi_{p,q})^{\otimes L_{p,q}}$ be an ITPFI_{m+2} factor written as in the proof of theorem 2.1. Let

$$M_1=\bigotimes_{p,q}[(M_{m+1}(C),\phi_p)^{\otimes K_{p,q}}\otimes(M_2(C),\phi_q)^{\otimes J_{p,q}}]$$

where $K_{p,q}=L_{p,q}-[\lambda_q L_{p,q}/\Lambda_{p,q}]$, $(\Lambda_{p,q}=1+\sum_{i=1}^m\lambda_{p_i}+\lambda_q=\Lambda_p+\lambda_q)$, $J_{p,q}=[((1+\lambda_q)/\Lambda_{p,q})L_{p,q}]$.

By the proof of theorem 2.1, M and M_1 are isomorphic. By induction, M_1 is isomorphic to $N_1=\bigotimes_k(M_2(C),\phi_k)^{\otimes S_k}$ where

$$S_k=\left[\sum_{i=1}^m\sum_{\substack{p,q\\p_i=k}}\frac{K_{p,q}(1+\lambda_k)}{\Lambda_p}\right]+\sum_p J_{p,k}.$$

As $K_{p,q}/\Lambda_p\geq L_{p,q}/\Lambda_{p,q}$, we have:

$$\begin{aligned}S_k&\geq\left[\sum_{i=1}^m\sum_{\{(p,q)|p_i=k\}}\frac{L_{p,q}(1+\lambda_k)}{\Lambda_{p,q}}\right]+\sum_p\left[\frac{L_{p,k}(1+\lambda_k)}{\Lambda_{p,k}}\right]\\&\geq R_k-\frac{(k+m)!}{k!m!},\end{aligned}$$

as

$$R_k=\left[\sum_{i=1}^m\sum_{\{(p,q)|p_i=k\}}\frac{L_{p,q}(1+\lambda_k)}{\Lambda_{p,q}}+\sum_p\frac{L_{p,k}(1+\lambda_k)}{\Lambda_{p,k}}\right].$$

On the other hand,

$$S_k\leq\left[\sum_{i=1}^m\sum_{\{(p,k)|p_i=k\}}\frac{L_{p,q}(1+\lambda_k)}{\Lambda_p}+\sum_p\frac{L_{p,k}(1+\lambda_k)}{\Lambda_{p,k}}\right].$$

Note that for $k\leq q$ we have

$$\frac{\Lambda_{p,q}}{\Lambda_{p,q}-\lambda_q}\leq\frac{\Lambda_{p,q}-\lambda_q+\lambda_k}{\Lambda_{p,q}-\lambda_q}\leq 1+\lambda_k.$$

Hence

$$S_k<(1+\lambda_k)(R_k+1).$$

We have $\lambda_k(R_k-S_k)\leq\lambda_k(k+m)!/k!m!$ and $(S_k-(R_k+1))/(R_k+1)<\lambda_k$. Therefore,

$$\left|\frac{\lambda_k(R_k-S_k)}{\lambda_k R_k+1}\right|\leq\lambda_k\frac{(k+m)!}{k!m!},$$

so that

$$\sum_{k=1}^{+\infty}\left|\frac{\lambda_k(R_k-S_k)}{\lambda_k R_k+1}\right|<+\infty.$$

By corollary 3.5, $N=\bigotimes_{k=1}^\infty(M_2(C),\psi_k)^{\otimes R_k}$ is isomorphic to N_1. □

Theorem 3.1 gives also a (negative) answer to problem 7 of [**16**, p. 37].

(3.7) COROLLARY. *Let L_k be a sequence of integers and λ, $0<\lambda<1$, a real number. Let ϕ_k be a state on $M_2(C)$ with eigenvalues $\{1/(1+\lambda^k),\ \lambda^k/(1+\lambda^k)\}$. Let M be the ITPFI_2 factor: $M=\bigotimes_{k\geq 1}(M_2(C),\phi_k)^{\otimes L_k}$.*

Then if M is not of type I, *there exists a sequence L'_k such that $\sum_{k\geq 1}\lambda^k|L'_k - L_k| = +\infty$ and M is isomorphic to the* ITPFI_2 *factor* $N = \bigotimes_{k\geq 1}(M_2(C), \phi_k)^{\otimes L'_k}$.

Proof. The factor M not being of type I, the sequence $(L_k\lambda^k)$ is not summable. Therefore there exists a sequence $(a_k)\in \ell_2(\mathbb{N})$ such that $\sum_{k\geq 1}(L_k\lambda^k)^{\frac{1}{2}}a_k = +\infty$. Let $L'_k = L_k + [a_k(L_k\lambda^{-k})^{\frac{1}{2}}]$ We have $\sum_k (L'_k - L_k)\lambda^k = +\infty$ and $\sum_k \lambda^k(L'_k - L_k)^2/L_k < +\infty$. This implies, by remark 3.4(i), that M and N are isomorphic. □

4. *An example*

Theorem 3.1 does not admit a converse in general (for example, the Powers factors of type III_λ or the Araki & Woods type III_1 factor can be written in many completely different ways).

However in some situations it does admit a converse (proposition 4.4). This yields a quite surprising example (4.5).

Let L_n be a sequence of positive integers and let λ_n be a sequence of positive real numbers. Let μ_n be the measure on $\mathbb{Z}$ (supported by $\{0, \ldots, L_n\}$) corresponding to (L_n, λ_n) and let $\mu = \bigotimes_{n\geq 1}\mu_n$ be the product measure on $\prod_{n\geq 1}\mathbb{Z}$.

(4.1) *Definition.* The sequence (L_n, λ_n) is said to *satisfy condition* C if there exists $\varepsilon > 0$ and sequences a_n, b_n of integers such that

(i) $\mu(\{x = (x_n)_{n\geq 1}\in \prod_{n\geq 1}\mathbb{Z} \mid a_n \leq x_n \leq b_n, \text{ for all } n\geq 1\}) > 0$;

(ii) for all $n \geq 1$, $-\log \lambda_n \geq \varepsilon - \sum_{k=1}^{n-1} 2(b_k - a_k)\log \lambda_k$.

A much stronger condition was used by Araki & Woods to prove the existence of type III_0 factors ([**1**, § 10], cf. also [**15**]).

(4.2) *Remark.* Let a_n, b_n be as in definition 4.1 and let $B = \{x\in \prod_{n\geq 1}\mathbb{Z} \mid a_n \leq x_n \leq b_n\}$. Using the description of the flow of weights given in the appendix, $B\times\mathbb{R}$ maps onto the flow space. Condition (ii) says that $B\times[0, \varepsilon)$ maps injectively to the flow space. In particular if $\sum_{n\geq 1} L_n\lambda_n = +\infty$ the corresponding factor is of type III_0.

(4.3) Lemma. *Let (L_n, λ_n), (L'_n, λ_n) satisfy condition* C. *Assume that the corresponding factors are isomorphic.*

Then, there exists a sequence of integers $(c_n)_{n\geq 1}$ such that $\phi_*\mu$ and μ' are not (mutually) singular, where $\phi: \prod_{n\geq 1}\mathbb{Z}\to\prod_{n\geq 1}\mathbb{Z}$ is defined by $\phi(x)_n = x_n + c_n$. (μ and μ' denote the measures on $\prod_{n\geq 1}\mathbb{Z}$ associated with (L_n, λ_n) and (L'_n, λ_n), as above.)

Proof. Let a_n, b_n, a'_n, b'_n, η $(= \inf(\varepsilon, \varepsilon'))$ be given by definition 4.1.

Let $B_0 = \prod_{n\geq 1}\{a_n, \ldots, b_n\}$, $B'_0 = \prod_{n\geq 1}\{a'_n, \ldots, b'_n\}$. Let $T_0: B_0\to B_0$, $T'_0: B'_0\to B'_0$ be the odometers. Let $x\in B_0$ and $p\in\mathbb{N}$. Then $(T_0^p x)_n = x_n$ for n large enough. Let

$$\xi(x, p) = -\sum_{n\geq 1}((T_0^p x)_n - x_n)\log\lambda_n$$

and

$$\xi'(x', p) = -\sum_{n\geq 1}((T_0'^p x')_n - x'_n)\log\lambda_n \qquad (x'\in B'_0).$$

As the corresponding factors are isomorphic, there exists an isomorphism between their flows of weights.

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These flows are constructed over the base transformations (B_0, μ, T_0), (B'_0, μ', T'_0) with the ceiling functions $\xi(x, 1)$ and $\xi'(x', 1)$. The isomorphism of the flows means that there exist: non-null subsets B_1, B'_1 of B_0 and B'_0; a measure-class preserving isomorphism $\phi: B_1 \to B'_1$ intertwining the induced transformations T_1 and T'_1; a measurable map $\alpha: B_1 \to \mathbb{R}$ such that if ξ_1 and ξ'_1 are the induced ceiling functions $(\xi_1(x) = \xi(x, m(x))$ if $T_1(x) = T_0^{m(x)}(x))$,

$$\xi'_1(\phi(x)) = \xi_1(x) + \alpha(T_1(x)) - \alpha(x)$$

for almost all x.

There exists an interval I in $\mathbb{R}$ of length smaller than η such that $\mu(\alpha^{-1}(I)) \neq 0$. Let $B = \alpha^{-1}(I)$, $B' = \phi(B)$. Let T and T' be the induced transformations in B and B'.

For $x \in B$ (resp. $x' \in B'$), let $p(x)$ (resp. $p'(x')$) be the integer satisfying $T(x) = T_0^{p(x)}(x)$ (resp. $T'(x')) = T_0'^{p'(x')}(x')$). Let $q(x)$ be defined by $T(x) = T_1^{q(x)}(x)$. Note that $T'(\phi(x)) = \phi(T(x)) = T_1'^{q(x)}(\phi(x))$.

Put $\xi(x) = \xi(x, p(x)) = \sum_{i=0}^{q(x)-1} \xi_1(T_1^i(x))$; $\xi'(x') = \xi'(x', p'(x'))$. Note that we still have $\xi'(\phi(x)) = \xi(x) + \alpha(Tx) - \alpha(x)$. (As $\xi'(\phi(x)) - \xi(x) = \sum_{i=0}^{q(x)-1} (\xi_1(\phi(T_1^i(x))) - \xi_1(T_1^i(x))) = \alpha(Tx) - \alpha(x)$.)

Now, for all x and p, x' and p', we have

$$|\xi(x, p) - \xi'(x', p')| \leq \eta \Rightarrow \xi(x, p) = \xi'(x', p')$$

(condition C). We derive $\xi'(\phi x)) = \xi(x)$. Hence,

$$-\sum_{n\geq 1} ((Tx)_n - x_n) \log \lambda_n = -\sum_{n\geq 1} ((T'\phi(x))_n - (\phi(x))_n) \log \lambda_n.$$

By condition C, we get $(Tx)_n - x_n = (T'\phi(x))_n - (\phi(x))_n$ for all n, which means $(\phi(x))_n - x_n = (\phi T(x))_n - (Tx)_n$. By ergodicity of T, we get that $(\phi(x))_n - x_n$ is essentially constant (equal to an integer c_n) for all n. Moreover, $\phi_*\mu|_B$ and $\mu'|_{B'}$ are equivalent. Therefore, if we define $\phi: \prod_{n\geq 1} \mathbb{Z} \to \prod_{n\geq 1} \mathbb{Z}$, by $(\phi(x))_n = x_n + c_n$, we get that $\phi_*\mu$ and μ' are not singular. □

The following proposition is a partial converse of theorem 3.1.

(4.4) Proposition. *Let (L_n, λ_n), (L'_n, λ_n) satisfy condition* C. *If the corresponding* ITPFI$_2$ *factors are isomorphic, then*

$$\sum_{n\geq 1} \frac{(e'_n - e_n - d_n)^2}{e'_n + e_n} + \left(\frac{d_n}{e'_n + e_n}\right)^2 < \infty,$$

where $e_n = L_n\lambda_n/(1+\lambda_n)$, $e'_n = L'_n\lambda_n/(1+\lambda_n)$ and d_n is the closest integer to $e'_n - e_n$.

Proof. By lemma 4.3, there exists a sequence $(c_n)_{n\geq 1}$ of integers such that $\phi_*\mu$ and μ' are not singular.

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers such that $\sum_{n\geq 1} a_n^2(e_n + e'_n)/(1+\lambda_n) < \infty$ and $a_n(e'_n - e_n - c_n) \geq 0$. As we have $\sigma_\mu^2(x_n) = e_n/(1+\lambda_n)$ and $\sigma_{\mu'}^2(x_n) = e'_n/(1+\lambda_n)$ and using a theorem of Kolmogorov (cf. theorem B of § 46 of [**9**]), we get: $\sum_{n\geq 1} a_n(x_n - e_n)$ converges μ-a.e., $\sum_{n\geq 1} a_n(x_n - c_n - e_n)$ converges $\phi_*\mu$-a.e., also $\sum_{n\geq 1} a_n(x_n - e'_n)$ converges μ'-a.e.

As $\phi_*\mu$ and μ' are not singular, we get

$$\sum_{n\geq 1} a_n(e'_n - c_n - e_n) < \infty.$$

If $(e'_n - e_n - c_n)/(e'_n + e_n)^{\frac{1}{2}} \notin \ell_2(\mathbb{N})$, there exists a sequence $(b_n)_{n\geq 1} \in \ell_2(\mathbb{N})$ such that $\sum_{n\geq 1} |b_n(e'_n - e_n - c_n)|/(e'_n + e_n)^{\frac{1}{2}} = \infty$. Setting $|a_n| = |b_n|/(e'_n + e_n)^{\frac{1}{2}}$, we get a contradiction. Therefore, $(e'_n - e_n - c_n)/(e'_n + e_n)^{\frac{1}{2}} \in \ell_2(\mathbb{N})$ and $\sum_{n\geq 1} (e'_n - e_n - d_n)^2/(e'_n + e_n) < \infty$, where d_n denotes the closest integer to $e'_n - e_n$.

Put $\delta_n = e'_n - e_n - c_n$, for all $n \geq 1$. We have:

$$\begin{aligned}\sigma^2_\mu\left(\left(x_n - e_n - \frac{\delta_n}{2}\right)^2\right) &= \frac{L_n\lambda_n}{(1+\lambda_n)^3}\left(\frac{2L_n\lambda_n}{1+\lambda_n} + \frac{1-4\lambda_n+\lambda_n^2}{1+\lambda_n}\right) \\ &\quad - 2\delta_n \frac{L_n\lambda_n(1-\lambda_n)}{(1+\lambda_n)^3} + \frac{\delta_n^2 L_n\lambda_n}{(1+\lambda_n)^2} \\ &\leq 2e_n^2 + e_n + 2|\delta_n| e_n + \delta_n^2 e_n.\end{aligned}$$

Also,

$$\sigma^2_{\mu'}\left(\left(x_n - e'_n + \frac{\delta_n}{2}\right)^2\right) \leq 2e'^2_n + e'_n + 2|\delta_n| e'_n + \delta_n^2 e'_n.$$

As $\sum_{n\geq 1} \delta_n^2/(e_n + e'_n) < +\infty$, we may assume $|\delta_n| \leq (e_n + e'_n)^{\frac{1}{2}}$ for all n. We get

$$\begin{aligned}&\sigma^2_\mu\left(\left(x_n - e_n - \frac{\delta_n}{2}\right)^2\right) + \sigma^2_{\mu'}\left(\left(x_n - e'_n + \frac{\delta_n}{2}\right)^2\right) \\ &\quad \leq 2(e_n + e'_n)^2 + e_n + e'_n + 2(e_n + e'_n)^{\frac{3}{2}} + (e_n + e'_n)^2 \\ &\quad \leq 4(e_n + e'_n)^2 + 2(e_n + e'_n) \\ &\quad \leq 4(e_n + e'_n + 1)^2.\end{aligned}$$

If $\sum_{n\geq 1} (e_n - e'_n)^2/(e_n + e'_n + 1)^2 = \infty$, there exists a sequence $(b_n)_{n\geq 1}$ such that $\sum_{n\geq 1} b_n^2 < \infty$, $b_n(e_n - e'_n) \geq 0$ and $\sum_{n\geq 1} b_n(e_n - e'_n)/(e_n + e'_n + 1) = \infty$. Put $a_n = b_n/(e_n + e'_n + 1)$.

As $\sum_{n\geq 1} a_n^2(\sigma^2_\mu((x_n - e_n - [\delta_n/2])^2) + \sigma^2_{\mu'}((x_n - e'_n + [\delta_n/2])^2)) < \infty$, by a theorem of Kolmogorov (cf. [**9**, § 46, theorem B]), we get

$$\textstyle\sum_{n\geq 1} a_n((x_n - e_n - (\delta_n/2))^2 - e_n/(1+\lambda_n) - \delta_n^2/4)$$

converges μ-a.e. Hence,

$$\textstyle\sum_{n\geq 1} a_n((x_n - e'_n + (\delta_n/2))^2 - e_n/(1+\lambda_n) - \delta_n^2/4)$$

converges $\phi_*\mu$-a.e. Also,

$$\textstyle\sum_{n\geq 1} a_n((x_n - e'_n + (\delta_n/2))^2 - e'_n/(1+\lambda_n) - \delta_n^2/4))$$

converges μ'-a.e. As $\phi_*\mu$ and μ' are not singular, we get

$$\textstyle\sum_{n\geq 1} a_n(e_n - e'_n)/(1+\lambda_n) < \infty,$$

which contradicts our assumption $\sum_{n\geq 1} b_n(e_n - e'_n)/(e_n + e'_n + 1) = \infty$.

The proposition now follows from remark 3.4. □

(4.5) *Example.* Let $\lambda \in (0, 1)$. Put $\lambda_k = \lambda^{(2k)!}$ $(k \geq 1)$. Let

$$L_k = \left[\left(\frac{1+\lambda_k}{\lambda_k}\right)k\right], \quad L'_k = \left[\left(\frac{1+\lambda_k}{\lambda_k}\right)(k+\tfrac{1}{2})\right], \quad L''_k = \left[\left(\frac{1+\lambda_k}{\lambda_k}\right)(k+1)\right].$$

One easily checks that the sequences (L_k, λ_k), (L'_k, λ_k), (L''_k, λ_k) satisfy condition C (taking $a_k = 0$, $b_k = k(k+1)$ and using Tchebyshev's inequality).

Let M, N, P be the corresponding factors. Using theorem 3.1, we get that M and P are isomorphic; using proposition 4.4, we get that M and N are not isomorphic. Note that, for all k, $L_k \leq L'_k \leq L''_k$.

Using theorem 3.1, again we get that $M \otimes M$ and $N \otimes N$ are isomorphic.

Let $\Lambda_k = [(1+\lambda_k)/2\lambda_k]$ and let Q be the factor corresponding to (Λ_k, λ_k). We have $M \otimes Q$ $(\cong N)$ is not isomorphic to M, though $M \otimes Q \otimes Q$ $(\cong P)$ is isomorphic to M.

Taking $\Lambda_k = [(1+\lambda_k)/q\lambda_k]$, where q is an integer, we get factors M and Q such that $M \otimes Q^{\otimes j}$, $j = 0, \ldots, q-1$, are pairwise non-isomorphic but $M \otimes Q^{\otimes q}$ is isomorphic to M.

Appendix

In §§ 2, 3 we use only a partial determination of the Connes-Takesaki flow of weights. In order to make the exposition reasonably self-contained, we outline the relevant aspects of this construction. In part (A) we set it in the general framework of equivalence relations. In part (B) we specialize it to the ITPFI case. Finally in parts (C) and (D) we indicate how this partial construction is used in §§ 3(C) and 2(D).

(A) Let $\mathcal{R}_0$ be a type III discrete ergodic equivalence relation on the Lebesgue measure space (Ω_0, P). Let $M = M(\mathcal{R}_0)$ be the corresponding factor.

By [**5**, theorem II 6.2] (cf. also [**4**], [**8**], [**10**], [**13**], [**14**]) the flow of weights of M is given by the action of $\mathbb{R}$ by translation in $\Omega_0 \times \mathbb{R}/\tilde{\mathcal{R}}_0$ (this quotient stands for the ergodic decomposition) where $\tilde{\mathcal{R}}_0$ is the equivalence relation in $(\Omega_0 \times \mathbb{R}, P \times m)$ given by $(x, t)\tilde{\mathcal{R}}_0(y, s)$ iff $x\mathcal{R}_0 y$ and $s = t - \log \delta(x, y)$ where $\delta(x, y)$ is the module of P (cf. [**4**], [**6**, p. 434]) (m is a finite measure on $\mathbb{R}$ equivalent to the Lebesgue measure). Note that as $(\Omega_0, P, \mathcal{R}_0)$ is weakly equivalent to its restriction $(A, P, \mathcal{R}_{0|A})$ for $A \subseteq \Omega_0$, $P(A) \neq 0$, the flow of weights may also be realized in $A \times \mathbb{R}/\tilde{\mathcal{R}}_{0|A\times\mathbb{R}}$.

This ergodic decomposition has explicitly been determined in some cases (cf. [**3**], [10, § I.6], it can also be obtained in § 4 of this paper). In general, however, it seems to be a problem.

Let $\tilde{\mathcal{R}}_1$ be another equivalence relation, $\tilde{\mathcal{R}}_1 \subseteq \tilde{\mathcal{R}}_0$. The ergodic decomposition $(\Omega_0 \times \mathbb{R})/\tilde{\mathcal{R}}_0$ can be obtained in two steps, computing first $(Y, \mu) = (\Omega_0 \times \mathbb{R}, P \times m)/\tilde{\mathcal{R}}_1$ and then $(Y, \mu)/\tilde{\mathcal{R}}$ where $\tilde{\mathcal{R}}$ is a discrete equivalence relation given below. The interest of this construction is that, in the cases we are interested in, we are able to compute both the quotient (Y, μ) and the equivalence relation $\tilde{\mathcal{R}}$.

The measure $P \times m$ admits the disintegration $P \times m = \int_Y \upsilon_y d\mu(y)$ where the υ_y are pairwise singular $\tilde{\mathcal{R}}_1$-quasi-invariant and ergodic measures on $\Omega \times \mathbb{R}$. Then $\tilde{\mathcal{R}}$ is defined by any of the equivalent conditions:

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$y\tilde{\mathscr{R}}y'$ iff (i) for all measurable subsets E, E' of $\Omega_0\times\mathbb{R}$ such that $\upsilon_y(E)\neq 0$, $\upsilon_{y'}(E')\neq 0$ there exist $x\in E$, $x'\in E'$ with $x\tilde{\mathscr{R}}_0x'$, or

(ii) there exists a partial borel transformation ϕ whose graph is contained in the graph of $\tilde{\mathscr{R}}_0$ such that $\upsilon_y(\operatorname{dom}\phi)=1$ and that $\phi_*(\upsilon_y)$ and $\upsilon_{y'}$ are not singular.

If $\tilde{\mathscr{R}}_0$ is defined by the countable set $(g_n)_{n\geq 1}$ of transformations of $\Omega_0\times\mathbb{R}$, we also have the condition

(iii) $m_y=\sum_{n\geq 1}2^{-n}g_{n*}(\upsilon_y)$ and $m_{y'}=\sum_{n\geq 1}2^{-n}g_{n*}(\upsilon_{y'})$ are equivalent.

(To see the equivalence of (i), (ii) and (iii) note that m_y and $m_{y'}$ are $\tilde{\mathscr{R}}_0$-quasi-invariant and ergodic, hence either singular or equivalent).

(B) In the ITPFI case, $(\Omega_0, P)=\prod_{n\geq 1}(X_n, P_n)$ where (X_n, P_n) is a finite probability space for all n. The equivalence relation $\mathscr{R}_0$ is given by $\omega\mathscr{R}_0\omega'$ iff $\omega_n=\omega'_n$ for all but finitely many n's. There may be some multiplicity in the eigenvalues, which means, in our setting, that there may be x, $x'\in X_n$ with $P_n(x)=P_n(x')$ $(x\neq x')$. Let $\mathscr{S}_n$ be an equivalence relation in X_n preserving P_n(i.e. if $x\mathscr{S}_nx'$, $P_n(x)=P_n(x')$).

Let $\mathscr{R}_1\subseteq\mathscr{R}_0$ be the equivalence relation given by $\omega\mathscr{R}_1\omega'$ iff $\omega\mathscr{R}_0\omega'$ and for all n, $\omega_n\mathscr{S}_n\omega'_n$. Let $\tilde{\mathscr{R}}_1\subseteq\tilde{\mathscr{R}}_0$ be given by $(\omega, t)\tilde{\mathscr{R}}_1(\omega', s)$ iff $\omega\mathscr{R}_1\omega'$ and $t=s$.

Clearly $\Omega_0\times\mathbb{R}/\tilde{\mathscr{R}}_1=\Omega_0/\mathscr{R}_1\times\mathbb{R}$ and $(\Omega_0, P)/\mathscr{R}_1=\prod_{n\geq 1}((X_n, P_n)/\mathscr{S}_n)$.

Put $(X_n, P_n)/\mathscr{S}_n=(\Omega_n, \mu_n)$. Let $\mathscr{R}$ be the equivalence relation on $(\Omega, \upsilon)=\prod_{n\geq 1}(\Omega_n, \mu_n)$ given by $x\mathscr{R}y$ iff there exists k such that $x_n=y_n$, $n\geq k$. Let $\pi\colon\Omega_0\to\Omega$ be the projection. If $x\mathscr{R}y$, let ω, $\omega'\in\Omega_0$ be such that $\pi(\omega)=x$, $\pi(\omega')=y$ and $\omega\mathscr{R}_0\omega'$. As $\mathscr{S}_n$ preserves P_n, we notice that

$$\delta_0(\omega, \omega')=\prod_{n\geq 1}P_n(\omega_n)P_n(\omega'_n)^{-1}$$

does not depend on the choice of ω, ω'. Put $\delta(x, y)=\delta_0(\omega, \omega')$. Then $\tilde{\mathscr{R}}$ is given by $(x, t)\tilde{\mathscr{R}}(y, s)$ iff there exists $k\geq 1$ such that $x_n=y_n$, $n\geq k$ and $s=t-\log\delta(x, y)$.

(C) In § 3, we have $(X_n, P_n)=(\{0, 1\}, \alpha_n)^{\otimes K_n}$ where $\alpha_n(0)=1/(1+\lambda_n)$, $\alpha_n(1)=\lambda_n/(1+\lambda_n)$. If $x=(x_l)_{l=1,\ldots,K_n}\in X_n$ put $k(x)=\#\{l\in\{1,\ldots, K_n\};\, x_l=1\}$. We have

$$P_n(x)=\frac{\lambda_n^{k(x)}}{(1+\lambda_n)^{K_n}}.$$

The equivalence relation $\mathscr{S}_n$ is here $x\mathscr{S}_nx'$ iff $k(x)=k(x')$. The quotient of X_n by $\mathscr{S}_n$ is $\{0,\ldots, K_n\}$ and the measure μ_n is given by

$$\begin{aligned}\mu_n(k)&=\sum_{x, k(x)=k}P_n(x)\\&=(\#\{x, k(x)=k\})\cdot\frac{\lambda_n^k}{(1+\lambda_n)^{K_n}}=\frac{K_n!}{(K_n-k)!k!}\cdot\frac{\lambda_n^k}{(1+\lambda_n)^{K_n}}.\end{aligned}$$

It is also convenient in § 3 to replace $\{0,\ldots, K_n\}$ by $\{-f_n, 1-f_n,\ldots, K_n-f_n\}$ considered as a subset of $\mathbb{Z}$. Note that if ω, $\omega'\in\Omega=\prod_{n\geq 1}\mathbb{Z}$ and if $\omega\mathscr{R}\omega'$ then $\delta(\omega, \omega')=\prod_{n\geq 1}\lambda_n^{\omega_n-\omega'_n}$ (if $\omega\mathscr{R}\omega'$, all but finitely many $\omega_n-\omega'_n$ are zero).

(D) In § 2, we have $(X_{p,q}, P_{p,q}) = (\{0, 1, \ldots, m, m+1\}, \alpha_{p,q})^{\otimes L_{p,q}}$ where $\alpha_{p,q}(0) = 1/\Lambda_{p,q}$, $\alpha_{p,q}(i) = \lambda_{p_i}/\Lambda_{p,q}$, $1 \le i \le m$, and $\alpha_{p,q}(m+1) = \lambda_q/\Lambda_{p,q}$. For $x \in X_{p,q}$, $i = 1, \ldots, m$, put $k_i(x) = \#\{l \in \{1, \ldots, L_{p,q}\};\ x_l = i]$ and $j(x) = \#\{l \in \{1, \ldots, L_{p,q}\};\ x_l = m+1\}$. The equivalence $\mathcal{S}_{p,q}$ is given by $x\mathcal{S}_{p,q}x'$ iff $k_i(x) = k_i(x')$ for all i and $j(x) = j(x')$. The quotient of $(X_{p,q}, P_{p,q})$ by $\mathcal{S}_{p,q}$ is the space of (k, j) considered in § 2, namely $X_{p,q}/\mathcal{S}_{p,q} = \{(k, j) \in \mathbb{N}^m \times \mathbb{N};\ |k| + j \le L_{p,q}\}$, and the quotient measure is the measure $\mu_{p,q}$ of § 2.

We also consider

$$(X'_{p,q}, P'_{p,q}) = (\{0, 1, \ldots, m\}, \beta_p)^{\otimes K_{p,q}} \times (\{0, 1\}, \gamma_q)^{\otimes J_{p,q}}$$

where $\beta_p(i) = \lambda_{p_i}/\Lambda_p$, $\gamma_q(0) = 1/(1+\lambda_q)$, $\gamma_q(1) = \lambda_q/(1+\lambda_q)$. An element $x \in X'_{p,q}$ is of the form (y, z), $y = (y_l)_{l=1,\ldots,K_{p,q}}$, $y_l \in \{0, 1, \ldots, m\}$, $z = (z_p)_{l=1,\ldots,J_{p,q}}$, $z_l \in \{0, 1\}$. If $x \in X'_{p,q}$ put $k_i(x) = \#\{l \in \{1, \ldots, K_{p,q}\};\ y_l = i\}$ and $j(x) = \#\{l \in \{1, \ldots, J_{p,q}\};\ z_l = 1\}$.

The equivalence relation $\mathcal{S}'_{p,q}$ is given by $x\mathcal{S}'_{p,q}x'$ iff $k_i(x) = k_i(x')$ for all i and $j(x) = j(x')$. The quotient of $(X'_{p,q}, P'_{p,q})$ by $\mathcal{S}'_{p,q}$ is the space $\{(k, j) \in \mathbb{N}^m \times \mathbb{N}, |k| \le K_{p,q}, j \le J_{p,q}\}$ and the measure $\mu'_{p,q}$ of § 2.

REFERENCES

[1] H. Araki & E. J. Woods. A classification of factors. *Publ. Res. Inst. Math. Sci. Ser.* A **4** (1968), 51–130.
[2] A. Connes. Une classification des facteurs de type III. *Ann Sci. E.N.S.* 4 *eme Série* **6** (1973), 133–252.
[3] A. Connes. On the hierarchy of W. Krieger. *Illinois J. Math.* **19**, (1975), 428–432.
[4] A. Connes. *Sur la Théorie non-commutative de l'Intégration.* Lecture Notes in Math. no 725. Springer Verlag, 1979.
[5] A. Connes & M. Takesaki. The flow of weights on factors of type III. *Tôhoku Math. J.* **29** (1977), 473–575.
[6] A. Connes, J. Feldman & B. Weiss. An amenable equivalence relation is generated by a single transformation. *Ergod. Th. & Dynam. Sys.* **1** (1981), 431–450.
[7] N. Draper & W. Lawrence. *Probability: an Introductory Course.* Markham: Chicago, 1970.
[8] P. Hahn. Regular representations of measure groupoids. *Trans. Amer. Math. Soc.* **242** (1978), 35–72.
[9] P. Halmos. *Measure Theory.* van Nostrand, 1950.
[10] T. Hamachi & M. Osikawa. Ergodic Groups of Automorphisms and Krieger's Theorems. Seminar on Math. Sci. Keio Univ. **3** (1981).
[11] E. Hewitt & K. Stromberg. *Real and Abstract Analysis.* Springer Verlag: New York, 1969.
[12] W. Krieger. On ergodic flows and the isomorphism of factors. *Math. Ann.* **223** (1976), 19–70.
[13] J. L. Sauvageot. Image d'un homomorphisme et flot des poids d'une relation d'équivalence mesurée. *C.R. Acad. Sci. Paris, Sér A.* **282** (1976), 619–621.
[14] C. Sutherland. Notes on orbit equivalence: Krieger's Theorem, Lecture Note series No. 23, Univ. i Oslo, 1976. Preprint.
[15] E. J. Woods. The classification of factors is not smooth. *Can. J. Math.* **25** (1973), 96–102.
[16] E. J. Woods. ITPFI factors—A survey. In *Proc. of Symposia in Pure Mathematics* **38** (1982), Part 2, 25–41.