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TWO CONSEQUENCES OF BRUNEL'S THEOREM

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ABSTRACT. In this note we observe two consequences of Brunel's recent theorem. If $T_1, \ldots, T_n$ are majorized by positive power-bounded operators $S_1, \ldots, S_n$ of L_p, $1 < p < \infty$, for which the ergodic theorem holds, then a multiple sequence ergodic theorem holds for $T_1, \ldots, T_n$. Further, the individual convergence for each T_k can be taken along uniform sequences.

1. **Introduction.** In what follows, we assume p fixed, $1 < p < \infty$. Let $(\underline{X}, \mathcal{P}, \mu)$ be a probability space, $\{T_k\}_{k=1}^n$ linear operators of $L_p(\underline{X}, \mathcal{P}, \mu) = L_p$. If T_k takes non-negative functions to non-negative functions, we say that T_k is *positive*. If there is a linear operator S_k such that $|T_k f| \leq S_k|f|$ for all f in L_p, we say that S_k *majorizes* T_k. If there is a constant B such that $\|T_k{}^n\|_p \leq B$ for all n, $n = 1, 2, \ldots$, we say that T_k is *power-bounded* with power bound B.

We put

$$A(m_1, \ldots, m_n; T_1, \ldots, T_n; f) = \frac{1}{m_1 \ldots m_n} \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_n=0}^{m_n-1} T_1{}^{k_1}, \ldots, T_n{}^{k_n} f$$

and

$$M(T_1, \ldots, T_n; f) = \sup_{m_1 \ldots m_n} |A(m_1 \ldots m_n; T_1, \ldots, T_n; f)|.$$

If there exists a constant C such that $\|M(T_1, \ldots, T_n; f)\|_p \leq C\|f\|_p$ for all $f \in L_p$, we say that $\{T_k\}_{k=1}^n$ admits a dominated estimate with constant C. The celebrated theorem of Brunel [3] states that if T_1 is a positive linear operator of L_p, T_1 admits a dominated estimate if and only if T_1 is Cesaro-bounded, (i.e., $\sup_n \|A(n, T_1, \cdot)\|_p < \infty$). Since a power-bounded operator is clearly Ceasaro-bounded, Brunel's theorem implies that positive, or positively dominated, power-bounded operators of L_p admit a dominated estimate, and, therefore,

$$\lim_{m \to \infty} A(m, T_1, f) \text{ exists a.s. for}$$

all f in L_p.

2. **A multiple sequence theorem.** Let $\{T_k\}_{k=1}^n$ be operators of L_p, each of which is majorized by a power-bounded operator S_k of L_p. Then each S_k admits a dominated estimate with constant M_k (and the average $A(m; S_k; f)$ converge a.s. as $m \to \infty$ for all $f \in L_p$). Then each T_k admits a dominated estimate with constant M_k, so the average $A(m; T_k; f)$ converge a.s. as $m \to \infty$ for all $f \in L_p$. We will now show that $\{T_k\}_{k=1}^n$ admits a dominated estimate and that the average $A(m_1, \ldots, m_n; T_1, \ldots, T_k; f)$ converse a.s. for all $f \in L_p$ as $m_1, \ldots, m_k$ tend to ∞ independently of each other.

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THEOREM 1. *Let $\{T_k\}_{k=1}^{n}$ be linear operators of L_p, for which each T_k is majorized by a power-bounded operator S_k of L_p. Further, suppose each S_k admits a dominated estimate with constant M_k (and hence $\lim_{m\to\infty} A(m; S_k; f)$ exists a.s. for all $f \in L_p$). Then $\{T_k\}_{k=1}^{n}$ admits a dominated estimate with constant $M_1 \cdot \ldots \cdot M_n$.*

PROOF. Follows by induction on the number of operators and noting that

$$|A(m_1, \ldots, m_n; T_1, \ldots T_n; f)| \leq A(m_1, \ldots, m_n; S_1, \ldots, S_n; |f|).$$

THEOREM 2. *Let $\{T_k\}_{k=1}^{n}$ be as in Theorem 1. Then*

$$\lim_{m_1, \ldots, m_n \to \infty} A(m_1, \ldots, m_n; T_1, \ldots, T_n; f)$$

exists a.s. for all $f \in L_p$. Here, $m_1, \ldots, m_n$ tend to infinity, independently of each other.

PROOF. We proceed again by induction, noting that the theorem is true for $n = 1$ by Brunel's result.

Assuming the therorem is true for any set $\{T_k\}_{k=1}^{n-1}$ of such operators $f \in L_p$, let $f = h + g - T_n g$, where $T_n h = h$ and $g \in L_p$. The set of such f's is dense in L_p by the Mean Ergodic Theorem, since T_n is power-bounded. Then

$$\begin{aligned} A(m_1, \ldots, m_n; T_1, \ldots, T_n; f) =& A\big(m_1, \ldots, m_n; T_1, \ldots, T_n; (h + g - T_n g)\big) \\ =& A(m_1, \ldots, m_n; T_1, \ldots, T_n; h) \\ &+ \frac{1}{n^1} A(m_1, \ldots, m_{n-1}; T_1, \ldots, T_{n-1}; g) \\ &- \frac{1}{m_n} A(m_1, \ldots, m_n; T_1, \ldots, T_{n-1}; T_n^{m_n} g). \end{aligned}$$

Now, the first two terms on the right of the last equality converge a.e. as $m_1, \ldots, m_n \longrightarrow \infty$ independently of each other (the second to zero), so we consider only the last.

$$\begin{aligned} &\left| \frac{1}{m_n} A\big(m_1, \ldots, m_n; T_1, \ldots, T_n; T_n^{m_n} g(x)\big) \right| \\ &\qquad \leq \frac{1}{m_n} M\Big(S_1, \ldots, S_{n-1}\big(S_n^{m_n} |g|(x)\big)\Big) \qquad \text{a.s., so} \end{aligned}$$

$$\begin{aligned} &\int \frac{1}{{m_n}^p} \left| A\big(m_1, \ldots, m_{n-1}; T_1, \ldots, T_{n-1}; T_n^{m_n} g(x)\big) \right|^p du \\ &\qquad \leq \int \frac{1}{{m_n}^p} \Big(M\big(S_1, \ldots, S_{n-1}; S_n^{m_n}(|g|)\big)\Big)^p du \\ &\qquad \leq \frac{1}{{m_n}^p} M_1 \cdot \ldots \cdot M_{n-1} \int \big(S_n^{m_n}(|g|)\big)^p du \\ &\qquad \leq \frac{1}{{m_n}^p} M_1 \cdot \ldots \cdot M_{n-1} B_n \int |g|^p du \end{aligned}$$

where $\|S_n^m\| \leq B_n$ for all m.

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Therefore

$$\sum_{n_k=1}^{\infty} \int \Big(\frac{1}{m_n} M\big(S_1, \ldots, S_{n-1}; S_n^{m_n}(|g|)\big)\Big)^p \, du$$

is finite, so

$$\lim_{m_n \to \infty} \frac{1}{m_n} M\big(S_1, \ldots, S_{n-1}; S_n^{m_n}(|g|)\big) = 0 \quad \text{a.s.},$$

and

$$\lim_{m_1, \ldots, m_n \to \infty} \left| \frac{1}{m_n} A\left(m_1, \ldots, m_{n-1}; T_1, \ldots, T_{n-1}; T_n^{m_n} g\right)\right| \\ \leq \lim_{m_n \to \infty} \frac{1}{m_n} M\big(S_1, \ldots, S_{n-1}\left(S_n^{m_n}|g|\right)\big) = 0 \quad \text{a.s.}$$

so

$$\lim_{m_1, \ldots, m_n \to \infty} \frac{1}{m_n} A\left(m_1, \ldots, m_{n-1}; T_1, \ldots, T_{n-1}; T_n^{m_n} g\right) = 0 \quad \text{a.s.}$$

The theorem now follows by the usual application of the Banach Principle (see [9] for example).

The proof is an adaptation of proof in [9] and [5]. The theorem also follows from 6.1 of [6]. The author is indebted to Prof. Sucheston for pointing this out. The above proof offers a different approach.

3. **Convergence along uniform sequences.** One question that arises for linear operators is: for which sequences of integers $\{n_k\}$ the average

$$\frac{1}{n} \sum_{k=0}^{n-1} T^{n_k} f$$

converge a.s. when f is in the domain and range of T. Recall that p is fixed, $1 < p < \infty$. The by now classical uniform sequences of Brunel-Keane [4] have been widely studied and these averages are known to converge a.s. when T is a positive contraction of L_p and $\{n_k\}$ is a uniform sequence (see [1], [7]). We paraphrase some of the results of [1].

THEOREM 3. *Let T be majorized by a power-bounded operator S from L_p to L_p. Suppose further that for every complex member $\lambda, |\lambda| = 1$,*

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k T^k f \quad \textit{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \lambda^k S^k f$$

exists a.s. for all $f \in L_p$. Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} T^{n_k} f \quad \textit{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} S^{n_k} f$$

exist a.s. for all $f \in L_p$ and uniform sequence $\{n_k\}$.

PROOF. This is essentially Corollary 6.2 of [1].

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THEOREM 4. *Let T be an operator of L_p that is majorized by a power-bounded operator S of L_p. The* $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^{n_k} f$ *exists a.s. for all* $f \in L_p$ *and uniform sequences* $\{n_k\}$.

PROOF. For all complex numbers $\lambda, |\lambda| = 1$, the operator $Uf = \lambda Tf$ is power-bounded from L_p to L_p (and hence to L_1) and is also majorized by S. S admits a dominated estimate, and so does U. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} U^k f$$

exists a.s. for all f in L_p. The same is true replacing T by S, allowing us to apply Theorem 3.

A multiparameter version of the above theorem also holds via the aforementioned result of Frangos-Sucheston. We omit the details, and refer the reader to [9], Theorem 2 and its corollary.

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