BULL. AUSTRAL. MATH. SOC. 20M20
VOL. 16 (1977), 147-149.

Abstracts of Australasian PhD theses

The isomorphism semigroup of a group

Ian Maurice Kirkwood

The isomorphism semigroup S_G of a group G is the set of all isomorphic mappings between subgroups of G. The main work done in this thesis has been to investigate what happens when two groups have isomorphic isomorphism semigroups.

Let G and H be groups such that S_G and S_H are isomorphic. Then it follows readily that G and H are lattice isomorphic. Hence, some results on lattice isomorphism can be used; in particular, those of Baer [1].

Suppose, in particular, G is a periodic abelian group, and H is a group. Rottlaender [2] showed that even if G and H are lattice isomorphic, they need not be isomorphic. However, what happens in the case when S_G and S_H are isomorphic?

THEOREM 1. *If G is a periodic group, and H is a group such that S_G and S_H are isomorphic, then G and H are isomorphic.*

Let G be a periodic group and let H be a group such that G and H are lattice isomorphic. According to Theorem 1, we cannot deduce that S_G and S_H are isomorphic. (If they were, then we could also deduce that G and H were necessarily isomorphic, contrary to Rottlaender's result [2].) A class of pairs of groups, G and H, are provided along these lines; that is, G and H are lattice isomorphic, but S_G and S_H are not isomorphic. This construction turns out to be a generalization of

Received 30 September 1976. Thesis submitted to Monash University, March 1976. Degree approved, August 1976. Supervisor: Professor G.B. Preston.

Rottlaender's result [2].

How does this generalize to non-periodic groups? According to Baer [1, Theorem 13.1], if G is an abelian group with a pair of independent elements, and H is a group which is lattice isomorphic with G, then G and H are isomorphic. Consequently, the only case that needed to be investigated was when G was of torsionfree rank 1 (that is, G is not periodic, but no pair of torsionfree elements are independent). A further restriction placed on G in the thesis was to make G a splitting group (that is, G can be written as the direct product of a periodic and a torsionfree group).

THEOREM 2. *Let G be a splitting abelian group of torsionfree rank 1, such that $G = G_1 \times F(G)$, where $F(G)$ is the subgroup of periodic elements of G; and let H be a group. Then S_G and S_H are isomorphic if and only if the following conditions are satisfied:*

(i) $F(G)$ *and* $F(H)$ *are isomorphic;*

(ii) H *is an abelian group of torsionfree rank* 1 ;

(iii) there is a multiplicative automorphism ψ on Q, the rationals, such that

A: ψ *induces a multiplicative automorphism on Z, the integers,*

B: *there is a non-identity element g_1 of G_1, an element h_1 of H, and a subgroup H_1 of H such that for any rational number r, g_1^r exists in G_1 if and only if $h_1^{r\psi}$ exists in H_1,*

C: *if x is the order of some element of $F(G)$, and m, n are integers, then $m \equiv n \pmod{x}$ if and only if $m\psi \equiv n\psi \pmod{x}$,*

D: *if x is the order of each of a pair of independent elements in $F(G)$, and if m is an integer, then $m\psi \equiv m \pmod{x}$.*

A consequence of Theorem 2 is the following:

https://doi.org/10.1017/S0004972700023091 Published online by Cambridge University Press

THEOREM 3. *Let G be a locally cyclic, torsionfree group, and let H be a group. Then the following statements are equivalent:*

(i) *S_G and S_H are isomorphic;*

(ii) *G and H are lattice isomorphic;*

(iii) *H is a torsionfree, locally cyclic group, and there is a permutation ψ_1 on the set of primes such that for some non-identity element g of G, and some element h of H, there are rational numbers r, s such that g^r exists in G if and only if h^s exists in H (where $r = p^{-m}$ and $s = (p\psi_1)^{-m}$) for every prime p and every integer m.*

Theorem 3 provides a fairly simple criterion for determining when two torsionfree abelian groups have isomorphic isomorphism semigroups, and when they are lattice isomorphic.

Some algebraic structure of the isomorphism semigroup was also briefly covered. In particular, Green's relations were discussed and characterized.

References

[1] Reinhold Baer, "The significance of a system of subgroups for the structure of the group", *Amer. J. Math.* 61 (1939), 1-44.

[2] Ada Rottlaender, "Nachweis der Existenz nicht-isomorpher Gruppen von gleicher Situation der Untergruppen", *Math. Z.* 28 (1928), 641-653.