its polar line (b_2), which must meet a_1, is then known. For a_3 choose any line that meets b_1 and b_2; its polar line (b_3) will meet a_1 and a_2 and will be known. For a_4 choose any line that meets b_1, b_2, b_3; its polar line (b_4) will meet a_1, a_2, a_3 and will be known. But at this point, since only two lines meet b_1, b_2, b_3, b_4, one of them must be a_5 and the other a_6; their polars b_5 and b_6 are the two lines which meet a_1, a_2, a_3, a_4: and *at this point*, since all but one of the conditions binding the lines (a) have been satisfied, the force of the invariant relation between Q_1 and Q_2 becomes effective and proves the last condition, viz. that a_5 meets the polar line of a_6 and vice versa. The twelve lines a and b thus constitute a double-six of lines. Moreover any double-six of lines may be thus obtained.

The evaluation of certain continued fractions

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1. If the approximate numerical value of e is expressed as a continued fraction the result is

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{6+}\,\frac{1}{1}\cdots \tag{1}$$

and it was in finding the proof that the sequence extends correctly to infinity that the following work was done. First the continued fraction may be simplified by setting down the difference equations for numerator and denominator as usual, and eliminating two out of every successive three equations. A difference equation is thus formed between the first, fourth, seventh, tenth .... convergents $\left(\text{counting the first as } 2 + \frac{1}{1}\right)$, and this equation will generate another continued fraction. After a little rearrangement of the first two members it appears that (1) implies

$$\frac{e-1}{e+1} = \frac{1}{2+}\,\frac{1}{6+}\,\frac{1}{10+}\cdots \tag{2}$$

2. We therefore consider the continued fraction

$$F\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} = \frac{\alpha}{\gamma+}\,\frac{\alpha+\beta}{\gamma+\delta+}\,\frac{\alpha+2\beta}{\gamma+2\delta+}\cdots \tag{3}$$

which includes (2), and also certain continued fractions which were discussed by Prof. Turnbull.[1] He evaluated them without solving the difference equations, and it is the purpose here to show how the difference equations may be solved completely both in his cases and in the different problem of (2). It will appear that the work is connected with certain types of hypergeometric function, but I shall not go into this deeply.

Both numerator and denominator of p_n/q_n the n^{th} convergent of (3) will obey the difference equation:

$$f_n = [\gamma + (n-1)\delta] f_{n-1} + [\alpha + (n-1)\beta] f_{n-2}. \tag{4}$$

Once the general solution of this is known, a complete solution of the problem follows easily. The equation (4) is solved by the substitution

$$f_n = \int \phi(t)\, t^n\, dt \tag{5}$$

with suitable choice of ϕ and of the range of integration. Substitute (5) in (4) and integrate by parts. The result is

$$[\phi(t)\{\delta t^n + \beta t^{n-1}\}] + \int \{\phi(t)[-t^n + (\gamma-\delta)t^{n-1} + \alpha t^{n-2}]$$
$$- \frac{d\phi}{dt}[\delta t^n + \beta t^{n-1}]\}\, dt = 0. \tag{6}$$

We choose ϕ so that

$$\frac{d}{dt}\log\phi = \frac{-t^2 + (\gamma-\delta)t + \alpha}{\delta t^2 + \beta t}. \tag{7}$$

Special cases arise when $\delta = 0$, or when $\beta = 0$, and for the present we exclude these. There is also a peculiar case when the numerator is divisible by $\delta t + \beta$, but this need not be excluded. The solution is

$$\phi = e^{-t/\delta}\, t^{\alpha/\beta}\, (\delta t + \beta)^{\lambda} \tag{8}$$

where
$$\lambda = \frac{\beta}{\delta^2} + \frac{\gamma}{\delta} - \frac{\alpha}{\beta} - 1. \tag{9}$$

There may be cases calling for special treatment such as those where δ is negative, or λ is less than -1, but into these we shall not enter; there should be no difficulty in discussing them if necessary. The

[1] H. W. Turnbull, *Math. Notes* 27 iv, 1932.

integrated terms in (6) will usually vanish at ∞, 0, and $-\beta/\delta$. We thus write

$$L_n = \int_0^{\infty} e^{-t/\delta}\, t^{n+\alpha/\beta}\, (\delta t + \beta)^{\lambda}\, dt \tag{10}$$

$$M_n = \int_{-\beta/\delta}^{0} e^{-t/\delta}\, t^{n+\alpha/\beta}\, (\delta t + \beta)^{\lambda}\, dt \tag{11}$$

and have

$$p_n = AL_n + BM_n;\ q_n = CL_n + DM_n$$

where $A \ldots . D$ are given by the initial values. These initial values would normally be taken as those of $p_1 \ldots . q_2$, but it is much simpler to extrapolate the difference equations backwards from (4) and take $p_0 = 0$, $p_{-1} = 1$; $q_0 = 1$, $q_{-1} = 0$. Then we have

$$\begin{aligned} p_n &= (L_0 M_n - M_0 L_n)/(L_0 M_{-1} - M_0 L_{-1}) \\ q_n &= (M_{-1} L_n - L_{-1} M_n)/(L_0 M_{-1} - M_0 L_{-1}). \end{aligned} \tag{13}$$

We require the ratio of these as n tends to infinity. Now L_n tends to become very large with n, after the general manner of a Γ function, whereas M_n does not.

Hence
$$F\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = -\frac{M_0}{M_{-1}} \tag{14}$$

$$= \int_0^{\beta/\delta} e^{s/\delta}\, s^{\alpha/\beta}\, (\beta - \delta s)^{\lambda}\, ds \Big/ \int_0^{\beta/\delta} e^{s/\delta}\, s^{\alpha/\beta - 1}\, (\beta - \delta s)^{\lambda}\, ds \tag{15}$$

where λ is given by (9).

As a simple example take $\alpha = \beta = \delta = 1$, $\gamma = 2$.

Then
$$\frac{1}{2+}\,\frac{2}{3+}\,\frac{3}{4+}\,\ldots . = \frac{3-e}{e-2}$$

or by a simple transformation

$$e = 2 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{2}{3+}\,\frac{3}{4+}\,\ldots . \tag{16}$$

In general α/β and λ are not integers and the integrations cannot be performed. For example

$$\frac{1}{2+}\,\frac{3}{4+}\,\frac{5}{6+}\,\ldots . = \int_0^1 e^{s/2}\, s^{\frac{1}{2}}\, du \Big/ \int_0^1 e^{s/2}\, s^{-\frac{1}{2}}\, du. \tag{17}$$

This is also an example of the peculiar case where a factor in numerator and denominator of (7) has cancelled out.

3. *There are still certain exceptional cases to be considered.* When $\delta = 0$, we have from (7)

$$\phi = e^{-t^2/2\beta + \gamma t/\beta}\, t^{\alpha/\beta}. \tag{18}$$

The proper ranges of integration are now $-\infty$, 0, ∞, and if we write

$$\left.\begin{aligned} L_n &= \int_0^\infty e^{-t^2/2\beta+\gamma t/\beta}\, t^{n+\alpha/\beta}\, dt \\ M_n &= \int_{-\infty}^0 e^{-t^2/2\beta+\gamma t/\beta}\, t^{n+\alpha/\beta}\, dt \end{aligned}\right\} \tag{19}$$

the solution (13) will satisfy the conditions. The ratio of $L_n : M_n$ tends to $e^{2\gamma\sqrt{n}/\beta}$ as n tends to infinity, as may be seen by locating the maxima of the integrands and the use of "steepest descents"; so again (14) is the answer but with the changed forms of integrals.

As example we have (with a small change of variable)

$$\frac{1}{1+}\,\frac{2}{1+}\,\frac{3}{1+}\,\frac{4}{1+}\,\cdots = \frac{\int_0^\infty e^{-s^2/2-s}\, s\, ds}{\int_0^\infty e^{-s^2/2-s}\, ds}. \tag{20}$$

4. The other exceptional case, with β vanishing, is very different, for it corresponds to a confluent hypergeometric function. From (7) we derive

$$\phi = t^{\gamma/\delta-1}\, e^{-t/\delta+\alpha/\delta t}.$$

This leads to a Bessel function and the simplest procedure is to quote the properties of those functions, and apply them directly to the difference equations. If the Bessel functions are defined in the manner of Whittaker and Watson's *Modern Analysis*, p. 353, it is there shown that

$$J_{m-1}(z) + J_{m+1}(z) = \frac{2m}{z}\, J_m(z)$$

whether m is an integer or not. If this is fitted into (4) we have:

$$f_n = i^n\, \alpha^{n/2}\, J_{n+\gamma/\delta}\left(\frac{2i\alpha^{\frac{1}{2}}}{\delta}\right). \tag{21}$$

As long as γ/δ is not an integer there is a second solution of similar form:—

$$f_n = i^n\, \alpha^{n/2}\, J_{-n-\gamma/\delta}\left(\frac{-2i\alpha^{\frac{1}{2}}}{\delta}\right). \tag{22}$$

The solution will be a sum of (21) and (22), and the same procedure will give the complete value for the continued fraction. There are a good many special points to discuss, such as the comparative asymptotic values of the two solutions and the case where γ/δ is an integer. As there is no serious difficulty in these, but a good deal of uninteresting detail, we shall here only consider a few special examples.

We first take (2) which is $F\begin{pmatrix}1 & 0\\ 2 & 4\end{pmatrix}$. By inspection of (21) the solutions will be imaginary half order functions. These are the simplest of all the Bessel functions and we may conveniently define

$$\left.\begin{aligned} L_n(z) &= z^n\left(\frac{1}{z}\,\frac{d}{dz}\right)^n\frac{e^z}{z}\\ M_n(z) &= z^n\left(\frac{1}{z}\,\frac{d}{dz}\right)^n\frac{e^{-z}}{z}\end{aligned}\right\}. \tag{23}$$

Then we have

$$\left.\begin{aligned} p_n &= A\,(-)^n L_n(\tfrac{1}{2}) + B\,(-)^n M_n(\tfrac{1}{2})\\ q_n &= C\,(-)^n L_n(\tfrac{1}{2}) + D\,(-)^n M_n(\tfrac{1}{2})\end{aligned}\right\}. \tag{24}$$

If we put in the values we find

$$\frac{p_n}{q_n} = \frac{e - (L_n(\tfrac{1}{2})/M_n(\tfrac{1}{2}))}{e + (L_n(\tfrac{1}{2})/M_n(\tfrac{1}{2}))} \tag{25}$$

and it is easily shown that $L_n\ M_n$ tend to equality as n tends to infinity.

Lastly consider

$$-\frac{1}{1}-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\cdots = F\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}. \tag{26}$$

By the same methods this is seen to depend on

$$\frac{p_n}{q_n} = \frac{Y_1(2)\,J_n(2) - J_1(2)\,Y_n(2)}{-Y_0(2)\,J_n(2) + J_0(2)\,Y_n(2)}.$$

Using the definition of Y_n on p. 363 *loc. cit.* we find that $Y_n(2)/J_n(2)$ tends to $-\log n$ as n becomes large. Hence

$$F\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix} = -\frac{J_1(2)}{J_0(2)}. \tag{27}$$

These examples will suffice to show how any continued fraction of the kind may be evaluated.