

FURTHER RESULTS ON THE CRITICAL GALTON-WATSON PROCESS WITH IMMIGRATION

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1. Introduction

Consider a Galton-Watson process in which each individual reproduces independently of all others and has probability a_j ($j = 0, 1, \dots$) of giving rise to j progeny in the following generation, and in which there is an independent immigration component where b_j ($j = 0, 1, \dots$) is the probability that j individuals enter the population at each generation. Defining X_n ($n = 0, 1, \dots$) to be the population size at the n -th generation, it is known that $\{X_n\}$ defines a Markov chain on the non-negative integers.

When $|x| < 1$, let $A(x) = \sum_{j=0}^{\infty} a_j x^j$, $B(x) = \sum_{j=0}^{\infty} b_j x^j$ and $P_i^{(n)}(x) = \sum_{j=0}^{\infty} p_{ij}^{(n)} x^j$ where $\{p_{ij}^{(n)}\}$ ($i, j, n = 0, 1, \dots$) are the n -step transition probabilities of the Markov chain $\{X_n\}$. We shall assume that $0 < a_0, b_0 < 1$. Denote the means of the offspring and immigration distributions by $\alpha = A'(1-)$ and $\beta = B'(1-)$ respectively. We always assume that $\beta < \infty$ and, unless otherwise stated, $\alpha = 1$. In this case the variance of the offspring distribution is given by $2\gamma = A''(1-)$ and we assume that $0 < \gamma < \infty$. Finally, let $\sigma = \beta/\gamma$.

Pakes [6] has shown that if $\sum_{j=1}^{\infty} a_j j^2 \log j$, $B''(1-) < \infty$ then $n^\sigma p_{00}^{(n)} \rightarrow \mu_0$, ($n \rightarrow \infty$) where $0 < \mu_0 < \infty$. For the case where $\{X_n\}$ is irreducible and aperiodic, this result shows it to be null-recurrent when $\sigma \leq 1$ and transient otherwise. In section 2 we shall show that $n^\sigma P_i^{(n)}(x)$ converges to a function $U(x)$ which is regular in the open unit disc and which generates the invariant measure, $\{\mu_j\}$, of $\{X_n\}$. Seneta [9] has demonstrated the existence and uniqueness (up to a constant multiple) of an invariant measure under very weak hypotheses. A discussion of the asymptotic behaviour of $\{p_{ij}^{(n)}\}$ and $\{\mu_j\}$ is given in section 2. Some results on the asymptotic behaviour of the Green's function $G_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)}$, which exists under the conditions of theorem 1 below and if $\sigma > 1$, are given in section 3.

It was shown in Pakes [6] that X_n/n converges weakly to a gamma distributed random variable. A related problem for $Y_n = \sum_{m=0}^n X_m$, the total number of individuals which have existed in the population up to the n -th generation, is considered in section 4. More specifically, we show that Y_n/n^2 converges weakly

to a random variable Y , where $E(e^{-\theta Y}) = [\operatorname{sech}((\gamma\theta)^{\frac{1}{2}})]^\sigma$. Finally, we briefly examine the rate of convergence of generating functions of the total population in a non-super-critical Galton-Watson process without immigration.

2. The asymptotic form of the transition probabilities

In the proof of theorem 1, we shall need the following result.

LEMMA 1. Let $K(x) = \sum_{j=0}^\infty k_j x^j$ be a probability generating function. If $0 < d_n < 1$ and $1 - d_n \sim a/n, 0 < a < \infty$, then

$$\sum_{n=1}^\infty (1 - K(d_n))/n < \infty \text{ iff } \sum_{j=1}^\infty k_j \log j < \infty.$$

PROOF. Since there exists finite positive constants c_1 and c_2 such that $c_1/n < 1 - d_n < c_2/n$ if $n > N$, it suffices to show that

$$S = \sum_{n=N}^\infty (1 - K(1 - c/n))/n < \infty \text{ iff } \sum_{j=1}^\infty k_j \log j < \infty,$$

where c is a finite positive constant and N is so large that $c/(N - 1) < 1$. Fubini's theorem yields

$$S = \sum_{j=0}^\infty k_j \sum_{n=N}^\infty (1 - (1 - c/n)^j)/n.$$

Let $S_j = \sum_{n=N}^\infty (1 - (1 - c/n)^j)/n$. For fixed j the terms of this series are monotone decreasing and so

$$0 \leq S_j - I_j \leq [1 - (1 - c/(N - 1))^j]/(N - 1) \leq 1.$$

where

$$\begin{aligned} I_j &= \int_N^\infty [1 - (1 - c/x)^j]/x \, dx = \int_{1 - c/N}^1 (1 - y^j)/(1 - y) \, dy \\ &= \log j + L_j - M_j, \end{aligned}$$

and where $L_j = \sum_{k=1}^j 1/k - \log j$ satisfies $0 < L_j < 1$ and

$$0 \leq M_j = \sum_{k=1}^j (1 - c/N)^k/k < (N/c) - 1.$$

The lemma now follows on observing that

$$0 \leq \sum_{j=0}^\infty k_j I_j \leq S \leq \sum_{j=0}^\infty k_j I_j + 1.$$

From the definition of the Markov chain $\{X_n\}$ it is easily seen that

$$(1) \quad P_i^{(n)}(x) = (A_n(x))^i \prod_{m=0}^{n-1} B(A_m(x))$$

where $A_0(x) = x$ and $A_{n+1}(x) = A(A_n(x))$.

THEOREM 1. *If, in addition to the conditions of section 1, $\sum_{j=1}^{\infty} a_j j^2 \log j$, $\sum_{j=1}^{\infty} b_j j \log j < \infty$, then the sequence of functions $\{n^\sigma P_i^{(n)}(x)\}$ converges to $U(x)$ ($|x| < 1$) where $U(x)$ satisfies the functional equation*

$$(2) \quad B(x)U(A(x)) = U(x).$$

The convergence is uniform over compact subsets of the open unit disc. Denoting the power series representation of $U(x)$ by $\sum_{j=0}^{\infty} \mu_j x^j$, the n -step transition probabilities are given by

$$(3) \quad p_{ij}^{(n)} = n^{-\sigma}(\mu_j + r_{ij}(n)) \quad (i, j = 0, 1, \dots)$$

where $r_{ij}(n) = o(1)$ ($n \rightarrow \infty$).

REMARK. Since completing this work, the author has found that Karlin and McGregor [4] have obtained the first part of this theorem but on assuming that $B(x)$ is regular at $x = 1$ and $A'''(1-) < \infty$. When $x = 0$, the above theorem slightly strengthens theorem 1 of Pakes [6].

PROOF. It is well known that under our hypotheses $A_n(x) \uparrow 1$ ($n \rightarrow \infty$); see Harris [3]. Thus we need only consider

$$(4) \quad D_n(x) \equiv n^\sigma P_0^{(n)}(x) = B(x) \prod_{m=1}^{n-1} (1 + 1/m)^\sigma B(A_m(x)).$$

For the present consider a fixed $x \in [0, 1]$. The existence of a finite positive limit of this sequence of functions is equivalent to the convergence of the series $\sum_{m=1}^{\infty} (d_m(x) - 1)$, where

$$d_m(x) = (1 + 1/m)^\sigma B(A_m(x)).$$

Using $(1 + 1/m)^\sigma = 1 + \sigma/m + r_m$ where $r_m = o(1/m^2)$ we have

$$(5) \quad \begin{aligned} d_m(x) - 1 &= \sigma/m - (1 - B(A_m(x)) - \sigma(1 - B(A_m(x))))/m + r_m B(A_m(x)) \\ &= \sigma/m - \beta(1 - A_m(x)) + (1 - A_m(x))(\beta - B'(\eta_m)) - \sigma(1 - B(A_m(x))/m \\ &\quad + r_m B(A_m(x))) \end{aligned}$$

where $A_m(x) < \eta_m < 1$, and we have used the mean value theorem to obtain the second equality. Theorem 1 of Kesten *et al.* [5] shows that $1 - A_n(x) \sim n/\gamma$ and so the third and fourth terms of equation (5) are $O(1/m^2)$. Clearly, the second term of (5) is non-negative and is dominated by $(1 - A_m(x))(\beta - B'(A_m(x)))$. Application of lemma 1 with $K(x) = B'(x)/\beta$ shows that

$$\sum_{m=1}^{\infty} (1 - A_m(x))(\beta - B'(\eta_m)) < \infty \text{ if } \sum_{j=1}^{\infty} b_j j \log j < \infty.$$

Writing

$$1 - A_m(x) = [m\gamma - h_m(x) + 1/(1-x)]^{-1}$$

we obtain

$$\sigma/m - \beta(1 - A_m(x)) = \sigma \cdot \frac{1/(1-x) - h_m(x)}{m(m - h_m(x) + 1/(1-x))} = -\sigma h_m(x)/\gamma m^2 + O(1/m^2)$$

where we have used $h_m(x) = o(m)$; see Kesten et al. [5]. Indeed it is shown in this reference that $h_m(x) = \sum_{k=0}^{m-1} \delta(A_k(x))$ where $\delta(x)$ satisfies the (corrected) inequality

$$(6) \quad -\gamma^2(1-x)/(1-A(0)) \leq \delta(x) \leq \varepsilon(x)$$

where $0 \leq \varepsilon(x) = \gamma - (A(x) - x)/(1-x)^2 \leq \gamma$, and $\varepsilon(x)$ is monotone non-increasing on $0 \leq x < 1$ and $\varepsilon(x) \downarrow 0(x \uparrow 1)$. Thus we have

$$(7) \quad -\frac{\gamma^2}{1-A(0)} \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{k=0}^{m-1} (1-A_k(0)) \leq \sum_{m=1}^{\infty} \frac{h_m(x)}{m^2} \leq \sum_{m=1}^{\infty} \frac{1}{m^2} \sum_{t=0}^{m-1} \varepsilon(A_k(0)).$$

For sufficiently large M there exist positive constants a, b such that $a/m < 1 - A_m(0) < b/m$ ($m \geq M$) and so we see that the terms of the series on the left of equation (7) are $O[(\log m)/m^2]$ for large m .

Observing that $\varepsilon(x) \leq \gamma - A''(x)$, use of lemma 1 with $K(x) = A''(x)/2\gamma$ shows that the series on the right of (7) will converge if $\sum_{j=0}^{\infty} a_j j^2 \log j < \infty$. Thus $D_n(x) \rightarrow U(x)$, say, ($n \rightarrow \infty$) and $0 < U(x) < \infty$ when $0 \leq x < 1$.

If $0 < R < 1$, equation (4) and the result of the last paragraph implies $|n^\sigma P_i^{(n)}(x)| \leq D_n(R) < M(R) < \infty$ ($|x| \leq R$) where $M(R)$ is a constant depending on R . Vitali's theorem (Titchmarsh [10]) shows that $n^\sigma P_i^{(n)}(x)$ converges uniformly over compact subsets of the disc $|x| < R$, and thus uniformly over compact subsets of the open unit disc, to a function $\sum_{j=0}^{\infty} \mu_j x^j$ which coincides with $U(x)$ for $0 \leq x < 1$ and thus defines $U(x)$ for all $|x| < 1$. It is clear that $\mu_j \geq 0$. Equation (3) follows from the uniform convergence.

From equation (4) we have

$$(1 + 1/n)^\sigma B(x) D_n(A(x)) = D_{n+1}(x)$$

and this implies equation (2), thus completing the proof.

Equation (3) clearly demonstrates the absence of the geometric ergodicity property; compare with the situation when $\alpha \neq 1$ in Pakes [7].

Theorem 1 allows us to write

$$(8) \quad P_i^{(n)}(x) = n^{-\sigma} [U(x) + r_i^{(n)}(x)] \quad (0 \leq x < 1; \quad n = 1, 2, \dots)$$

where $r_i^{(n)}(x) = o(1)$ ($n \rightarrow \infty$). We can obtain some information on the asymptotic form of $r_i^{(n)}(x)$. Equation 1, theorem 1 and the fact that $1 - A_n(x) \sim n/\gamma$ (Kesten et al. [5]) shows that $n^{\sigma+1}(P_0^{(n)}(x) - P_i^{(n)}(x)) = iU(x)(1 + \zeta_i^{(n)}(x))/\gamma$ where $\zeta_i^{(n)}(x) = o(1)$ ($n \rightarrow \infty$). This yields

$$P_i^{(n)}(x) = n^{-\sigma}(U(x) + r_0^{(n)}(x)) - iU(x)n^{-\sigma-1}(1 + \zeta_i^{(n)}(x))/\gamma.$$

We shall now show that, in general, $n^{-\sigma} r_0^{(n)}(x)$ tends to zero much less rapidly than does $n^{-\sigma-1}$. This is evident from the following:

COROLLARY. *If we assume that $B''(1-), A^{iv}(1-) < \infty$ then*

$$r_0^{(n)}(x) = -(\Omega U(x) \log n) n + O(1/n)$$

where

$$\Omega = \beta(\gamma^2 - A'''(1-)/6)\gamma^3.$$

PROOF. From theorem 1 we have

$$r_0^{(n)}(x) = D_n(x)(1 - \prod_{m=n}^{\infty} (1 + 1/m)^\sigma B(A_m(x))).$$

Now

$$(9) \quad V_n(x) = 1 - \prod_{m=n}^{\infty} (1 + 1/m)^\sigma B(A_m(x)) = \sum_{m=n}^{\infty} (1 - d_m(x)) + W_n(x)$$

where $W_n(x)$ is an error term which will be examined subsequently. The proof of theorem 1 and the finiteness of $B''(1-)$ show that

$$1 - d_m(x) = \sigma h_m(x)/\gamma m^2 + O(1/m^2).$$

Since $A^{iv}(1-) < \infty$ it follows from lemma 10.1 and case c of the proof of theorem 11.1 in Harris [3] that

$$(10) \quad 1 - d_m(x) = -\Omega[\log(1 + m\gamma(1-x))]/m^2 + O(1/m^2) \quad (0 \leq x < 1).$$

If m is sufficiently large, then $[\log(1 + am)]/m^2$ ($0 < a < \infty$) becomes monotone non-increasing and so

$$\int_n^\infty [\log(1 + ay)]/y^2 dy \leq \sum_{m=n}^\infty [\log(1 + am)]/m^2 \leq \int_{n-1}^\infty [\log(1 + ay)]/y^2 dy$$

for n sufficiently large. This shows, after some manipulation, that

$$\sum_{m=n}^\infty (1 - dm(x)) = -(\log n)/n + O(1/n).$$

Equation (10) shows that for $\Omega \neq 0$ and m sufficiently large, $1 - d_m(x)$ is either positive or negative, the sign being determined by that of Ω . Use of the appropriate one of the inequalities

$$\begin{aligned} \sum_{m=n}^\infty x_m - (\sum_{m=n}^\infty x_m)^2/2 &\leq 1 - \prod_{m=n}^\infty (1 - x_m) \leq \sum_{m=n}^\infty x_m + \sum_{m=n}^\infty [x_m^2/(1 - x_m)] \quad (0 \leq x_m < 1) \\ \sum_{m=n}^\infty x_m - (\sum_{m=n}^\infty x_m)^2/2(1 + \sum_{m=n}^\infty x_m) &\leq 1 - \prod_{m=n}^\infty (1 - x_m) \leq \sum_{m=n}^\infty x_m + (\sum_{m=n}^\infty x_m^2)/2 \\ &\quad (-1 < x_m \leq 0) \end{aligned}$$

where $0 < \prod_{m=n}^\infty (1 - x_m) < \infty$, shows that $W_n(x) = O(1/n)$.

If $\Omega = 0$, then applying the last two inequalities to the bounds of $1 - d_m(x)$ implied by (10) proves the corollary directly. This completes the proof.

Observe that this corollary gives the rate at which $n^\sigma P_i^{(n)}(x)$ approaches $U(x)$. We now briefly examine the asymptotic behaviour of the invariant measure.

THEOREM 2. *If the conditions of theorem 1 are satisfied, then*

$$\sum_{i=0}^j \mu_j \sim j^\sigma / \Gamma(\sigma + 1) \quad (j \rightarrow \infty)$$

where $\Gamma(\cdot)$ is the Gamma function.

PROOF. From theorem 1 we have

$$(11) \quad U(x) = B(x) \prod_{m=1}^{\infty} (1 + 1/m)^\sigma B(A_m(x)) \quad (0 \leq x < 1).$$

Following Kesten et al. [5] we consider

$$\begin{aligned} U(A_n(0)) &= B(A_n(0)) \prod_{m=1}^{\infty} (1 + 1/m)^\sigma B(A_{m+n}(0)) \\ &= B(A_n(0)) \prod_{m=1}^n (1 + 1/m)^\sigma \prod_{m=n+1}^{\infty} (1 + 1/m)^\sigma B(A_m(0)). \end{aligned}$$

The convergence of the infinite product (11) implies that the second product in the last expression $\rightarrow 1$ ($n \rightarrow \infty$) thus giving

$$U(A_n(0)) \sim (n + 1)^\sigma \quad (n \rightarrow \infty).$$

Since $U(x)$ is monotone on $0 \leq x \leq 1$, we have

$$U(x) \sim (1 - x)^{-\sigma} \quad (x \uparrow 1)$$

and the theorem now follows from Karamata's theorem (Feller [2]).

3. Asymptotic properties of the Green's functions

We define the Green's functions by

$$(12) \quad G_{ij} = \sum_{n=0}^{\infty} p_{ij}^{(n)} \quad (i, j = 0, 1, \dots)$$

where, by equation (3), the series converges if $\sigma > 1$. In this section we shall obtain some information on the behaviour of G_{ij} when i is fixed and $j \rightarrow \infty$ and when j is fixed and $i \rightarrow \infty$. We turn now to the first case, the approach being similar to that used in the proof of theorem 2.

THEOREM 3. *If $\sigma > 1$ and the conditions of theorem 1 are fulfilled then $\sum_{k=0}^j G_{ik} \sim j/(\sigma - 1)$ ($j \rightarrow \infty$).*

PROOF. Equation (12) yields

$$(13) \quad G_i(x) = \sum_{j=0}^{\infty} G_{ij} x^j = \sum_{n=0}^{\infty} P_i^{(n)}(x) \quad (i = 0, 1, \dots; 0 \leq x < 1).$$

which implies, on using (1),

$$G_i(A_m(0)) = \sum_{n=0}^{\infty} p_{i0}^{(n+m)} / p_{00}^{(m)}.$$

Since $n^\sigma p_{i0}^{(n)} \rightarrow \mu_0 > 0$ ($n \rightarrow \infty; i = 0, 1, \dots$), then for each $\varepsilon > 0$ there exists $M(\varepsilon)$ such that

$$(14) \quad (1 - \varepsilon)(m/(m+n))^\sigma < p_{i0}^{(n+m)} / p_{00}^{(m)} < (1 + \varepsilon)(m/(m+n))^\sigma \quad (m > M(\varepsilon); n = 0, 1, \dots).$$

Observing that the terms of the series $\sum_{n=0}^{\infty} (m+n)^{-\sigma}$ are monotone decreasing, comparison with $\int_0^\infty (x+m)^{-\sigma} dx$ eventually yields

$$1 - \varepsilon \leq (\sigma - 1)G_i(A_m(0))/m \leq (1 + \varepsilon)(m/(m-1))^\sigma < (1 + \varepsilon)^\sigma$$

when $m > \max(M(\varepsilon), 1 + 1/\varepsilon)$. Thus $G_i(A_m(0)) \sim m/(\sigma - 1)$ and since $G_i(x)$ is monotone on $0 \leq x < 1$,

$$G_i(x) \sim 1/(\sigma - 1)(1 - x) \quad (x \uparrow 1)$$

and the theorem now follows on applying Karamata's theorem.

In considering the behaviour of G_{ij} for large i , we need the following result generalising lemma 1 of [5] (which was stated without proof).

LEMMA 2. If $0 < \alpha_n < 1$, $\alpha_n \sim c/n$ and $\beta_n \sim bn^{-\sigma}$ where $0 < b, c < \infty$ and $\sigma > 1$, then

$$\lim_{i \rightarrow \infty} \sum_{n=1}^{\infty} (1 - \alpha_n)^i i^{\sigma-1} \beta_n = bc^{1-\sigma} \Gamma(\sigma - 1).$$

PROOF. For each $\varepsilon > 0$ we have

$$(15) \quad (b - \varepsilon)n^{-\sigma} \leq \beta_n \leq (b + \varepsilon)n^{-\sigma} \quad (n \geq N = N(\varepsilon))$$

so that

$$\begin{aligned} (b - \varepsilon) \left\{ (1/i) \sum_{n=N}^{\infty} (i/n)^\sigma e^{-ic/n} \right\} &\leq \sum_{n=N}^{\infty} i^{\sigma-1} \beta_n e^{-ic/n} \\ &\leq (b + \varepsilon) \left\{ (1/i) \sum_{n=N}^{\infty} (i/n)^\sigma e^{-ic/n} \right\}. \end{aligned}$$

The expression in curly brackets on the left and right hand side of this inequality tends to $\int_0^\infty x^{-\sigma} e^{-c/x} dx = c^{1-\sigma} \Gamma(\sigma - 1)$ and since ε is arbitrary we have

$$(16) \quad \lim_{i \rightarrow \infty} \sum_{n=1}^{\infty} i^{\sigma-1} \beta_n e^{-ic/n} = bc^{1-\sigma} \Gamma(\sigma - 1)$$

Thus it suffices to show that

$$\lim_{i \rightarrow \infty} \sum_{n=1}^{\infty} (e^{-ic/n} - (1 - \alpha_n)^i) i^{\sigma-1} \beta_n = 0.$$

For each $\varepsilon > 0$ there exists $M = M(\varepsilon)$ such that both (15) and

$$(17) \quad 0 < 1 - (1 + \varepsilon)c/n \leq 1 - \alpha_n \leq 1 - (1 - \varepsilon)c/n$$

hold for $n \geq M$. The last inequality gives

$$\begin{aligned} e^{-ic/n} - (1 - \alpha_n)^i &\leq [e^{-c/n} - (1 - (1 + \varepsilon)c/n)] \sum_{k=0}^{i-1} e^{-kc/n} (1 - \alpha_n)^{i-1-k} \\ &\leq (\varepsilon c n + c^2/2n^2) \sum_{k=0}^{i-1} e^{-kc/n} (1 - (1 - \varepsilon)c/n)^{i-1-k} \\ &\leq J i (\varepsilon c/n + c^2/2n^2) e^{-ic/n} \end{aligned}$$

where J is a positive constant. Thus we obtain

$$\begin{aligned} \limsup_{i \rightarrow \infty} \sum_{n=1}^{\infty} (e^{-ic/n} - (1 - \alpha_n)^i) i^{\sigma-1} \beta_n \\ \leq J(b + \varepsilon) \lim_{i \rightarrow \infty} \sum_{n=M}^{\infty} (\varepsilon c/n - c^2/2n^2) e^{-ic/n} (i/n)^\sigma \\ = \varepsilon c^{1-\sigma} \Gamma(\sigma) J(b + \varepsilon). \end{aligned}$$

But since ε is arbitrary the limit superior is non-positive. Similarly, use of the left hand sides of (15) and (17) shows that the limit inferior is non-negative, thus completing the proof.

THEOREM 4. *Let the conditions of theorem 1 be satisfied and let $\sigma > 1$. Then for each $0 \leq x < 1$,*

$$(18) \quad G_i(x) \sim (\gamma/i)^{\sigma-1} \Gamma(\sigma-1) U(x) \quad (i \rightarrow \infty),$$

and for each $j = 0, 1, \dots$,

$$(19) \quad G_{ij} \sim (\gamma/i)^{\sigma-1} \Gamma(\sigma-1) \mu_j \quad (i \rightarrow \infty).$$

PROOF. Recalling that $P_0^{(n)}(x) \sim n^{-\sigma} U(x)$ and that $A_n(x) = 1 - \alpha_n$ where $\alpha_n \sim 1/(n\gamma)$ we see, using equation (1), that lemma 2 can be applied to equation (13) thus yielding (18).

Setting $x = 0$ in (18) gives (19) when $j = 0$ and this yields $G_i(x)/G_{i0} \rightarrow U(x)/\mu_0$ ($i \rightarrow \infty$). Recalling that $\mu_0 > 0$,

$$G_{ij}/G_{i0} \leq x^{-j} G_i(x)/G_{i0} < M < \infty.$$

This enables us to apply the dominated convergence theorem to the series $G_i(y)/G_{i0}$ with $0 \leq y < x$ and obtain $G_{ij}/G_{i0} \rightarrow \mu_j/\mu_0$ and this implies (19).

4. A limit theorem for the total number of individuals

Let $Y_n = \sum_{m=0}^n X_m$ be the total number of individuals that have existed up to and including the n -th generation in our Galton-Watson process with immigration. Assuming $X_0 = i$, it is not difficult to show that

$$(20) \quad Q_i^{(n)}(x) = E(x^{Y_n}) = (g_n(x))^i \prod_{m=0}^{n-1} B(g_m(x))$$

where $g_n(x)$ is the generating function of the total number of individuals up to and including the n -th generation that are descended from a single ancestor (and including this individual) in the zero-th generation of an ordinary Galton-Watson process. It is well known that

$$(21) \quad g_{n+1}(x) = xA(g_n(x)), \quad g_0(x) = x \quad (n = 0, 1, \dots)$$

and that if $\alpha \leq 1$,

$$(22) \quad g_n(x) \downarrow g(x) = xA(g(x)) \quad (n \rightarrow \infty)$$

where $g(x)$ generates an honest probability distribution; see [3] p. 32. (Monotone convergence is not proven in [3], however see equation (32) below).

To see that (20) is true, consider a sequence $\{\mathcal{B}_n\}_0^\infty$, of independent Galton-Watson processes having an offspring distribution generated by $A(x)$ and initial distributions generated by x^i when $n = 0$, and $B(x)$ when $n = 1, 2, \dots$. Defining $y_{n,m}$ to be the total number of individuals that have existed up to and including the m -th generation of \mathcal{B}_n , it is clear that

$$(23) \quad Y_n = \sum_{m=0}^n y_{m, n-m}$$

and this is equivalent to (20).

Logarithmic differentiation of (20) and use of (21) yields

$$E(Y_n) = i(n+1) + \beta n(n+1)/2,$$

which suggests considering the convergence of Y_n/n^2 in some sense.

THEOREM 5. *Under the assumptions made in section 1, the random variable Y_n/n^2 converges weakly to a random variable Y whose distribution is defined by the Laplace-Stieltjes transform*

$$(24) \quad E(e^{-\theta Y}) = [\operatorname{sech}(\gamma\theta)^{\frac{1}{2}}]^\sigma.$$

REMARKS. When $\sigma = 1$, the distribution defined by (24) arises in the context of first passage time distributions for the Wiener-Lévy process, see for example Bharucha-Reid [1] p. 152. It is clear from (23) that Y is the weak limit of a system of infinitesimal random variables and, as such, it has an infinitely divisible distribution. The Kolmogorov canonical representation of the characteristic function of Y is given by

$$i\beta t/2 + \sigma \int_0^\infty (e^{iut} - 1 - iut)\theta_2(0, e^{i\pi u/\gamma})/(2u) du$$

where $\theta_2(\cdot, \cdot)$ is the second theta function; the example on p. 534 of [2] is relevant to the derivation of this result.

We now obtain two results which are needed for the proof of theorem 5.

LEMMA 3. Let $u \equiv u(x) = xA'(g(x))$. Then,

$$x - g(x) \sim [(1-x)/\gamma]^{\frac{1}{2}}$$

and

$$1 - u(x) \sim 2[\gamma(1-x)]^{\frac{1}{2}}, \quad (x \uparrow 1).$$

PROOF. Using a four term Taylor expansion of the right hand side of (22) we have

$$g(x) = \frac{\gamma x(1-g(x))^2}{1-x} + \frac{(1-g(x))^3}{1-x} \frac{A'''(\eta)}{6} \quad (g(x) < \eta < 1).$$

But $g(x) \uparrow 1(x \uparrow 1)$ and so the first part of the lemma follows. The second part follows from

$$1 - A'(g(x)) = 2\gamma(1-g(x)) - A''(\zeta)(1-g(x))^2/2 \quad (g(x) < \zeta < 1).$$

LEMMA 4. Let $\theta_n = e^{-\theta/n^2}$ ($\theta > 0$) and let $\phi(x) \rightarrow 1$ as $x \uparrow 1$. Then

$$(25) \quad \lim_{n \rightarrow \infty} (1 - u(\theta_n)) \sum_{m=1}^{n-1} \frac{\phi(\theta_n)(u(\theta_n))^m}{1 + \phi(\theta_n)(u(\theta_n))^m} = -\log \left(\frac{1 + e^{-2(\gamma\theta)^{1/2}}}{2} \right).$$

PROOF. By hypothesis we have

$$(26) \quad \phi(\theta_n) = 1 + t_n, \quad t_n = o(1) \quad (n \rightarrow \infty)$$

and using the second part of lemma 3 it is easily seen that

$$(27) \quad 1 - u(\theta_n) = 2(\gamma\theta)^{\frac{1}{2}}/n + r_n/n, \quad r_n = o(1) \quad (n \rightarrow \infty).$$

Writing $\delta = 2(\gamma\theta)^{\frac{1}{2}}$ and $\eta_{m,n} = (1 - \delta/n)^m$, it is not difficult to show that

$$(28) \quad \left| \frac{\phi(\theta_n)(u(\theta_n))^m}{1 + \phi(\theta_n)(u(\theta_n))^m} - \frac{\exp(-\delta m/n)}{1 + \exp(-\delta m/n)} \right| \leq |\eta_{m,n} - (u(\theta_n))^m| + |t_n| + |\exp(-\delta m/n) - \eta_{m,n}|.$$

Observing that (27) is equivalent to $u(\theta_n) = \eta_{1,n} - r_n/n$, we have

$$|\eta_{m,n} - (u(\theta_n))^m| \leq m|r_n|/n.$$

Noting that $e^{-\delta/n} = \eta_{1,n} + v_n$ where $|v_n| \leq \delta/2n^2$, yields

$$|\exp(-\delta m/n) - \eta_{m,n}| \leq \delta m/2n^2.$$

Summing (28) from $m = 0$ to $m = n - 1$, and using the last two estimates proves

the equality of the left hand side of (25) (if it exists) and

$$\lim_{n \rightarrow \infty} \frac{\delta}{n} \sum_{m=1}^{n-1} \frac{\exp(-\delta m/n)}{1 + \exp(-\delta m/n)}.$$

However this limit does exist and is clearly equal to

$$\delta \int_0^1 e^{-\delta x} / (1 + e^{-\delta x}) dx$$

thus completing the proof.

PROOF OF THEOREM 5. Recalling that $\theta_n = e^{-\theta/n^2}$ we have

$$E(\exp(-\theta Y_n/n^2)) = Q_i^{(n)}(\theta_n)$$

and $g_n(\theta_n) \rightarrow 1$ ($n \rightarrow \infty$), and so it suffices to consider the case $i = 0$. We have

$$(29) \quad \log Q_0^{(n)}(\theta_n) = - \sum_{m=0}^{n-1} (1 - B(g_m(\theta_n))) - \sum_{m=0}^{n-1} R_{m,n}(\theta)$$

where

$$0 \leq R_{m,n}(\theta) \leq [1 - B(g_m(\theta_n))]^2 / B(g_m(\theta_n)) \\ \leq [1 - B(g(\theta_n))] [1 - B(g_m(\theta_n))] / B(g(\theta_n)).$$

If the first sum in (29) is bounded as $n \rightarrow \infty$, then the second sum, denoted by $R_1^{(n)}(\theta)$, tends to zero.

Using the expansion $B(x) = 1 - \beta(1-x) + (1-x)r(x)$, where $r(x)$ is monotone non-increasing on $[0, 1]$ and $r(x) \downarrow 0$ as $x \uparrow 1$, we obtain

$$(30) \quad \log Q_0^{(n)}(\theta_n) = -\beta \sum_{m=0}^{n-1} (1 - g_m(\theta_n)) + R_1^{(n)}(\theta) + R_2^{(n)}(\theta)$$

where

$$0 \leq R_2^{(n)}(\theta) = \sum_{m=0}^{n-1} (1 - g_m(\theta_n)) r(g_m(\theta_n)) \\ \leq r(g(\theta_n)) \sum_{m=0}^{n-1} (1 - g_m(\theta_n))$$

and the last quantity is $o(1)$ if the sum in (30) remains bounded as $n \rightarrow \infty$. Putting the sum in (30) into the form

$$(31) \quad - \sum_{m=0}^{n-1} (1 - g_m(\theta_n)) = - \sum_{m=0}^{n-1} (1 - g(\theta_n)) + \sum_{m=0}^{n-1} [g_m(\theta_n) - g(\theta_n)]$$

and using the first half of lemma 3 we see that the limit of the first expression on the right hand side of (31) equals $-(\theta/\gamma)^{\frac{1}{2}}$. Thus we need only find the limit of the second sum on the right hand side of (31). We shall do this by obtaining upper and lower bounds for $g_n(x) - g(x)$ which will enable us to invoke lemma 4. Our approach is based on that of Seneta [8].

Equation (21) and the mean value theorem yields, for $0 \leq x < 1$

$$(32) \quad 0 < g_{n+1}(x) - g(x) = x(g_n(x) - g(x))A'(\eta_n) \\ (n = 0, 1, \dots; g(x) < \eta_n < g_n(x)).$$

The left hand inequality is clearly true for $n = 0$, and so for all n , by induction. The same argument shows that $g_{n+1}(x) \leq g_n(x) (0 \leq x \leq 1)$ and thus if we let h be a fixed non-negative integer and write $b_n = 1/(g_n(x) - g(x))$, then the following inequality obtains for $0 \leq h \leq n$,

$$(33) \quad u = xA'(g(x)) \leq b_n/b_{n+1} \leq xA'(g_h(x)).$$

Moreover,

$$g_{n+1}(x) = x[A(g(x)) + (g_n(x) - g(x))A'(g(x)) + (g_n(x) - g(x))^2 A''(\delta_n)]$$

where $g(x) < \delta_n < g_n(x)$, and this yields

$$b_{n+1} = \frac{b_n}{u} - \frac{x A''(\delta_n)}{2u} \cdot \frac{b_{n+1}}{b_n}.$$

Noting that $A''(g(x)) < A''(\delta_n) \leq A''(g_n(x)) \leq A''(g_h(x))$, and combining the last equation with (33) finally yields

$$\frac{b_n}{u} - \frac{x A''(g_h(x))}{2u^2} \leq b_{n+1} \leq \frac{b_n}{u} - \frac{A''(g(x))}{2u A'(g_h(x))}.$$

Proceeding us in [8], we iterate this inequality and invert the result to obtain finally,

$$(34) \quad \frac{2A'(g_h(x))(1-u)}{A''(g(x))} \cdot \frac{\phi u^n}{1+\phi u^n} \leq g_{h+n}(x) - g(x) \\ \leq \frac{2u(1-u)}{xA''(g_h(x))} \cdot \frac{\tau u^n}{1+\tau u^n}$$

where

$$\phi = \phi(x) = \frac{A''(g(x))}{2A''(g_h(x))b_h(1-u) - A''(g(x))}, \\ \tau = \tau(x) = \frac{x A''(g(x))}{2u(1-u)b_h - x A''(g_h(x))}.$$

We now take $h = 0$. Using lemma 3, it is clear that $\phi(x), \tau(x) \rightarrow 1 (x \uparrow 1)$, and so lemma 4 is applicable and we obtain

$$\lim_{n \rightarrow \infty} \sum_{m=0}^{n-1} (g_m(\theta_n)) - g(\theta_n) = -\frac{1}{\gamma} \log \left(\frac{1 + \exp[-2(\gamma\theta)^\frac{1}{2}]}{2} \right).$$

Thus our boundedness requirements are satisfied and the error terms in equation (30) are $o(1)$. The theorem now follows on applying the continuity theorem for Laplace-Stieltjes transforms; see Feller [2].

5. The rate of convergence of $g_n(x)$

We shall briefly look at the rate at which $g_n(x) \rightarrow g(x)$ in a non-supercritical Galton-Watson process, that is, $\alpha \leq 1$. Iteration of equation (32) gives

$$g_n(x) - g(x) = (x - g(x))x^n \prod_{m=0}^{n-1} A'(\eta_m) \leq (x - g(x))(\alpha x)^n$$

which yields

$$(35) \quad u^{-n}(g_n(x) - g(x)) = (x - g(x)) \prod_{m=0}^{n-1} [1 - x(A'(g(x)) - A'(\eta_m))/u].$$

The left hand side of this equation will have a limit iff the series $\sum_{m=0}^{\infty} (A'(\eta_m) - A'(g(x)))$ is convergent. We have

$$\begin{aligned} 0 &\leq A'(\eta_m) - A'(g(x)) \leq A'(g_m(x)) - A'(g(x)) \\ &\leq (g_m(x) - g(x))A''(g_m(x)) \\ &\leq (x - g(x))(\alpha x)^m A''(g_m(x)). \end{aligned}$$

Thus, even if $A''(1-) = \infty$, the series converges for $0 \leq x \leq 1$ and uniformly so on $0 \leq x \leq a < 1$. If $A''(1-) < \infty$ then the convergence is uniform on $[0, 1]$. The same applies to the limit of the product in equation (35), and so we have

$$\lim_{n \rightarrow \infty} (xA'(g(x)))^{-n}(g_n(x) - g(x)) = G(x)$$

where $G(x)$ is continuous and non-negative for $0 \leq x < 1$. Equation (34) yields bounds for $G(x)$ namely

$$\frac{2\phi(x)A'(g_h(x))(1-u)}{u^h A''(g(x))} \leq G(x) \leq \frac{2\tau(x)u(1-u)}{xu^h A''(g_h(x))}.$$

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Added in proof: The density function in the statement of Theorem 3 (page 480) is incorrect, substitute γ for β .
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