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INVERSE SEMIGROUP C^*-ALGEBRAS ASSOCIATED WITH LEFT CANCELLATIVE SEMIGROUPS

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Abstract To each discrete left cancellative semigroup S one may associate an inverse semigroup $I_l(S)$, often called the left inverse hull of S. We show how the full and reduced C^*-algebras of $I_l(S)$ are related to the full and reduced semigroup C^*-algebras for S, recently introduced by Li, and give conditions ensuring that these algebras are isomorphic. Our picture provides an enhanced understanding of Li's algebras.

Keywords: C^*-algebra; left cancellative semigroup; inverse semigroup; amenability

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1. Introduction

In [**17**], Li proposed a construction for the full C^*-algebra $C^*(S)$ of a discrete left cancellative semigroup S. For a semigroup S that embeds into a group he also constructed a related C^*-algebra called $C^*_s(S)$. The reason one considers left cancellative semigroups is that these are the semigroups that can be faithfully represented as semigroups of isometries on Hilbert spaces. For instance, one can represent S on $\ell^2(S)$ by isometries in this case. This representation is called the left regular representation of S and generates what is called the Toeplitz algebra or reduced C^*-algebra of S, denoted by $C^*_r(S)$. (One could of course consider right cancellative semigroups instead.)

Murphy previously constructed C^*-algebras of left cancellative semigroups, but these turned out to be very large. For instance, his C^*-algebra associated to $(\mathbb{Z}^+)^2$ is non-nuclear [**21**] (see [**17**] for more references). Li added a few extra restrictions that make the algebras behave better. In particular, he showed that $C^*(S)$ generalizes two important types of C^*-algebras: Nica's C^*-algebras for quasi-lattice ordered groups from [**22**], and the Toeplitz algebras associated with the ring of integers in a number field [**7**].

Li also showed that a cancellative left reversible semigroup (a semigroup S is left reversible if for any $s, t \in S$, $sS \cap tS \neq \emptyset$; this is also called the Ore condition) S is left amenable if and only if $C^*_s(S)$ and $C^*_r(S)$ are canonically isomorphic, but only given that the constructible right ideals of S satisfy a certain technical requirement called independence. Note that Li's proof uses the fact that left reversibility of a cancellative semigroup S implies that S embeds into a group and that there exists a character on $C_s^*(S)$.

Let $I(S)$ be the inverse semigroup of all partial bijections on S. For each $s \in S$, let $\lambda_s \colon S \to sS$ be given by $\lambda_s(t) = st$. Since S is left cancellative, each λ_s is a partial bijection. The set $\{\lambda_s\}_{s\in S}$ generates an inverse subsemigroup $I_l(S) \subset I(S)$ called the left inverse hull of S. We show that $I_l(S)$ is isomorphic to an inverse semigroup $V(S)$ of partial isometries generating $C_r^*(S)$. By considering the full and reduced C^*-algebras of $I_l(S)$ as defined, for instance, in Paterson's book [**27**], we get the surjective $*$-homomorphisms

$$C^*(S) \xrightarrow{\eta} C_0^*(I_l(S)) \xrightarrow{\Lambda_0} C_{r,0}^*(I_l(S)) \xrightarrow{h} C_r^*(S).$$

Here, $C_0^*(I_l(S))$ and $C_{r,0}^*(I_l(S))$ are the quotients of $C^*(I_l(S))$ and $C_r^*(I_l(S))$, respectively, by the ideal generated by the 0-element of $I_l(S)$, if it has one. The composition of these $*$-homomorphisms is the canonical $*$-homomorphism $C^*(S) \to C_r^*(S)$. The question of whether this is an isomorphism splits into three separate problems. When S embeds into a group, we get the decomposition

$$C^*(S) \xrightarrow{\pi_s} C_s^*(S) \xrightarrow{\simeq} C_0^*(I_l(S)) \xrightarrow{\Lambda_0} C_{r,0}^*(I_l(S)) \xrightarrow{h} C_r^*(S).$$

In particular, $C_s^*(S)$ and $C_0^*(I_l(S))$ are canonically isomorphic.

A semigroup S is said to satisfy Clifford's condition if, for all $s, t \in S$, either $sS \cap tS = \emptyset$ or $sS \cap tS = rS$ for some $r \in S$. Any semigroup that is the positive cone in one of Nica's quasi-lattice ordered groups satisfies Clifford's condition. The $ax + b$ semigroup over an integral domain R satisfies Clifford's condition if and only if every pair of elements in R has a least common multiple. If S satisfies Clifford's condition, η is an isomorphism and the constructible right ideals of S are independent. If S is cancellative and satisfies Clifford's condition, or if S embeds into a group and the constructible right ideals of S are independent, then h is an isomorphism.

Using Milan's work [**19**] on weak containment for inverse semigroups we show that when S embeds into a group G, Λ_0 is an isomorphism if and only if a certain Fell bundle over G associated with $I_l(S)$ is amenable. In the special case when S is left reversible, Λ_0 is an isomorphism if and only if S is left amenable if and only if $C_0^*(I_l(S))$ is nuclear.

In the first part of the paper we recall the algebraic theory of semigroups and inverse semigroups, and also look at an algebraic partial order and see how it is related to Nica's quasi-lattice ordered groups. We show that many of the properties of the positive cone in these groups can be defined in a more general context, and remark that the algebraic order is not essential for the theory to work.

In the second part of the paper, we introduce the C^*-algebras associated with S and $I_l(S)$, and prove the above-stated results. In addition, we show that our construction generalizes a method used by Nica in [**23**] to construct the C^*-algebra of a quasi-lattice ordered group from a certain inverse semigroup called a Toeplitz inverse semigroup.

We also prove a functoriality result for the construction $S \mapsto G(S)$ when S is left reversible. Here, $G(S)$ is the maximal group homomorphic image of $I_l(S)$. We use this to show that the construction $S \mapsto C^*(I_l(S))$ is functorial for homomorphisms into groups when S is left reversible. The construction $S \mapsto G(S)$ originates from Rees's proof of Ore's theorem: all cancellative left reversible semigroups are group embeddable. An account of this theorem can be found in [**6**, p. 35] or in [**15**, Chapter 2.4].

2. Semigroups

2.1. Semigroups and algebraic orders

There are many sources on the algebraic theory of semigroups (see, for example, [**6**] or [**18**] and the references therein).

Definition 2.1. A *semigroup* is a set S together with an associative binary operation $\cdot\colon S \times S \to S$, written $(s,t) \mapsto st$, and an identity element $1 \in S$. (Usually, semigroups are not required to have identities, and semigroups with identities are called *monoids*. We will, however, only talk about monoids in this article, and we prefer to call them semigroups.) That is, for all $s,r,t \in S$, $s(rt) = (sr)t$ and $1s = s1 = s$. Sometimes we write $1 = 1_S$.

If S has an element $z \in S$ such that $zs = sz = z$ for all $s \in S$, we write that $z = 0 = 0_S$. If S is a semigroup, define $S^0 = S$ if S already has a 0 element, and otherwise let S^0 be the semigroup $S \cup \{0\}$ with the extended multiplication rule $s0 = 0s = 0$ for all $s \in S^0$.

This choice of notation can be confusing, for instance in the case of $(\mathbb{Z}^+, +)$ where we have $1_{\mathbb{Z}^+} = 0$, and where $\mathbb{Z}^+$ does not have an element $0_{\mathbb{Z}^+}$ in the sense of the above definition, but the notation is otherwise very convenient. (In our notation, $\mathbb{Z}^+$ denotes $\{0,1,2,\dots\}$, while $\mathbb{N}$ denotes $\{1,2,\dots\}$.)

Definition 2.2. A *homomorphism* between semigroups S, S' is a function $f\colon S \to S'$ such that, for all $s,t \in S$, $f(st) = f(s)f(t)$ and $f(1_S) = 1_{S'}$. The homomorphism f is a 0-*homomorphism* if, in addition, $f(0_S) = 0_{S'}$ (and this term is only defined for homomorphisms between semigroups with zeroes).

Definition 2.3. A semigroup S is *left cancellative* if, for every $s,r,t \in S$, $sr = st$ implies that $r = t$. Equivalently, for every $s \in S$, the map $\lambda_s\colon S \to sS$ given by $\lambda_s(t) = st$ is bijective. In a left cancellative semigroup, if $ss' = 1$, then $ss's = 1s = s1$, so $s's = 1$, that is, every element with a left (or right) inverse is invertible. One can similarly define right cancellativity. S is *cancellative* if it is both left and right cancellative.

Definition 2.4. A *congruence* on a semigroup S is an equivalence relation $\sim$ such that, for all $s,t,r,\in S$, $s \sim t$ implies that $sr \sim tr$ and $rs \sim rt$. One can show that $S/\sim$ is a semigroup and that there exists a homomorphism $S \to S/\sim$ sending elements to equivalence classes. In fact, the homomorphism theorems for semigroups state that every surjective homomorphism can be constructed in this way.

Definition 2.5. A subset $X \subset S$ is a *right ideal* if, for all $t \in X$ and $s \in S$, $ts \in X$.

For $X \subset S$ and $s \in S$, define $s^{-1}(X) = \{t\colon st \in X\}$ and $sX = \{st\colon t \in X\}$. For simplicity, we sometimes write $s^{-1}X$ for $s^{-1}(X)$. If $X \subset S$ is a right ideal, then so are sX and $s^{-1}X$. The right ideals of the form sS are called the *principal right ideals* of S.

Let $\preceq$ be the relation on S given by $s \preceq t$ if there exists an $r \in S$ such that $s = tr$. This relation is reflexive and transitive, so it gives a preorder on S. If it is antisymmetric, then it is a partial order called the *algebraic order* on S and we say that S is *algebraically ordered*. Note that $\preceq$ is often written with the opposite symbol $\succeq$ or $\geqslant$ (such as in Nica's work [**22**]), but this is just a matter of convenience. For instance, we have $5 \preceq 4$ in $(\mathbb{Z}^+, +)$ with our notation.

For $s, t \in S$, $s \preceq t$ is easily seen to be equivalent to $s \in tS$ and to $sS \subset tS$. It is also equivalent to $t^{-1}(\{s\}) \neq \emptyset$, and if $S \subset G$, where G is a group, it is equivalent to $t^{-1}s \in S$. Note that 1_S is a maximal element for $\preceq$ and that if 0_S exists, it is a minimal element.

Lemma 2.6. *Let S be a left cancellative semigroup. Then, S is algebraically ordered if and only if 1 is the only invertible element in S.*

Proof. Suppose that $rS = tS$ for some $r, t \in S$. There then exist $s, s' \in S$ such that $rs = t$ and $ts' = r$. So $ts's = t$. By left cancellation with t, this gives us that $s's = 1$. Then, s has a left inverse, so it is invertible since S was left cancellative. If 1 is the only invertible element in S, then $s = 1$, and this implies that $r = t$.

On the other hand, suppose that there exist $s, s' \in S$, with $s's = 1$. Then, $s'sS \subset s'S \subset S = s'sS$, so, if S is algebraically ordered, $s' = 1$. □

For instance, when S is a subsemigroup of a group G, S is algebraically ordered if and only if $S \cap S^{-1} = \{1\}$.

2.2. Inverse semigroups

Inverse semigroups are a large topic. See, for example, [**15**] or [**27**] and the references therein for a review of the literature. In this section we just give a short overview of the main concepts that we need.

Definition 2.7. A semigroup P is an *inverse semigroup* if, for every $p \in P$, there exists a unique element $p^* \in P$ such that $pp^*p = p$ and $p^*pp^* = p^*$.

It follows from the uniqueness of p^* that, for any semigroup homomorphism $f\colon P \to Q$ between inverse semigroups, $f(p^*) = f(p)^*$ for any $p \in P$. Let $L(P)$ be the set of idempotents in the inverse semigroup P. Then, $L(P) = \{p^*p\colon p \in P\} = \{pp^*\colon p \in P\}$. One can show that $L(P)$ is a commutative subsemigroup of P, so $L(P)$ is what is called a semilattice.

Definition 2.8. A *semilattice* is a commutative semigroup where every element is idempotent.

Lemma 2.9. *Let L be a semilattice, and let $a, b \in L$. Then, $a \preceq b$ if and only if $ba = a$. Hence, $\preceq$ is a partial order on L.*

Proof. If $a = ba$, then $a \in bL$, so $a \preceq b$. Suppose that $a \in bL$, so $a = bc$ for some $c \in L$. Then $a = aa = bca$. So $bca = bbca = ba$, which implies that $a = ba$. If $a \preceq b$ and $b \preceq a$, then $a = ba = ab = b$. So $\preceq$ is a partial order. □

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It also follows that, for $a, b \in L$, ab is the greatest lower bound of a and b. On the other hand, if L is a partially ordered set where any finite subset has a unique greatest lower bound and one defines ab to be the greatest lower bound of $\{a, b\}$, then L is a semilattice with the product $(a, b) \mapsto ab$. We later study partially ordered semigroups S, and for this it is useful to let $s \wedge t$ mean the greatest lower bound of s and t if it exists, while st means the already existing semigroup product of s and t. These two products only coincide if S is a semilattice.

Remark 2.10. Each inverse semigroup admits a natural partial order, which is generally not the same as the partial order described in the preceding paragraphs. We do not use the natural partial order explicitly in this paper.

Perhaps the most important example of an inverse semigroup is the semigroup $I(X)$ of all partially defined bijective maps on some set X. By a partially defined bijective map on X, we mean a bijective function $f\colon \operatorname{dom}(f) \to \operatorname{ran}(f)$, where $\operatorname{dom}(f)$ and $\operatorname{ran}(f)$ are subsets of X. The product fg of $f, g \in I(X)$ is defined such that $\operatorname{dom}(fg) = g^{-1}(\operatorname{dom}(f))$ and $fg(x) = f(g(x))$ for all $x \in \operatorname{dom}(fg)$. Note that this product can result in the empty function, which acts as a 0 for $I(X)$. The $*$-operation is given by function inversion. For any $f \in I(X)$, $f^*f = i_{\operatorname{dom}(f)}$, where $i_{\operatorname{dom}(f)}\colon \operatorname{dom}(f) \to \operatorname{dom}(f)$ is the identity map.

The Wagner–Preston theorem states that any inverse semigroup P can be faithfully represented as a subsemigroup of $I(P)$ as follows. Let $\tau\colon P \to I(P)$ be the map such that, for $p \in P$, $\operatorname{dom}(\tau(p)) = \{q \in P\colon p^*pq = q\}$, and $\tau(p)(q) = pq$ for all $q \in \operatorname{dom}(\tau(p))$.

Another important class of inverse semigroups are semigroups of partial isometries in a C^*-algebra. Note that, in general, the product of two partial isometries does not have to be a partial isometry. Two partial isometries can be part of the same inverse semigroup if and only if their initial and final projections commute.

The following concepts are very important in the theory of inverse semigroups.

Definition 2.11. An inverse semigroup P is *E-unitary* if, for every $p, q \in P$, $pq = q$ implies that $p \in L(P)$. It is *E^*-unitary* (also-called 0-E-unitary) if, for every $p, q \in P$, $pq = q$ and $q \neq 0$ implies that $p \in L(P)$.

Note that if P is an E-unitary inverse semigroup with 0, then it is a semilattice. Note also that, if we want to, we can assume without loss of generality that the q in either definition is idempotent. Multiply the equation $pq = q$ on the right by q^*. This gives us that $pqq^* = qq^*$, where qq^* is idempotent. Recall that, for any semigroup S, $S^0 = S$ if S already has a 0 element, and, otherwise, S^0 is the semigroup $S \cup \{0\}$ with the extended multiplication rule $s0 = 0s = 0$ for all $s \in S^0$.

Definition 2.12. A *grading* of the inverse semigroup P is a map $\varphi\colon P^0 \to G^0$, where G is a group, such that $\varphi^{-1}(\{0\}) = \{0\}$ and, for all $p, q \in P$, $\varphi(pq) = \varphi(p)\varphi(q)$ as long as $pq \neq 0$. P is *strongly E^*-unitary* if it has a grading φ such that $\varphi^{-1}(\{1_G\}) = L(P) \setminus \{0\}$. Such a grading is sometimes said to be *idempotent pure*.

Note that if $\varphi\colon P^0 \to G^0$ is a grading of P, then $L(P) \setminus \{0\} \subset \varphi^{-1}(\{1_G\})$. Note also that if P is strongly E^*-unitary, then it is E^*-unitary. It turns out that if P does not have a 0, all these concepts are equivalent.

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Definition 2.13. Define a relation $\sim$ on P by $p \sim q$ if $pr = qr$ for some $r \in P$ (if and only if $pr = qr$ for some $r \in L(P)$). Then $\sim$ is a congruence, and $P/\sim$ is a group denoted by $G(P)$. Let $\alpha_P \colon P \to G(P)$ be the quotient homomorphism. Then, P is E-unitary if and only if $\alpha_P^{-1}(1_{G(P)}) = L(P)$. The group $G(P)$ is often called the *maximal group homomorphic image* of P.

We will need the following lemma later in the paper.

Lemma 2.14. *Let $f \colon P \to Q$ be a surjective homomorphism between inverse semigroups. Suppose that the restriction of f to $L(P)$ is an isomorphism onto $L(Q)$. Then f is an isomorphism if and only if $f^{-1}(L(Q)) = L(P)$.*

Proof. The 'only if' part is trivial. Suppose that $f^{-1}(L(Q)) = L(P)$. Let $p, q \in P$ with $f(p) = f(q)$. Then, $f(pq^*) = f(qq^*) \in L(Q)$, so by assumption pq^* is idempotent. Since f is an isomorphism restricted to $L(P)$, $pq^* = qq^*$, so $q^*pq^* = q^*$. Similarly, $f(q^*) = f(p^*)$, so $q^*p = p^*p$ and $pq^*p = p$. Thus, $p = q$ by the uniqueness property for these relations in an inverse semigroup. □

2.3. The semilattice $J(S)$, Clifford's condition and the independence of constructible right ideals

We are interested in the semilattice $J(S)$ of constructible right ideals in the left cancellative semigroup S given by

$$J(S) = \left\{ \bigcap_{j=1}^{N} t_{j1}^{-1} s_{j1} \cdots t_{jn_j}^{-1} s_{jn_j} S \colon N, n_j \in \mathbb{N},\ s_{jk}, t_{jk} \in S \right\}.$$

We actually see in Lemma 3.9 that

$$J(S) = \{ t_1^{-1} s_1 \cdots t_n^{-1} s_n S \colon n \in \mathbb{N},\ s_i, t_i \in S \}.$$

Here, the semilattice product on $J(S)$ is given by set intersection. To motivate this study, we can reveal that $J(S)$ is isomorphic to a semilattice of projections generating the diagonal subalgebra of $C_r^*(S)$. It is also the semilattice of idempotents in the left inverse hull of S. We establish these facts later. This semilattice plays an important part in Li's theory [**17**]. Li's $\mathcal{J}$ is the same as our $J(S) \cup \{\emptyset\} \simeq J(S)^0$.

Lemma 2.15. *Let S be an algebraically ordered semigroup and let $s, t \in S$. If $sS \cap tS = rS$ for some $r \in S$, then $s \wedge t$ exists and equals r. Conversely, if $s \wedge t$ exists, then $(s \wedge t)S = sS \cap tS$.*

Proof. First, suppose that $r' \preceq s, t$. Then, $r'S \subset sS \cap tS = rS$, so $r' \preceq r$ and, therefore, r is the greatest lower bound of s and t, i.e. $r = s \wedge t$.

Next, if $s \wedge t$ exists, then by definition $(s \wedge t)S \subset sS \cap tS$. Let $r \in sS \cap tS$. Then, $r \preceq s, t$, so $r \preceq s \wedge t$, and $r \in (s \wedge t)S$, so $(s \wedge t)S = sS \cap tS$. □

Lemma 2.16. *Let S be a semigroup, and let $s_1, \ldots, s_n \in S$. If $\bigcup_{i=1}^{n} s_i S = rS$ for some $r \in S$, then $rS = s_i S$ for at least one $1 \leqslant i \leqslant n$.*

Proof. We have that $rS \subset \bigcup_{i=1}^n s_n S$ is equivalent to $r \in \bigcup_{i=1}^n s_n S$, which implies that $r \in s_j S$ for some j. Then, $rS \subset s_j S \subset \bigcup_{i=1}^n s_i S = rS$, so $rS = s_j S$. □

Definition 2.17. We say that a semigroup S satisfies *Clifford's condition* if, for any $s, t \in S$, either $sS \cap tS = \emptyset$ or there exists an $r \in S$ such that $sS \cap tS = rS$. (This is not the same concept as a Clifford semigroup. Clifford's condition is a term coined by Lawson [**14**] because it plays an important role in the construction of 0-bisimple inverse semigroups, and Clifford was the first to use this in [**5**]. See [**14**] for more on this.)

For instance, all free or free abelian semigroups satisfy Clifford's condition. We see more examples below.

Definition 2.18. Following Li [**17**], we say that $J(S)$ is *independent* or that *the constructible right ideals of S are independent* if, for any $X_1, \dots, X_n, Y \in J(S)$, $\bigcup_{i=1}^n X_n = Y$ implies that $X_i = Y$ for at least one $1 \leqslant i \leqslant n$.

Proposition 2.19. *Let S be a left cancellative semigroup. The following two conditions are equivalent.*

(i) *S satisfies Clifford's condition.*

(ii) *For every $s, t \in S$ with $t^{-1}(sS)$ non-empty, there exists some $r \in S$ such that $t^{-1}(sS) = rS$.*

These conditions imply that $J(S) \cup \{\emptyset\} = \{sS\colon s \in S\} \cup \{\emptyset\}$ and that $J(S)$ is independent. If S is algebraically ordered, (i) is equivalent to the following statement.

(iii) *Every pair of elements in S that have a common lower bound have a greatest lower bound.*

This implies that when S is an algebraically ordered semigroup satisfying Clifford's condition, $(S^0, \wedge)$ is a semilattice and is isomorphic as a semilattice to $J(S) \cup \{\emptyset\} \simeq J(S)^0$.

Proof. (i) ⇒ (ii). Since S is left cancellative, then for any $X \subset S$ we have that $tt^{-1}(X) = tS \cap X$ and $t^{-1}(tX) = X$. If $t^{-1}(sS)$ is non-empty, then so is $tt^{-1}(sS) = sS \cap tS$. Let $q \in S$ be such that $qS = sS \cap tS$. Since $q \in tS$, $t^{-1}(\{q\})$ is non-empty and contains a unique element r, since S was left cancellative. We now have that

$$\begin{aligned} rS = (t^{-1}\{q\})S &= \{uv\colon u, v \in S,\ tu = q\} \\ &= t^{-1}\{tuv\colon u, v \in S,\ tu = q\} \\ &= t^{-1}\{qv\colon v \in S\} = t^{-1}(qS) \\ &= t^{-1}(sS \cap tS) = t^{-1}(sS). \end{aligned}$$

(ii) ⇒ (i). If $tS \cap sS$ is non-empty, then so are $tt^{-1}(sS)$ and $t^{-1}(sS)$. By assumption, $t^{-1}(sS) = rS$, so $tS \cap sS = (tr)S$.

That (i)+(ii) implies that $J(S) \cup \{\emptyset\} = \{sS\colon s \in S\} \cup \{\emptyset\}$ is proved by simple induction. That this implies that $J(S)$ is independent follows from Lemma 2.16: if $\bigcup_{i=1}^n s_i S = Y \in J(S)$, $Y = rS$ for some $r \in S$, and Lemma 2.16 gives that $rS = s_i S$ for at least one i.

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(i) $\Leftrightarrow$ (iii). Let $s, t \in S$. Then, $sS \cap tS \neq \emptyset$ if and only if there exists some $r \in S$ such that $r \preceq s, t$ if and only if s and t have a common lower bound. By Lemma 2.15, s and t have a greatest lower bound $s \wedge t$ if and only if $sS \cap tS = (s \wedge t)S$.

By going to S^0, we have that $sS^0 \cap tS^0 = \{0\} = 0S^0$ if and only if $sS \cap tS = \emptyset$. Otherwise, $sS^0 \cap tS^0 = rS^0$ for some $r \in S$. The isomorphism from $(S^0, \wedge)$ to $J(S) \cup \{\emptyset\}$ is then constructed by sending s to sS for $s \in S$ and 0 to $\emptyset$. This is injective since S was algebraically ordered. □

Definition 2.20. Let G be a group and let $S \subset G$ be a subsemigroup. If S is algebraically ordered and generates G, it induces a partial order on all of G by $g \leqslant h$ if and only if $g^{-1}h \in S$. Nica [**22**] calls (G, S) *quasi-lattice ordered* if, in addition, any finite family of elements in G that has a common upper bound in S has a least common upper bound in S. In this case S is called the *positive cone* in (G, S).

Note that, when restricted to S, $\leqslant$ is the same as our $\succeq$. This shows that if S is a positive cone in a quasi-lattice ordered group, any pair in S that have a common lower bound in S with respect to $\preceq$ have a greatest lower bound in S. So S satisfies Clifford's condition by Proposition 2.19, and it follows that $J(S)$ is independent. Note that Li proved in [**17**] that the positive cones of the quasi-lattice ordered groups have independent constructible right ideals.

We can give a description of when the $ax + b$ semigroup over an integral domain R satisfies Clifford's condition. The $ax+b$ semigroup over R, denoted by $R \rtimes R^{\times}$, is defined to be the set $R \times R^{\times}$ with product $(b, a)(d, c) = (b + ad, ac)$. Here, $R^{\times} = R \setminus \{0\}$. The reason one considers integral domains is that the $ax + b$ semigroups over these are left cancellative.

Consider first the multiplicative semigroup $(R^{\times}, \cdot)$. This is a semigroup, since R has no zero divisors. We see that, for $a, b \in R^{\times}$, $a \succeq b$ if and only if a divides b.

Definition 2.21. A *common multiple* of $a, b \in R^{\times}$ is an element c of $R^{\times}$ that is divided by a and b. A *least common multiple* of a and b is a common multiple c such that if c' is a common multiple of a and b, then c divides c'.

It follows by a similar argument to that in Lemma 2.15 that $aR^{\times} \cap bR^{\times} = cR^{\times}$ if and only if c is a least common multiple of a and b. Note that since R is commutative, $ab \in aR^{\times} \cap bR^{\times} \neq \emptyset$. So $R^{\times}$ satisfies Clifford's condition if and only if every pair in $R^{\times}$ has a least common multiple (see also [**4**, Theorem 2.1]). Such an integral domain R is often called a GCD domain because one can show that every pair has a greatest common divisor if and only if every pair has a least common multiple. See [**4**] for a detailed discussion of GCD domains; they are also discussed in [**2**], where they are called pseudo-Bezout domains. The next lemma is stated without proof in Li's paper, but we include it for completeness.

Lemma 2.22. *Let R be a ring. For any subrings $I, J \subset R$ and $b, d \in R$, either $(b + I) \cap (d + J) = \emptyset$ or there exists some $x \in R$ such that $(b + I) \cap (d + J) = x + I \cap J$.*

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Proof. Suppose that $(b+I)\cap(d+J)\neq\emptyset$. There then exist $y\in I$ and $z\in J$ such that $b+y=d+z$. Write $x=b+y=d+z$. Then, $x+I=b+y+I=b+I$ and $x+J=d+z+J=d+J$. So $(b+I)\cap(d+J)=(x+I)\cap(x+J)=x+I\cap J$. □

Proposition 2.23. *Let R be an integral domain. Then, $R\rtimes R^\times$ satisfies Clifford's condition if and only if R is a GCD domain.*

Proof. We show that $R\rtimes R^\times$ satisfies Clifford's condition if and only if $R^\times$ satisfies Clifford's condition. Suppose that $R^\times$ does satisfy Clifford's condition. Note that since R is a commutative ring, aR is an ideal of R for every $a\in R$. Let $a,b,c,d\in R$. Then,

$$\begin{aligned}(b,a)(R\rtimes R^\times)\cap(d,c)(R\rtimes R^\times)&=[(b+aR)\times aR^\times]\cap[(d+cR)\times cR^\times]\\&=[(b+aR)\cap(d+cR)]\times[aR^\times\cap cR^\times].\end{aligned}$$

If this set is non-empty, $(b+aR)\cap(d+cR)$ is non-empty, so by the previous lemma there exists some $x\in R$ satisfying $(b+aR)\cap(d+cR)=x+aR\cap cR$. Moreover, $aR^\times\cap cR^\times\neq\emptyset$, so, since $R^\times$ satisfies Clifford's condition, there exists some $y\in R^\times$ satisfying $aR^\times\cap cR^\times=yR^\times$. This also implies that $aR\cap cR=yR$. So we get that

$$(b,a)(R\rtimes R^\times)\cap(d,c)(R\rtimes R^\times)=(x+yR)\times yR^\times=(x,y)(R\rtimes R^\times).$$

Suppose that $R\rtimes R^\times$ satisfies Clifford's condition and let $a,c\in R^\times$. Since $aR^\times\cap cR^\times\neq\emptyset$,

$$(0,a)(R\rtimes R^\times)\cap(0,c)(R\rtimes R^\times)=(aR\cap cR)\times(aR^\times\cap cR^\times)\neq\emptyset.$$

It follows that there exist $y\in R^\times$ and $x\in R$ (one may take $x=0$) such that

$$(0,a)(R\rtimes R^\times)\cap(0,c)(R\rtimes R^\times)=(x,y)(R\rtimes R^\times).$$

This implies that $aR^\times\cap cR^\times=yR^\times$. □

Li showed that when R is a Dedekind domain, $J(R\rtimes R^\times)$ is independent. Every Dedekind domain that is also a GCD domain is a principal ideal domain. One way to see this is to use that every non-trivial ideal in a Dedekind domain R is of the form $c^{-1}(aR)$ for some $c,a\in R$. This is proved, for instance, in [**17**]. Note that Li denoted $c^{-1}(aR)$ as $((c^{-1}a)\cdot R)\cap R$. This comes from viewing $c^{-1}a$ as an element of the field of fractions of R. Applying Proposition 2.19 (ii) to the semigroup $R^\times$ one can deduce that, if R is also a GCD domain, any non-trivial ideal in R is of the form aR for some $a\in R$. This is the definition of a principal ideal domain.

There exist Dedekind domains that are not principal ideal domains. An example of this is $\mathbb{Z}[\sqrt{10}]$ as seen in [**12**, p. 407]. This shows that not every left cancellative semigroup with independent constructible right ideals satisfies Clifford's condition. On the other hand, every Dedekind domain is noetherian (see [**12**, Theorem 6.10]), but not every GCD domain is noetherian. So the integral domain R does not have to be a Dedekind domain for $J(R\rtimes R^\times)$ to be independent. Examples of non-noetherian GCD domains can be found in [**4**].

3. C^*-theory

3.1. The C^*-algebras of an inverse semigroup

Let P be an inverse semigroup. We recall some common constructions for C^*-algebras that are generated by representations of P by partial isometries. This is a short account of the theory. A more thorough account can be found, for instance, in [**27**] or [**8**]. One may construct such C^*-algebras by associating them with certain groupoids, but we do not use this approach in the present paper.

Let $\{\delta_p\}_{p\in P}$ be the canonical basis of $\ell^2(P)$ such that $\delta_p(q) = 1$ if $p = q$, and $\delta_p(q) = 0$ otherwise. Let $\mathbb{C}P$ be the vector space consisting of the formal sums

$$\sum_{i=1}^{n} a_i p_i$$

for any $n \in \mathbb{N}$, $a_i \in \mathbb{C}$ and $p_i \in P$. Define an involution on $\mathbb{C}P$ by

$$\left(\sum_{i=1}^{n} a_i p_i\right)^* = \sum_{i=1}^{n} \overline{a_i} p_i^*$$

and a product by

$$\left(\sum_{i=1}^{n} a_i p_i\right)\left(\sum_{j=1}^{m} b_j q_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j p_i q_j.$$

With these operations, $\mathbb{C}P$ is a $*$-algebra. The left regular representation of $\mathbb{C}P$ is defined to be the map $\Lambda\colon \mathbb{C}P \to B(\ell^2(P))$ given by

$$\Lambda(p)\delta_q = \begin{cases} \delta_{pq} & \text{if } p^*pq = q, \\ 0 & \text{otherwise.} \end{cases}$$

Then, Λ can be shown to be a faithful $*$-representation of $\mathbb{C}P$. Define $C_r^*(P)$ to be the closure of the image of Λ with respect to the operator norm.

One way to construct the full C^*-algebra of P is to show that $\mathbb{C}P$ is dense in the convolution algebra $\ell^1(P)$. One then lets $C^*(P)$ be the certain C^*-completion of the Banach $*$-algebra $\ell^1(P)$. The left regular representation Λ extends to a $*$-homomorphism $\Lambda\colon C^*(P) \to C_r^*(P)$.

$C^*(P)$ is universal for representations of P by partial isometries. If A is a C^*-algebra, $\mathrm{Piso}(A)$ is the set of partial isometries in A and $f\colon P \to \mathrm{Piso}(A)$ is a homomorphism onto a subsemigroup of $\mathrm{Piso}(A)$, then there exists a $*$-homomorphism $\pi\colon C^*(P) \to A$ such that $\pi(p) = f(p)$ for each $p \in P$. This implies that if P, Q are two inverse semigroups, then every homomorphism $f\colon P \to Q$ extends to a $*$-homomorphism $\pi_f\colon C^*(P) \to C^*(Q)$.

Note that if P has a 0, then $\Lambda(0)\delta_0 = \delta_0$ and $\Lambda(0)\delta_p = 0$ for $p \neq 0$. So $\Lambda(0) \neq 0$ is a one-dimensional projection. This is undesirable in some of our later applications, but it is not too difficult to work around the problem. If 0_P exists, then $\mathbb{C}0_P$ is an ideal in $\mathbb{C}P$, so it is an ideal in $C^*(P)$, and $\Lambda(\mathbb{C}0_P)$ is an ideal in $C_r^*(P)$. Let $C_0^*(P) = C^*(P)/\mathbb{C}0_P$ and let $C_{r,0}^*(P) = C_r^*(P)/\Lambda(\mathbb{C}0_P)$.

Since Λ sends $a \in C^*(P)$ to $\Lambda(\mathbb{C}0_P)$ if and only if $a \in \mathbb{C}0_P$, Λ defines a $*$-homomorphism $\Lambda_0\colon C_0^*(P) \to C_{r,0}^*(P)$. Moreover, if P and Q are inverse semigroups and $f\colon P^0 \to Q^0$ is a 0-homomorphism, then π_f pushes down to $\pi_{f,0}\colon C_0^*(P) \to C_0^*(Q)$. It is important to note that if P is an inverse semigroup without 0, then $C_0^*(P^0) \simeq C^*(P)$ and $C_{r,0}^*(P^0) \simeq C_r^*(P)$.

Definition 3.1. The inverse semigroup P is said to have *weak containment* if $\Lambda\colon C^*(P) \to C_r^*(P)$ is an isomorphism. Clearly, Λ is an isomorphism if and only if Λ_0 is an isomorphism. See [**19**] for a recent study of weak containment for inverse semigroups.

Proposition 3.2. *Let P be a commutative inverse semigroup. Then P has weak containment.*

Proof. This follows from, for example, Paterson's results in [**25**], since every commutative inverse semigroup P is a so-called Clifford semigroup and any subgroup of P has to be amenable. □

Corollary 3.3. *Let P be an inverse semigroup. Let D be the C^*-subalgebra of $C_r^*(P)$ generated by $\Lambda(L(P))$. Then, D is canonically isomorphic to $C_r^*(L(P))$ and $C^*(L(P))$. A similar result holds if we look at the subalgebra generated by $\Lambda_0(L(P))$ in $C_{r,0}^*(P)$.*

Proof. Since $L(P)$ is commutative it has weak containment, so $C_r^*(L(P)) \simeq C^*(L(P))$ is universal for representations of $L(P)$. Thus, the norm that $\mathbb{C}L(P)$ obtains from its representation on $\ell^2(L(P))$ is greater than or equal to that it obtains from $\ell^2(P)$. But we also have that, for $a \in \mathbb{C}L(P)$,

$$\begin{aligned}\|\Lambda(a)\|_{C_r^*(P)} &= \sup\{\|\Lambda(a)\xi\|\colon \xi \in \ell^2(P)\}\\ &\geqslant \sup\{\|\Lambda(a)\xi\|\colon \xi \in \ell^2(L(P))\}\\ &= \|\Lambda(a)\|_{C_r^*(L(P))}.\end{aligned}$$

□

Let L be a semilattice with 0 and let S be a set such that there exists an injective 0-homomorphism $f\colon L \to 2^S$. Here, $2^S = \{X\colon X \subset S\}$ is given the structure of a semilattice by saying that the semigroup product is given by set intersection. This gives a representation $\mu\colon L \to \ell^\infty(S)$ by $\mu(a) = \chi_{f(a)}$, where χ_X is the characteristic function of $X \subset S$. Let $C^*(L, f)$ be the C^*-algebra generated by the image of μ. By the universality of $C_0^*(L)$, for 0-representations of L by commuting projections, there exists a $*$-homomorphism $\pi\colon C_0^*(L) \to C^*(L, f)$ such that $\pi(a) = \mu(a)$ for each $a \in L$. We say that f is a *maximal* representation of L if π is an isomorphism.

Before we investigate this we discuss filters, which is a concept that is important in the representation theory of semilattices in the same way that characters are important in the representation theory of abelian C^*-algebras. See, for instance, [**11**, **16**, **27**] for examples.

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Definition 3.4. Let L be a semilattice with 0. A *filter* on L is a 0-homomorphism $\phi\colon L \to \{0,1\}$. Here, $\{0,1\}$ is given the structure of a semilattice with $1 \cdot 0 = 0$. An alternative view of filters on L is to define them to be subsets $\phi \subset L$ such that, for all $a, b \in L$, the following hold:

(i) if $a \in \phi$, then $a \preceq b$ implies that $b \in \phi$;

(ii) if $a, b \in \phi$, then $ab \in \phi$;

(iii) $0 \notin \phi$ and $1 \in \phi$.

Through the correspondence $a \in \phi \Leftrightarrow \phi(a) = 1$, one sees that these are equivalent definitions. The latter picture is the more traditional one.

Each character ψ on $C^*(L, f)$ defines a filter ϕ on L by $\phi(a) = \psi(\mu(a))$ for each $a \in L$. Since $\mu(a)$ is always an idempotent, we have that $\psi(\mu(a)) \in \{0,1\}$, so ϕ is well defined. Moreover, since $\mu(L)$ generates $C^*(L, f)$, two characters on $C^*(L, f)$ are equal if and only if their associated filters are equal. So the characters on $C^*(L, f)$ are completely determined by their associated filters. In general, not every filter on L will extend to a character on $C^*(L, f)$.

Proposition 3.5. *Let the set-up be as above. The following conditions are equivalent:*

- *$f\colon L \to 2^S$ is a maximal representation, i.e. $\pi\colon C_0^*(L) \to C^*(L, f)$ is an isomorphism;*
- *for every filter $\phi\colon L \to \{0,1\}$, there exists a character ψ on $C^*(L, f)$ such that $\psi(\mu(a)) = \phi(a)$ for each $a \in L$;*
- *for all $a_1, \dots, a_n \in L$, if there exists some $b \in L$ such that $\bigcup_{i=1}^n f(a_i) = f(b)$, then $a_i = b$ for at least one $1 \leqslant i \leqslant n$.*

Proof. (i) $\Rightarrow$ (ii). Let $\phi\colon L \to \{0,1\}$ be a filter. If $C^*(L, f)$ is isomorphic to $C_0^*(L)$, then by the universal properties of this C^*-algebra there exists a non-zero $*$-homomorphism $C^*(L, f) \to C_0^*(\{0,1\}) \simeq \mathbb{C}$ extending ϕ. A non-zero $*$-homomorphism to $\mathbb{C}$ is the definition of a character.

(ii) $\Rightarrow$ (iii). Let $a_1, \dots, a_n, b \in L$ be such that $\bigcup_{i=1}^n f(a_i) = f(b)$. Note that $\mu(b) \leqslant \sum_{i=1}^n \mu(a_i)$. Let $\phi = \{c \in L\colon b \preceq c\}$. This is a filter on L. Let ψ be the extending character. Then

$$1 = \psi(\mu(b)) \leqslant \sum_{i=1}^n \psi(\mu(a_i)).$$

Then $\psi(\mu(a_i)) = \phi(a_i) = 1$ for at least one i. We have that $f(a_i b) = f(a_i) \cap f(b) = f(a_i)$, so $a_i b = a_i$, that is, $a_i \preceq b$. If $\phi(a_i) = 1$, then $b \preceq a_i$ by the definition of ϕ, so $a_i = b$.

(iii) $\Rightarrow$ (i). This can be proved just like (i) $\Rightarrow$ (ii) in [**17**, Proposition 2.24], so we skip the proof. □

Corollary 3.6. *Let S be a left cancellative semigroup. $J(S)$ is then independent (see Definition 2.18) if and only if the inclusion $\iota\colon J(S)\cup\{\emptyset\}\to 2^S$ is a maximal representation.*

Proof. This follows from (iii) above and the definition of the independence of $J(S)$. □

We will need the following proposition later.

Proposition 3.7. *Let P be an E^*-unitary inverse semigroup, and let L be its subsemilattice of idempotents. There exists a faithful conditional expectation $E_{r,0}\colon C^*_{r,0}(P)\to C^*_0(L)$ such that $E_{r,0}(\Lambda_0(p))=p$ if $p\in L$, and $E_{r,0}(\Lambda_0(p))=0$ otherwise.*

Proof. Let $E_r\colon B(\ell^2(P))\to\ell^\infty(P)$ be the usual faithful conditional expectation given by

$$\langle E_r(a)\delta_q,\delta_q\rangle=\langle a\delta_q,\delta_q\rangle.$$

Here $\ell^\infty(P)$ is viewed as a subalgebra of $B(\ell^2(P))$ represented by pointwise multiplication. First, if $p\in L$, then $\langle\Lambda(p)\delta_q,\delta_r\rangle\neq 0$ if and only if $p^*pq=pq=q$ and $pq=r$, which implies that $r=q$. So $\Lambda(p)\in\ell^\infty(P)$, and $E_r(\Lambda(p))=\Lambda(p)$.

In general, let $p\in P$ and suppose that $\langle\Lambda(p)\delta_q,\delta_q\rangle\neq 0$ for some $q\in P\setminus\{0\}$. Then $pq=q$, so, since P is E^*-unitary, $p\in L$.

Due to Corollary 3.3, we now identify $C^*(L)$ with the closure of $\mathbb{C}L$ inside $C^*_r(P)$. We have $E_r\colon C^*_r(P)\to C^*(L)$, and, since $E_r(\Lambda(0))=\Lambda(0)$, $E_{r,0}\colon C^*_{r,0}(P)\to C^*_0(L)$ can be defined with the desired properties.

For any $a\in C^*_r(P)$, let $[a]$ denote its image in $C^*_{r,0}(P)$. We have that $E_{r,0}([a^*a])=0$ if and only if $E_r(a^*a)=\alpha\Lambda(0_P)$ for some $\alpha\in\mathbb{C}$ if and only if $\langle a^*a\delta_{0_P},\delta_{0_P}\rangle=\|a\delta_{0_P}\|^2=\alpha$ and $\|a\delta_q\|^2=0$ for all $q\in P\setminus\{0_P\}$. This implies that $a^*a=\alpha\Lambda(0_P)$ and that $[a^*a]=0$, so $E_{r,0}$ is faithful. □

On the other hand, we have the following lemma, which is also interesting.

Lemma 3.8. *Let A be a C^*-algebra generated by an inverse semigroup P of partial isometries. Let L be the semilattice of idempotents in P, and let D be the subalgebra of A generated by L. If there exists a conditional expectation $E\colon A\to D$ such that $E(p)=p$ if $p\in L$ and $E(p)=0$ otherwise, then P is E^*-unitary.*

Proof. Suppose that $p\in P$ and $q\in L\setminus\{0\}$ satisfy $pq=q$. Then, $pq=q=E(q)=E(pq)=E(p)q$ since E is a conditional expectation. This implies that $E(p)\neq 0$, so $p\in L$. This shows that P is E^*-unitary. □

3.2. The left regular representation and the left inverse hull of a left cancellative semigroup

From now on, S will always be a left cancellative semigroup unless otherwise stated. Let $\{\varepsilon_s\}_{s\in S}$ be the orthogonal basis of $\ell^2(S)$, where $\varepsilon_s(t)=1$ if $s=t$, and 0 otherwise. The left regular representation of S is the semigroup homomorphism $s\mapsto V_s$, where $V_s\colon\ell^2(S)\to\ell^2(S)$ is given by $V_s\varepsilon_t=\varepsilon_{st}$.

Now, $\langle V_s^* \varepsilon_t, \varepsilon_r \rangle = 1$ if $t = sr$ and 0 otherwise, so

$$V_s^* \varepsilon_t = \sum_{r \in t^{-1}(\{s\})} \varepsilon_r.$$

Since S is left cancellative, $t^{-1}(\{s\})$ is either a singleton or empty. It follows readily that V_s is an isometry for each $s \in S$.

Let $E\colon B(\ell^2(S)) \to \ell^\infty(S)$ be the conditional expectation given by $\langle E(a)\varepsilon_s, \varepsilon_s \rangle = \langle a\varepsilon_s, \varepsilon_s \rangle$ for each $s \in S$. Here we view $\ell^\infty(S)$ as a subalgebra of $B(\ell^2(S))$ represented by pointwise multiplication. For a subset $X \subset S$, let $\chi_X \in \ell^\infty(S)$ be the associated characteristic function. It is easy to check that, for all $s \in S$ and $X \subset S$,

$$V_s \chi_X V_s^* = \chi_{sX}, \qquad V_s^* \chi_X V_s = \chi_{s^{-1}(X)}. \tag{3.1}$$

In particular, $V_s V_s^* = \chi_{sS}$.

We let $C_r^*(S)$ be the C^*-algebra generated by $\{V_s\colon s \in S\}$, and let $D_r(S)$ be the commutative C^*-algebra generated by $\{\chi_X\colon X \in J(S)\}$. Note that $C_r^*(S)$ is the closed linear span of the set

$$V(S) = \{V_{t_1}^* V_{s_1} \cdots V_{t_n}^* V_{s_n} \colon n \in \mathbb{N},\ s_1, \dots, s_n, t_1, \dots, t_n \in S\}.$$

$V(S)$ is itself a semigroup under composition of operators, and it is an inverse semigroup since it consists of partial isometries with commuting initial and final projections. To see this, note that if $V = V_{t_1}^* V_{s_1} \cdots V_{t_n}^* V_{s_n}$, then by repeatedly applying the relations in (3.1) we get that $VV^* = \chi_X$, with $X = t_1^{-1}s_1 \cdots t_n^{-1}s_n S$. Similarly, $V^*V = \chi_Y$, with $Y = s_n^{-1}t_n \cdots s_1^{-1}t_1 S$. The second assertion of the next lemma is also proved in [**17**].

Lemma 3.9. *The semilattice of idempotents in $V(S)$ is isomorphic to the $\cap$-semilattice*

$$J := \{t_1^{-1}s_1 \cdots t_n^{-1}s_n S\colon n \in \mathbb{N},\ s_i, t_i \in S\}.$$

Moreover, $J = J(S)$.

Proof. We saw in the previous paragraph that the semilattice of idempotents in $V(S)$ is $\{\chi_X\colon X \in J\}$. For any $X, Y \in J$, $\chi_X \chi_Y = \chi_{X \cap Y}$, so J has to be a $\cap$-semilattice and be isomorphic to $\{\chi_X\colon X \in J\}$. Since J is therefore closed under $\cap$, it must be equal to $J(S)$ as defined in § 2.3. □

The inverse semigroup $V(S)$ plays an important role in the following. As we noted in § 1, it can be given a purely algebraic description. Let $I(S)$ be the inverse semigroup of all bijective partially defined functions $S \to S$. Define $I_l(S)$ to be the inverse subsemigroup of $I(S)$ generated by the partial bijections $\{\lambda_s\}_{s \in S}$, where $\lambda_s\colon S \to sS$ is given by $\lambda_s(t) = st$.

Lemma 3.10. *There exists a faithful representation $\omega\colon I_l(S) \to B(\ell^2(S))$ such that $\omega(\lambda_s) = V_s$ for each $s \in S$. So ω is an isomorphism of $I_l(S)$ onto $V(S)$.*

Proof. For any $f \in I_l(S)$, define $\omega(f)\colon \ell^2(S) \to \ell^2(S)$ by

$$\omega(f)\varepsilon_s = \begin{cases} \varepsilon_{f(s)} & \text{if } s \in \operatorname{dom}(f), \\ 0 & \text{otherwise,} \end{cases}$$

where $\operatorname{dom}(f)$ is the domain of f. Then, $\omega(f) \in B(\ell^2(S))$ since f is injective, and, for any $s \in S$, $\omega(\lambda_s) = V_s$. For any $f, g \in I_l(S)$ we have that $f = g$ if and only if $\operatorname{dom}(f) = \operatorname{dom}(g)$ and $f(s) = g(s)$ for all $s \in \operatorname{dom}(f)$. This happens if and only if $\ker \omega(f) = \ker \omega(g)$ and $\omega(f)\varepsilon_s = \omega(g)\varepsilon_s$ for all $s \in S$. This is equivalent to $\omega(f) = \omega(g)$. So ω is an injective map. Now, for any $f, g \in I_l(S)$ and $s \in S$,

$$\omega(f)\omega(g)\varepsilon_s = \begin{cases} \varepsilon_{f(g(s))} & \text{if } s \in g^{-1}(\operatorname{dom}(f)), \\ 0 & \text{otherwise,} \end{cases}$$

so $\omega(f \cdot g) = \omega(f)\omega(g)$. This shows that ω is a surjective homomorphism of $I_l(S)$ onto $V(S)$, and thus it is an isomorphism. □

$I_l(S)$ is often called the *left inverse hull* of S and has been previously studied in several settings. Some recent information on it can be found in [**13**–**15**, **18**]. Using ω, one can translate most statements about $V(S)$ into statements about $I_l(S)$ and vice versa. Note that the semilattice of idempotents in $I_l(S)$ is $\{i_X \colon X \in J(S)\} \simeq J(S)$, where $i_X \colon X \to X$ is the identity map on X. We occasionally identify $J(S)$ with $\{i_X \colon X \in J(S)\}$ in what follows.

Many of the ideas of this subsection are present in [**17**], but they are not expressed in terms of $V(S)$ as an inverse semigroup. Using a simple induction argument (or deducing it from the proof of Lemma 3.10) we know that, for any $V \in V(S)$ and $s \in S$, $V\varepsilon_s$ is either 0 or ε_t for some $t \in S$.

Lemma 3.11. *Let $V \in V(S)$. Then, $E(V) = V$ if and only if V is idempotent.*

Proof. By Lemma 3.9, V is idempotent if and only if $V = E_X$ for some $X \in J(S)$. This implies that $E(V) = V$. Let $V \in V(S)$, and suppose that $E(V) = V$. For every $s \in S$, $\langle V\varepsilon_s, \varepsilon_s\rangle$ is either 0 or 1, so $V = E(V) = \chi_X$, where $X = \{s \in S \colon \langle V\varepsilon_s, \varepsilon_s\rangle = 1\}$. Hence, $V = V^2$. □

Corollary 3.12. *Let $a \in C_r^*(S)$. Then, $E(a) = a$ if and only if $a \in D_r(S)$.*

Proof. $V(S)$ spans a dense subset of $C_r^*(S)$, and $\{E_X \colon X \in J(S)\}$ spans a dense subset of $D_r(S)$. The result follows by the linearity and continuity of E. □

Lemma 3.13. *Let ρ be the right action of S on itself given by $\rho_r(s) = sr$ for $s, r \in S$. Then, for every $f \in I_l(S)$, $s \in \operatorname{dom}(f)$ and $r \in S$,*

$$\rho_r(f(s)) = f(\rho_r(s)). \tag{3.2}$$

Proof. Note that since $\mathrm{dom}(f) = \mathrm{dom}(f^*f) \in J(S)$ is a right ideal in S, $s \in \mathrm{dom}(f)$ implies that $sr \in \mathrm{dom}(f)$ for all $r \in S$. Equation (3.2) clearly holds when $f = \lambda_t$ for some $t \in S$. Suppose now that $f = \lambda_t^*$. Let $s \in \mathrm{dom}(\lambda_t^*) = tS$. There then exists some $q \in S$ such that $tq = s$ and $\lambda_t^*(s) = q$. For any $p \in S$, $\lambda_t^*(\rho_r(s)) = p$ if and only if $sr = \rho_r(s) = tp$. On the other hand, $\rho_r(\lambda_t^*(s)) = qr = p$ if and only if $tqr = tp$. Since $s = tq$, this happens if and only if $sr = tp$. So, for any $p \in S$, $\rho_r(\lambda_t^*(s)) = p$ if and only if $\lambda_t^*(\rho_r(s)) = p$. This shows that $\rho_r(\lambda_t^*(s)) = \lambda_t^*(\rho_r(s))$.

Let $f \in I_l(S)$ be arbitrary. There then exist $n \in \mathbb{N}$ and $s_1, \dots, s_n, t_1, \dots, t_n \in S$ such that $f = \lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}$. Since $s \in \mathrm{dom}(f)$,

$$\lambda_{t_j}^* \lambda_{s_j} \cdots \lambda_{t_n}^* \lambda_{s_n}(s) \in \mathrm{dom}(\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_{j-1}}^* \lambda_{s_{j-1}})$$

for all $1 \leqslant j \leqslant n$. So,

$$\rho_r(\lambda_{t_j}^* \lambda_{s_j} \cdots \lambda_{t_n}^* \lambda_{s_n}(s)) \in \mathrm{dom}(\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_{j-1}}^* \lambda_{s_{j-1}})$$

for all $1 \leqslant j \leqslant n$, since $\mathrm{dom}(\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_{j-1}}^* \lambda_{s_{j-1}})$ is a right ideal in S. Starting with

$$\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}(\rho_r(s)),$$

we can then move ρ_r to the left one step at a time until we get

$$\rho_r(\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}(s)).$$

□

Corollary 3.14. *Let $f \in I_l(S)$. For any $s \in \mathrm{dom}(f)$, $f\lambda_s = \lambda_{f(s)}$.*

Proof. Note first that $\mathrm{dom}\, f\lambda_s = \mathrm{dom}\, \lambda_{f(s)} = S$. Lemma 3.13 implies that, for any $r \in S$,

$$f\lambda_s(r) = f(sr) = f(\rho_r(s)) = \rho_r(f(s)) = f(s)r = \lambda_{f(s)}(r).$$

So $f\lambda_s(r) = \lambda_{f(s)}(r)$ for all $r \in S$; hence, $f\lambda_s = \lambda_{f(s)}$. □

Lemma 3.15. *$V(S)$ is E^*-unitary if and only if for every $V \in V(S)$ we have that $E(V) = V$ or $E(V) = 0$.*

Proof. Assume that $V(S)$ is E^*-unitary and let $V \in V(S)$. We show that if $V\varepsilon_r = \varepsilon_r$ for some $r \in S$, then $E(V) = V$. Otherwise $E(V)$ is of course 0. Suppose that $V\varepsilon_r = \varepsilon_r$. Using ω we can get from Corollary 3.14 that $VV_r = V_r$. Since $V(S)$ is E^*-unitary this implies that V is idempotent, so $E(V) = V$ by Lemma 3.11. The converse statement follows from Lemma 3.8. □

Note that this implies that if $V(S)$ is E^*-unitary, then $E(C_r^*(S)) = D_r(S)$. Of course, $V(S)$ is E^*-unitary if and only if $I_l(S)$ is E^*-unitary.

Lemma 3.16. *Let $s, t \in S$. The following conditions are equivalent:*

(i) $s = t$;

(ii) $\lambda_s \lambda_t^*$ *is idempotent;*

(iii) $\lambda_t^* \lambda_s = 1$.

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Proof. (i) ⇒ (ii). This is trivial.

(ii) ⇒ (iii). We have that $\lambda_s\lambda_t^*\lambda_s\lambda_t^* = \lambda_s\lambda_t^*$. Left multiplying with λ_s^* and right multiplying with λ_t gives the desired equality.

(iii) ⇒ (i). This implies that $\lambda_s\lambda_t^*\lambda_s = \lambda_s$ and $\lambda_t^*\lambda_s\lambda_t^* = \lambda_t^*$. Since these relations are unique for λ_s^*, we get that $\lambda_t^* = \lambda_s^*$. So $\lambda_s = \lambda_t$ and $s = \lambda_s(1) = \lambda_t(1) = t$. □

Corollary 3.17. *If $I_l(S)$ is E^*-unitary, S is cancellative.*

Proof. Let $s, t, r, p \in S$ and assume that $sr = tr = p$. Since $\lambda_s\lambda_r = \lambda_p$, $\lambda_s^*\lambda_p = \lambda_r$. So $\lambda_t\lambda_s^*\lambda_p = \lambda_t\lambda_r = \lambda_p$. Since $I_l(S)$ is E^*-unitary, $\lambda_t\lambda_s^*$ is idempotent, so by the previous lemma $s = t$. Hence, S is (right) cancellative. □

Lemma 3.18. *Assume that S is a subsemigroup of a group G. Let $m, n \in \mathbb{N}$ and let $s_i, t_i, p_j, q_j \in S$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$. Set $f = \lambda_{t_1}^*\lambda_{s_1}\cdots\lambda_{t_n}^*\lambda_{s_n}$ and $f' = \lambda_{q_1}^*\lambda_{p_1}\cdots\lambda_{q_m}^*\lambda_{p_m}$, and assume that $f, f' \neq 0$. If $f = f'$, then the equality*

$$t_1^{-1}s_1\cdots t_n^{-1}s_n = q_1^{-1}p_1\cdots q_m^{-1}p_m \tag{3.3}$$

holds in G, where $(\cdot)^{-1}$ means taking inverses in G.

Proof. Since $f, f' \neq 0$ we can pick some $r \in \operatorname{dom}(f)$. The equality $f(r) = f'(r)$ then gives that

$$t_1^{-1}s_1\cdots t_n^{-1}s_n r = q_1^{-1}p_1\cdots q_m^{-1}p_m r,$$

where $(\cdot)^{-1}$ denotes the preimage by left multiplication in S. However, this implies that the same relation holds in G, where $(\cdot)^{-1}$ now stands for the inverse operation in G. Cancelling with r, we get (3.3). □

The proof of the next proposition uses techniques similar to those employed by Jiang in [**13**].

Proposition 3.19. *Let S be a left cancellative semigroup. Then, S embeds into a group if and only if $I_l(S)$ is strongly E^*-unitary.*

Proof. Suppose first that S embeds into a group G. We omit the embedding homomorphism, and instead view S as a subsemigroup of G. Define a grading $\varphi\colon I_l(S)^0 \to G^0$ by

$$\varphi(0) = 0,$$
$$\varphi(\lambda_{t_1}^*\lambda_{s_1}\cdots\lambda_{t_n}^*\lambda_{s_n}) = t_1^{-1}s_1\cdots t_n^{-1}s_n \quad \text{when } \lambda_{t_1}^*\lambda_{s_1}\cdots\lambda_{t_n}^*\lambda_{s_n} \neq 0.$$

This is well defined because of Lemma 3.18. Suppose that $\varphi(f) = 1$ for some $f \in I_l(S)$. Then, if $f = \lambda_{t^1}^*\lambda_{s_1}\cdots\lambda_{t_n}^*\lambda_{s_n}$, $t_1^{-1}s_1\cdots t_n^{-1}s_n = 1$. So $f(r) = t_1^{-1}s_1\cdots t_n^{-1}s_n r = r$ for all $r \in \operatorname{dom}(f)$. Hence, f is idempotent, and φ is idempotent pure. This shows that $I_l(S)$ is strongly E^*-unitary.

Now suppose that $I_l(S)$ is strongly E^*-unitary by some idempotent pure grading $\varphi\colon I_l(S)^0 \to G^0$. For any $t \in S$, $1 = \varphi(\lambda_t^*\lambda_t) = \varphi(\lambda_t^*)\varphi(\lambda_t)$, so $\varphi(\lambda_t)^{-1} = \varphi(\lambda_t^*)$.

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For any $s, t \in S$, if $\varphi(\lambda_s) = \varphi(\lambda_t)$, then $\varphi(\lambda_s\lambda_t^*) = 1$, so $\lambda_s\lambda_t^*$ is idempotent and, by Lemma 3.16, $s = t$. This implies that the homomorphism $S \to G$ given by $s \mapsto \varphi(\lambda_s)$ is injective. □

We want to find a relation between $C^*_{r,0}(I_l(S))$ and $C^*_r(S)$.

Lemma 3.20. *Let $T\colon \ell^2(S) \to \ell^2(I_l(S))$ be the isometry defined by*

$$T\varepsilon_s = \delta_{\lambda_s}, \quad s \in S.$$

Let $\omega\colon I_l(S) \to B(\ell^2(S))$ be the map defined in Lemma 3.10. Then $T^*\Lambda(f)T = \omega(f)$ for all $f \in I_l(S)$.

Proof. Let $f \in I_l(S)$ and let $s \in \mathrm{dom}(f)$. Then $s \in \mathrm{dom}(f^*f)$, so $f^*f(s) = s$. By Corollary 3.14, $f^*f\lambda_s = \lambda_s$ and $f\lambda_s = \lambda_{f(s)}$. Now, by the definition of Λ,

$$T^*\Lambda(f)T\varepsilon_s = T^*\Lambda(f)\delta_{\lambda_s} = T^*\delta_{f\lambda_s} = T^*\delta_{\lambda_{f(s)}} = \varepsilon_{f(s)} = \omega(f)\varepsilon_s.$$

On the other hand, if $s \notin \mathrm{dom}(f)$, then $s \notin \mathrm{dom}(f^*f)$. Thus, $f^*f\lambda_s \neq \lambda_s$ and we get that $T^*\Lambda(f)T\varepsilon_s = T^*\Lambda(f)\delta_{\lambda_s} = 0$. So $T^*\Lambda(f)T\varepsilon_s = \omega(f)\varepsilon_s$ for any $s \in S$. This shows that $T^*\Lambda(f)T = \omega(f)$ for any $f \in I_l(S)$. □

Corollary 3.21. *There exists a surjective $*$-homomorphism $h\colon C^*_{r,0}(I_l(S)) \to C^*_r(S)$ such that $h(\Lambda_0(f)) = \omega(f)$ for all $f \in I_l(S)$.*

Proof. Define $h'\colon C^*_r(I_l(S)) \to C^*_r(S)$ by

$$h'(a) = T^*aT.$$

Then, h' is a $*$-homomorphism on the span of $\Lambda(I_l(S))$. Since this span is dense in $C^*_r(I_l(S))$ and since h' is continuous, it has to be a $*$-homomorphism on all of $C^*_r(I_l(S))$. Since h' sends $\Lambda(0)$ to 0 whenever $0 \in I_l(S)$, it descends to a $*$-homomorphism $h\colon C^*_{r,0}(I_l(S)) \to C^*_r(S)$ with the desired properties. □

Theorem 3.22. *Suppose that $I_l(S)$ is E^*-unitary. The map*

$$h\colon C^*_{r,0}(I_l(S)) \to C^*_r(S)$$

is then an isomorphism if and only if $J(S)$ is independent.

Proof. Recall that $D_r(S)$ is the diagonal subalgebra of $C^*_r(S)$ generated by $J(S)$. Then $D_r(S)$ is $C^*(J(S)^0, \iota)$ as described in Proposition 3.5 and the paragraphs before it. Here $\iota\colon J(S)^0 \to 2^S$ is the inclusion map.

The restriction of h to $C^*_0(J(S))$ (which we can identify with the subalgebra generated by the image of $J(S)$ in $C^*_{r,0}(I_l(S))$ by Corollary 3.3) maps onto $D_r(S)$, and this restriction must necessarily be equal to the map π as described in and before Proposition 3.5. According to this proposition and Corollary 3.6, $\pi = h|_{C^*_0(J(S))}$ is an isomorphism if and only if $J(S)$ is independent. This means that, if $J(S)$ is not independent, h is not injective.

Suppose that $J(S)$ is independent and $I_l(S)$ is E^*-unitary. By Proposition 3.7 there exists a faithful conditional expectation $E_{r,0}\colon C^*_{r,0}(I_l(S)) \to C^*_0(J(S))$. As a consequence of Lemma 3.15, $E(C^*_r(S)) = D_r(S)$. Moreover, for any $V \in V(S)$, $E(V) = V$ if and only if V is idempotent, and $E(V) = 0$ otherwise. Using the properties of $E_{r,0}$ given in Proposition 3.7, it follows that $E \circ h = h \circ E_{r,0}$.

Now, assume that $h(a) = 0$ for some $a \in C^*_{r,0}(I_l(S))$. Then $h(a^*a) = 0$, so $E(h(a^*a)) = 0 = h(E_{r,0}(a^*a))$. Since h is an isomorphism on the image of $E_{r,0}$, $E_{r,0}(a^*a) = 0$, so $a^*a = a = 0$ since $E_{r,0}$ is faithful. This shows that h is an isomorphism. □

We can use the equality $D_r(S) = C^*(J(S)^0, \iota)$ to describe the characters on $D_r(S)$. Proposition 3.5 and Corollary 3.6 imply that when $J(S)$ is independent, the characters on $C^*(J(S)^0, \iota)$ are uniquely determined by the filters on $J(S)^0$. When S is algebraically ordered and satisfies Clifford's condition, Proposition 2.19 tells us that $(S^0, \wedge) \simeq J(S)^0$. So, in this case, the characters on $D_r(S)$ correspond to the filters on $(S^0, \wedge)$. It is not difficult to see that the filters on $(S^0, \wedge)$ are exactly what Nica, in [**22**, § 6.2], called non-void hereditary directed subsets of S. This is sometimes called the Nica spectrum of S. In general, the set of characters on $D_r(S)$ corresponds to some subset of the set of filters on $J(S)^0$, but it is not always obvious what this subset is.

Performing computations in $I_l(S)$ can be difficult, but if S satisfies Clifford's condition, it becomes easier. Note that S satisfies Clifford's condition exactly when $I_l(S)$ is 0-bisimple. We will, however, not use this fact explicitly in this paper. See, for example, [**5**, **14**] or [**13**] for more information on this.

Proposition 3.23. *The following conditions are equivalent:*

(i) *S satisfies Clifford's condition;*

(ii) *for all $s, t \in S$ such that $\lambda_t^*\lambda_s \neq 0$ there exist $p, q \in S$ such that $\lambda_t^*\lambda_s = \lambda_p\lambda_q^*$;*

(iii) *$I_l(S) \setminus \{0\} = \{\lambda_p\lambda_q^*\colon p, q \in S\}$.*

Proof. (i) ⇒ (ii). Let $s, t \in S$. If $\lambda_t^*\lambda_s \neq 0$, then $sS \cap tS \neq \emptyset$, so $sS \cap tS = rS$ for some $r \in S$. Since $r \in sS \cap tS$, $p := t^{-1}(r)$ and $q := s^{-1}(r)$ exist, and $\lambda_t\lambda_p = \lambda_r$, so we get that $\lambda_t^*\lambda_r = \lambda_t^*\lambda_t\lambda_p = \lambda_p$. Similarly, $\lambda_s^*\lambda_r = \lambda_q$. By the definition of r we have that

$$\lambda_t\lambda_t^*\lambda_s\lambda_s^* = i_{tS}i_{sS} = i_{rS} = \lambda_r\lambda_r^*.$$

So, multiplying from the left by λ_t^* and from the right by λ_s, we get

$$\lambda_t^*\lambda_s = (\lambda_t^*\lambda_r)(\lambda_r^*\lambda_s) = \lambda_p\lambda_q^*.$$

(ii) ⇒ (i). Let $s, t \in S$ such that $sS \cap tS \neq \emptyset$. Then $\lambda_t^*\lambda_s \neq 0$, so there exist $p, q \in S$ with $\lambda_t^*\lambda_s = \lambda_p\lambda_q^*$. This means that $\lambda_t\lambda_t^*\lambda_s\lambda_s^* = \lambda_{tp}\lambda_{sq}^*$. This element is idempotent, so $tp = sq$ by Lemma 3.16. Write $r = tp = sq$. This gives that $\lambda_t\lambda_t^*\lambda_s\lambda_s^* = \lambda_r\lambda_r^*$, which is equivalent to $sS \cap tS = rS$.

(iii) ⇒ (ii). This is trivial.

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It remains to prove (ii) $\Rightarrow$ (iii). Let $f \in I_l(S) \setminus \{0\}$. There then exist $n \in \mathbb{N}$ and $s_1, \dots, s_n, t_1, \dots, t_n \in S$ such that $f = \lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}$. By condition (ii) we have that $\lambda_{t_n}^* \lambda_{s_n} = \lambda_{p_n} \lambda_{q_n}^*$ for some $p_n, q_n \in S$. Assume that for a given $1 \leqslant j \leqslant n$ there exist $p_j, q_j \in S$ such that

$$\lambda_{t_j}^* \lambda_{s_j} \cdots \lambda_{t_n}^* \lambda_{s_n} = \lambda_{p_j} \lambda_{q_j}^*.$$

Then

$$\lambda_{t_{j-1}}^* \lambda_{s_{j-1}} \lambda_{t_j}^* \lambda_{s_j} \cdots \lambda_{t_n}^* \lambda_{s_n} = \lambda_{t_{j-1}}^* \lambda_{s_{j-1}} \lambda_{p_j} \lambda_{q_j}^*.$$

Now, by condition (ii), there exist $p, q \in S$ such that

$$\lambda_{t_{j-1}}^* \lambda_{s_{j-1}} \lambda_{p_j} = \lambda_{t_{j-1}}^* \lambda_{s_{j-1} p_j} = \lambda_p \lambda_q^*.$$

Setting $p_{j-1} = p$ and $q_{j-1} = q q_j$, we get that

$$\lambda_{t_{j-1}}^* \lambda_{s_{j-1}} \lambda_{p_j} \lambda_{q_j}^* = \lambda_{p_{j-1}} \lambda_{q_{j-1}}^*.$$

By induction on j,

$$\lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n} = \lambda_{p_1} \lambda_{q_1}^*.$$

This shows that any $f \in I_l(S) \setminus \{0\}$ is of the form $\lambda_p \lambda_q^*$ for some $p, q \in S$. □

Corollary 3.24. *Suppose that S satisfies Clifford's condition. $I_l(S)$ is then E^*-unitary if and only if S is cancellative.*

Proof. One implication was proved in Corollary 3.17. Suppose that S is cancellative, and that $f i_X = i_X$ for some non-zero $f \in I_l(S)$ and some non-empty $X \in J(S)$. Then, $f \lambda_r = \lambda_r$ for any $r \in X$ since X is a right ideal. Write $f = \lambda_s \lambda_t^*$, with $s, t \in S$. Then $\lambda_t^* \lambda_r = \lambda_s^* \lambda_r$. Let $p \in \operatorname{dom} \lambda_t^* \lambda_r$ and define $q = \lambda_t^* \lambda_r(p) = \lambda_s^* \lambda_r(p)$. Then $tq = sq = rp$. By right cancellativity this gives that $t = s$, so f is idempotent. □

Corollary 3.25. *If S is cancellative and satisfies Clifford's condition, $h \colon C_{r,0}^*(I_l(S)) \to C_r^*(S)$ is an isomorphism.*

Proof. By the previous corollary, $I_l(S)$ is E^*-unitary. By Proposition 2.19, $J(S)$ is independent, so Theorem 3.22 implies that h is an isomorphism. □

Any semigroup that is the positive cone in a quasi-lattice ordered group satisfies these conditions. Note, however, that a semigroup satisfying Clifford's condition is allowed to have non-trivial invertible elements. For instance, $(\mathbb{Z} \times \mathbb{Z}^+, +)$ satisfies Clifford's condition, but it is not algebraically ordered.

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3.3. Li's constructions of full C^*-algebras for a left cancellative semigroup

In [**17**], Li defined the full C^*-algebra $C^*(S)$ of a left cancellative semigroup S. The construction is as follows. $C^*(S)$ is the universal C^*-algebra generated by the isometries $\{v_s\colon s \in S\}$ and projections $\{e_X\colon X \in J(S)^0\}$ such that, for all $s, t \in S$ and $X, Y \in J(S)^0$,

$$v_{st} = v_s v_t, \qquad v_s e_X v_s^* = e_{sX},$$
$$e_S = 1, \qquad e_\emptyset = 0, \qquad e_{X\cap Y} = e_X e_Y.$$

Li also defined a C^*-algebra $C_s^*(S)$ when S embeds into a group G. The C^*-algebra $C_s^*(S)$ is the universal C^*-algebra generated by the isometries $\{v_s\colon s \in S\}$ and projections $\{e_X\colon X \in J(S)^0\}$ such that, for all $s, t \in S$,

$$v_{st} = v_s v_t,$$
$$e_\emptyset = 0,$$

and, whenever $s_1, \dots, s_n, t_1, \dots, t_n \in S$ satisfy $t_1^{-1} s_1 \cdots t_n^{-1} s_n = 1$ in G,

$$v_{t_1}^* v_{s_1} \cdots v_{t_n}^* v_{s_n} = e_{(t_1^{-1} s_1 \cdots t_n^{-1} s_n S)}.$$

Li then showed that $\{v_s\colon s \in S\} \subset C_s^*(S)$ and $\{e_X\colon X \in J(S)^0\} \subset C_s^*(S)$ satisfy the relations defining $C^*(S)$, so there exists a surjective $*$-homomorphism $\pi_s\colon C^*(S) \to C_s^*(S)$ that sends $v_s \in C^*(S)$ to $v_s \in C_s^*(S)$. The universal property of $C_s^*(S)$ also gives a canonical $*$-homomorphism $C_s^*(S) \to C_r^*(S)$ that sends v_s to V_s for all $s \in S$. Moreover, we have the following.

Proposition 3.26. *Suppose that S embeds into a group G. There exists a $*$-isomorphism $\kappa\colon C_s^*(S) \to C_0^*(I_l(S))$ such that $\kappa(v_s) = \lambda_s$ for each $s \in S$.*

Proof. The existence of a surjective $*$-homomorphism $\kappa\colon C_s^*(S) \to C_0^*(I_l(S))$ follows from the universality of $C_s^*(S)$. $C^*(I_l(S))$ is generated by $\{\lambda_s\colon s \in S\}$ and $\{i_X\colon X \in J(S)\}$, and these satisfy the given relations when projected down to $C_0^*(I_l(S))$. In particular, if $s_1, \dots, s_n, t_1, \dots, t_n \in S$ satisfy $t_1^{-1} s_1 \cdots t_n^{-1} s_n = 1$ in G, then, by the proof of Proposition 3.19, $f := \lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}$ is idempotent. So $f = ff^* = i_X$, with $X = t_1^{-1} s_1 \cdots t_n^{-1} s_n S$.

Let $\mathcal{V}'(S)$ be the subset of $C_s^*(S)$ given by

$$\mathcal{V}'(S) = \{v_{t_1}^* v_{s_1} \cdots v_{t_n}^* v_{s_n}\colon n \in \mathbb{N},\ t_1, \dots, t_n, s_1, \dots, s_n \in S\}.$$

Li's relations guarantee that $\mathcal{V}'(S)$ is actually an inverse semigroup. Using [**17**, Lemma 2.8] (and mapping down to $C_s^*(S)$), we get that, for any $v \in \mathcal{V}'(S)$, $v^*v = e_X$ and $vv^* = e_Y$ for some $X, Y \in J(S)$, so v is a partial isometry. Moreover, any $v, w \in \mathcal{V}'(S)$ have commuting initial and final projections.

Comparing the universal properties of $C_s^*(S)$ and $C_0^*(\mathcal{V}'(S))$ yields that these two C^*-algebras are canonically isomorphic. Hence, κ is an isomorphism if and only if its

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restriction to $\mathcal{V}'(S)$ gives a semigroup isomorphism $\mathcal{V}'(S) \to I_l(S)$ (note that the restriction of κ to $\mathcal{V}'(S)$ is automatically a surjective semigroup homomorphism onto $I_l(S)$). Let $J'(S)$ be the semilattice of idempotents in $\mathcal{V}'(S)$. For any $v \in J'(S)$, $v^*v = v$, so $v = e_X$ for some $X \in J(S)$, i.e. $J'(S) = \{e_X : X \in J(S)\}$. Hence, κ restricts to an isomorphism $J'(S) \to J(S)$ since it is injective on this set.

Let $v = v_{t_1}^* v_{s_1} \cdots v_{t_n}^* v_{s_n} \in \mathcal{V}'(S)$. Suppose that $\kappa(v) = \lambda_{t_1}^* \lambda_{s_1} \cdots \lambda_{t_n}^* \lambda_{s_n}$ is idempotent. Then $t_1^{-1} s_1 \cdots t_n^{-1} s_n = 1$ in G. This implies that v is idempotent. Thus, $\kappa^{-1}(J(S)) = J'(S)$, so $\kappa|_{\mathcal{V}'(S)}$ is an isomorphism by Lemma 2.14. □

Proposition 3.27. *Let S be any left cancellative semigroup. There exists a surjective $*$-homomorphism $\eta\colon C^*(S) \to C_0^*(I_l(S))$ such that $\eta(v_s) = \lambda_s$ for each $s \in S$. If S satisfies Clifford's condition, then η is an isomorphism.*

Proof. The existence of η follows, as before, from the universality of $C^*(S)$. Define $\mathcal{V}(S) \subset C^*(S)$ to be

$$\mathcal{V}(S) = \{v_{t_1}^* v_{s_1} \cdots v_{t_n}^* v_{s_n} : n \in \mathbb{N},\ t_1, \dots, t_n, s_1, \dots, s_n \in S\}.$$

As in the previous proposition, $\mathcal{V}(S)$ is an inverse semigroup and it is sufficient to show that the restriction of η to $\mathcal{V}(S)$ is a semigroup isomorphism onto $I_l(S)$. Let $J''(S) = \{e_X : X \in J(S)\}$. Then, $J''(S)$ is the semilattice of idempotents in $\mathcal{V}(S)$ and $\eta|_{J''(S)}$ is an isomorphism onto $J(S)$.

We first show that $\mathcal{V}(S) \setminus \{0\} = \{v_p v_q^* : p, q \in S\}$. The proof is almost identical to that in Proposition 3.23, and we only show that, for any $s, t \in S$ with $v_t^* v_s \neq 0$, there exist $p, q \in S$ with $v_t^* v_s = v_p v_q^*$. If $v_t^* v_s \neq 0$, then $v_t v_t^* v_s v_s^* \neq 0$, so since $\eta|_{J''(S)}$ is an isomorphism onto $J(S)$, and since S satisfies Clifford's condition, there exists some $r \in S$ with $sS \cap tS = rS$ and

$$v_t v_t^* v_s v_s^* = v_r v_r^*. \tag{3.4}$$

Since $r \in sS \cap tS$, there exist $p, q \in S$ such that $sq = r$ and $tp = r$. Then $v_s v_q = v_r$, so $v_q = v_s^* v_r$. Similarly, $v_p = v_t^* v_r$. Now, by multiplying both sides of (3.4) on the left with v_t^*, then multiplying both sides on the right with v_s, we get that

$$v_t^* v_s = v_t^* v_r v_r^* v_s = v_p v_q^*.$$

Consider $v \in \mathcal{V}(S) \setminus \{0\}$, and suppose that $\eta(v)$ is idempotent. There exist $p, q \in S$ with $v = v_p v_q^*$. Now $\eta(v) = \lambda_p \lambda_q^*$, so $p = q$ by Lemma 3.16. Thus, $v = v_p v_q^* = v_p v_p^*$, which is idempotent. It follows from Lemma 2.14 that $\eta|_{\mathcal{V}(S)}$ is an isomorphism. □

It is now clear that the canonical map $C^*(S) \to C_r^*(S)$ factors as

$$C^*(S) \xrightarrow{\eta} C_0^*(I_l(S)) \xrightarrow{\Lambda_0} C_{r,0}^*(I_l(S)) \xrightarrow{h} C_r^*(S).$$

When S embeds into a group we get the following factorization:

$$C^*(S) \xrightarrow{\pi_s} C_s^*(S) \quad \underset{\simeq}{\xrightarrow{\kappa}} \quad C_0^*(I_l(S)) \xrightarrow{\Lambda_0} C_{r,0}^*(I_l(S)) \xrightarrow{h} C_r^*(S).$$

Note that in this case $\eta = \kappa \circ \pi_s$, so π_s is an isomorphism if and only if η is an isomorphism.

Li asked when a semigroup homomorphism $\phi\colon S \to R$ of left cancellative semigroups induces a $*$-homomorphism $C^*(S) \to C^*(R)$ by the formula $v_s \mapsto v_{\phi(s)}$. We can give a partial answer. It induces a $*$-homomorphism $C_0^*(I_l(S)) \to C_0^*(I_l(R))$ given by $\lambda_s \mapsto \lambda_{\phi(s)}$ if and only if ϕ extends to a 0-homomorphism $I_l(S)^0 \to I_l(R)^0$. Of course, determining when this is the case may not be easy. See Corollary 3.39 for a result in this direction when S is left reversible and R is a group.

$C^*(S)$ has the nice feature that it can be described as a crossed product by endomorphisms (see [**17**, Lemma 2.14]). Li also showed that $C^*(S)$ generalizes Nica's C^*-algebras for quasi-lattice ordered groups as well as for the Toeplitz algebras associated with rings of integers [**7**]. Nica proved in [**23**] that his C^*-algebra for the quasi-lattice ordered group (G, S) can be constructed as a C^*-algebra of the Toeplitz inverse semigroup $\mathcal{T}(G, S)$. This can be explained by the next lemma as well as Proposition 3.27. For each $g \in G$, define

$$\beta_g\colon \{s \in S\colon gs \in S\} \to \{s \in S\colon g^{-1}s \in S\}, \quad \beta_g(s) = gs.$$

Then, $\mathcal{T}(G, S)$ is defined to be the inverse subsemigroup of $I(S)$ generated by $\{\beta_g\}_{g\in G}$.

Lemma 3.28. *Let (G, S) be a quasi-lattice ordered group. Then $I_l(S)^0 = \mathcal{T}(G, S)^0$.*

Proof. Let $g \in S$. Then $\{s \in S\colon gs \in S\} = S$ and $\{s \in S\colon g^{-1}s \in S\} = gS$, so $\beta_g = \lambda_g$. This shows that $\mathcal{T}(G, S)$ contains $I_l(S)$.

Note that, for any $g \in G$, $\beta_{g^{-1}} = \beta_g^*$. If $\beta_g \neq 0$, $\operatorname{dom}\beta_g^* = \{s \in S\colon g^{-1}s \in S\} = gS \cap S$ is non-empty, so $g^{-1}s = t \in S$ for some $s, t \in S$. Then $g \leqslant s$, as defined in Definition 2.20. Moreover, $1^{-1}s \in S$, so $1 \leqslant s$. Thus, s is a common upper bound for g and 1 in S. Then, g and 1 have a least common upper bound $r \in S$ since (G, S) is quasi-lattice ordered.

Let $p \in gS \cap S$. Using the same arguments as we did for s, we get that $g \leqslant p$ and $1 \leqslant p$, so $r \leqslant p$ since r was a least common upper bound for g and 1. Then $r \succeq p$, so $p \in rS$. This shows that $gS \cap S \subset rS$. However, since $g \leqslant r$, $g^{-1}r = u$ for some $u \in S$. Then $r = gu$, so $rS = guS \cap S \subset gS \cap S$. Now,

$$\operatorname{dom}\beta_g^* = gS \cap S = rS = \operatorname{dom}\lambda_u\lambda_r^*.$$

Moreover, for any $v \in gS \cap S$,

$$\beta_g^*(v) = \beta_{g^{-1}}(v) = g^{-1}v = ur^{-1}v = \lambda_u\lambda_r^*(v).$$

So $\beta_g^* = \lambda_u\lambda_r^*$, and $\beta_g = \lambda_r\lambda_u^*$. This shows that $\mathcal{T}(G, S)^0 \subset I_l(S)^0$. □

A more detailed discussion on the relationship between $I_l(S)$ and $\mathcal{T}(G, S)$ can be found in [**13**]. Nica's construction of a C^*-algebra for $\mathcal{T}(G, S)$ uses a groupoid, but Milan explained in [**19**, § 5] why this C^*-algebra is isomorphic to $C_0^*(\mathcal{T}(G, S))$.

3.4. Left reversible semigroups, left amenability and functoriality

We still consider a left cancellative semigroup S unless otherwise stated. Recall that S is *left reversible* if, for any $s, t \in S$, $sS \cap tS \neq \emptyset$. The next lemma is a slightly stronger version of [**15**, Lemma 2.4.8].

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Lemma 3.29. *S is left reversible if and only if $0 \notin I_l(S)$ if and only if $\emptyset \notin J(S)$.*

Proof. Clearly, $0 \notin I_l(S)$ if and only if $\emptyset \notin J(S)$. Moreover, $sS \cap tS \in J(S)$ for all $s, t \in S$, so $\emptyset \notin J(S)$ implies that S is left reversible. Suppose that S is left reversible, and let $X, Y \subset S$ be non-empty right ideals. If $s \in X$ and $t \in Y$, then $sS \subset X$ and $tS \subset Y$, so $sS \cap tS \subset X \cap Y$ and $X \cap Y \neq \emptyset$. Moreover, for any $t \in S$, $tt^{-1}X = X \cap tS$, so $t^{-1}X$ is non-empty. It follows by a simple induction argument that $\emptyset \notin J(S)$. □

We include a short proof of the well-known fact that left amenable semigroups are left reversible. See also [**26**, Proposition 1.23]. To be formal, a *left invariant mean* on S is a state μ on $\ell^\infty(S)$ such that, for any $s \in S$ and $\xi \in \ell^\infty(S)$, $\mu(\xi \circ \lambda_s) = \mu(\xi)$. The semigroup S is *left amenable* if it has a left invariant mean. Right amenability is similarly defined. It is not difficult to show that a group or an inverse semigroup is left amenable if and only if it is right amenable, so left amenable groups and inverse semigroups are often just called *amenable*.

Lemma 3.30. *Let S be left amenable. Then, for any left invariant mean μ on S, $\mu(\chi_X) = 1$ for all $X \in J(S)$. This implies that S is left reversible.*

Proof. For convenience, we set $\mu(X) = \mu(\chi_X)$ for any $X \subset S$. Note that, for any $t \in S$, $\chi_X \circ \lambda_t = \chi_{t^{-1}X}$. Thus, $\mu(t^{-1}X) = \mu(X)$. Since $t^{-1}tX = X$, we also have that $\mu(tX) = \mu(X)$. By Lemma 3.9,

$$J(S) = \{t_1^{-1}s_1 \cdots t_n^{-1}s_n S \colon n \in \mathbb{N},\ s_i, t_i \in S\}.$$

So an induction shows that $\mu(X) = \mu(S) = 1$ for all $X \in J(S)$. As $\mu(\emptyset) = 0$, this shows that $\emptyset \notin J(S)$. □

It is also well known that every cancellative left reversible semigroup embeds into a group G. This is one formulation of Ore's theorem [**24**]. The proof of Theorem 3.31 we present below is basically the same as Rees's proof, found in [**6**, p. 35]. The reason we repeat it here is that it illustrates how $I_l(S)$ is related to G. See also [**15**, Chapter 2.4], where Lawson gives an account of this proof and shows that $I_l(S)$ is E-unitary when S is left reversible and cancellative.

Suppose that $0 \notin I_l(S)$. We can then construct the maximal group homomorphic image $G(I_l(S))$ of $I_l(S)$ as described in Definition 2.13. For simplicity we write that $G(S) = G(I_l(S))$. Let $\alpha_S \colon I_l(S) \to G(S)$ denote the quotient homomorphism. Let $\gamma_S \colon S \to G(S)$ be given by $\gamma_S(s) = \alpha_S(\lambda_s)$. $G(S)$ is then generated by the cancellative semigroup $\gamma_S(S)$.

Theorem 3.31. *Let S be left reversible. The following conditions are equivalent:*

(i) *S is cancellative;*

(ii) *$\gamma_S \colon S \to G(S)$ is injective;*

(iii) *S embeds into a group;*

(iv) *$I_l(S)$ is E-unitary.*

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Proof. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial. (iii) $\Leftrightarrow$ (iv) follows from Proposition 3.19 and the fact that an inverse semigroup without 0 is E-unitary if and only if it is strongly E^*-unitary.

(i) $\Rightarrow$ (ii). Since the map $S \hookrightarrow I_l(S)$ is injective, we have only to prove that the homomorphism $I_l(S) \to G(S)$ is injective on the set $\{\lambda_s \colon s \in S\}$. Let $s, t \in S$ and suppose that λ_s and λ_t map to the same element. Then, by the definition of the congruence that was used to construct $G(S)$ there exists an $X \subset J(S)$ such that $\lambda_s i_X = \lambda_t i_X$. Hence, for any $r \in X$, $sr = \lambda_s(r) = \lambda_t(r) = tr$. By cancelling with r we get that $s = t$. □

Definition 3.32. Let P be a semigroup. Recall that a subset $X \subset P$ is said to be *left thick* if, for any finite sequence $s_1, \dots, s_n \in P$,

$$\bigcap_{i=1}^{n} s_i X \neq \emptyset.$$

The next proposition is related to [**26**, Proposition 1.27].

Proposition 3.33. *Let S be a subsemigroup of a group G such that S generates G. S is then left reversible if and only if S is a left thick subset of G.*

Proof. For $t \in S$ and $X \subset S$, $t^{-1}(X) = (t^{-1})X \cap S$, where $t^{-1}(X)$ is the preimage inside S and $(t^{-1})X$ is defined by multiplication inside G. So, for any $n \in \mathbb{N}$ and $s_i, t_i \in S$,

$$t_1^{-1}s_1 \cdots t_n^{-1}s_n S = S \cap \bigcap_{j=1}^{n} (t_1^{-1}s_1 \cdots t_j^{-1}s_j)S.$$

If S is left thick, this set is never empty, nor is any finite intersection of sets of this type, so $\emptyset \notin J(S)$. On the other hand, for any $g_1 \cdots g_m \in G$, write that $g_i = t_{i,1}^{-1}s_{i,1} \cdots t_{i,n_i}^{-1}s_{i,n_i}$, with $s_{i,j}, t_{i,j} \in S$. Then,

$$\bigcap_{i=1}^{m} g_i S \supset S \cap \bigcap_{i=1}^{m} \bigcap_{j=1}^{n_i} (t_{i,1}^{-1}s_{i,1} \cdots t_{i,j}^{-1}s_{i,j})S.$$

If $\emptyset \notin J(S)$, the right-hand side is non-empty, and so is the left-hand side, so S is a left thick subset of G. □

By a theorem of Mitchell [**20**], if S' is a left thick subsemigroup of a semigroup S, then S' is left amenable if and only if S is left amenable. Hence, we get the following.

Corollary 3.34. *Let S be cancellative and left reversible. S is then left amenable if and only if $G(S)$ is amenable.*

We now show that the assumption that S is right cancellative is redundant in the statement of Corollary 3.34. If P is any semigroup, let $\approx$ (or $\approx_P$) be the relation on P given by $s \approx t$ if there exists some $r \in P$ with $sr = tr$. From [**6**, p. 35] we have that if P is left reversible, $\approx$ is a congruence and $P/{\approx}$ is a right cancellative semigroup. Proposition 1.25 of [**26**] states that P is left amenable if and only if $P/{\approx}$ is left amenable. Note that $\approx_{I_l(S)}$ is exactly the congruence on $I_l(S)$ one takes the quotient with to create $G(S)$.

Lemma 3.35. *Let S be left reversible. $\gamma_S(S)$ is then isomorphic to $S/{\approx_S}$*

Proof. We show that $\gamma_S(s) = \gamma_S(t)$ if and only if there exists some $r \in S$ such that $sr = tr$. First, if $sr = tr$, then $\lambda_s\lambda_r = \lambda_t\lambda_r$, so $\gamma_S(s) = \gamma_S(t)$. On the other hand, if $\gamma_S(s) = \gamma_S(t)$, there exists some $X \in J(S)$ such that $\lambda_s i_X = \lambda_t i_X$. Let $r \in X$. Then, $\lambda_s i_X \lambda_r = \lambda_s\lambda_r = \lambda_t i_X \lambda_r = \lambda_t\lambda_r$, so by evaluating at 1 we get that $sr = tr$. □

Corollary 3.36. *Let S be a left cancellative, left reversible semigroup. The following statements are equivalent:*

(i) *S is left amenable;*

(ii) *$\gamma_S(S)$ is left amenable;*

(iii) *$G(S)$ is amenable;*

(iv) *$I_l(S)$ is amenable.*

Proof. To show that the amenability of $G(S)$ is equivalent to the left amenability of $\gamma_S(S)$, we need to show that $\gamma_S(S)$ is left reversible. We have that, for any $s, t \in S$,

$$\gamma_S(s)\gamma_S(S) \cap \gamma_S(t)\gamma_S(S) = \gamma_S(sS) \cap \gamma_S(tS) \supset \gamma_S(sS \cap tS).$$

The right-hand side is non-empty, so the left-hand side must be non-empty as well. This proves that $\gamma_S(S)$ is left reversible, since any $p \in \gamma_S(S)$ is of the form $\gamma_S(s)$ for some $s \in S$. All the other equivalences are taken care of by the results we have developed so far. Since $G(S) = I_l(S)/{\approx_{I_l(S)}}$, $G(S)$ is amenable if and only if $I_l(S)$ is amenable. Since $\gamma_S(S) \simeq S/{\approx_S}$, $\gamma_S(S)$ is left amenable if and only if S is left amenable. □

We conclude this subsection by showing that when S is left reversible the construction $S \mapsto G(S)$ is a generalization of the Grothendieck construction in that it is functorial. This is probably already known by specialists, but we give a proof here for completeness. Another way to prove it is to show that any homomorphism of S to a group can be extended to define a group homomorphic image of $I_l(S)$, and then use that $G(S)$ is the maximal group homomorphic image of $I_l(S)$.

Lemma 3.37. *Let S be a subsemigroup of a group G such that S generates G, and let H be a group. If S is left thick in G, then every homomorphism $\phi\colon S \to H$ uniquely extends to a homomorphism $\phi'\colon G \to H$.*

Proof. Let $t_1, \dots, t_n, s_1, \dots, s_n \in S$. We want to define

$$\phi'(t_1^{-1}s_1 \cdots t_n^{-1}s_n) = \phi(t_1)^{-1}\phi(s_1)\cdots\phi(t_n)^{-1}\phi(s_n),$$

so we need to show that this is a consistent definition. Let $q_1 \cdots q_m, p_1 \cdots p_m \in S$ be such that

$$t_1^{-1}s_1 \cdots t_n^{-1}s_n = q_1^{-1}p_1 \cdots q_m^{-1}p_m.$$

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Since S is left thick in G,

$$S \cap \bigcap_{i=1}^{n}(s_n^{-1}t_n \cdots s_i^{-1}t_i)S \cap \bigcap_{j=1}^{m}(p_m^{-1}q_m \cdots p_j^{-1}q_j)S \neq \emptyset.$$

So there exists an $r \in S$ such that

$$u_i := t_i^{-1}s_i \cdots t_n^{-1}s_n r \in S,$$
$$v_j := q_j^{-1}p_j \cdots q_m^{-1}s_m r \in S$$

for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. First, $t_n^{-1}s_n r = u_n$, so $s_n r = t_n u_n$, which implies that $\phi(s_n)\phi(r) = \phi(t_n)\phi(u_n)$ and $\phi(t_n)^{-1}\phi(s_n)\phi(r) = \phi(u_n)$. Suppose now that, for some $1 \leqslant k \leqslant n$,

$$\phi(t_k)^{-1}\phi(s_k)\cdots\phi(t_n)^{-1}\phi(s_n)\phi(r) = \phi(u_k).$$

Then, since

$$s_{k-1}t_k^{-1}s_k \cdots t_n^{-1}s_n r = s_{k-1}u_k = t_{k-1}u_{k-1},$$

we get that

$$\phi(t_{k-1})^{-1}\phi(s_{k-1})\phi(u_k) = \phi(u_{k-1}).$$

Using induction, this implies that $\phi(t_1)^{-1}\phi(s_1)\cdots\phi(t_n)^{-1}\phi(s_n)\phi(r) = \phi(u_1)$. Similarly, $\phi(q_1)^{-1}\phi(p_1)\cdots\phi(q_m)^{-1}\phi(p_m)\phi(r) = \phi(v_1) = \phi(u_1)$, so by cancelling with $\phi(r)$ we see that ϕ' is well defined. The uniqueness of ϕ' is trivial since S generates G. $\square$

Theorem 3.38. *Let S be left reversible and let H be a group. Every homomorphism $\phi\colon S \to H$ then gives rise to a unique homomorphism $\phi'\colon G(S) \to H$ such that $\phi' \circ \gamma_S = \phi$.*

Moreover, for any left cancellative left reversible semigroup R and homomorphism $\phi\colon S \to R$, there exists a unique homomorphism $\phi'\colon G(S) \to G(R)$ such that $\phi' \circ \gamma_S = \gamma_R \circ \phi$.

Proof. First we need to show that $\phi\colon S \to H$ can be pushed down to a homomorphism $\gamma_S(S) \to H$. If $s, t, r \in S$, with $sr = tr$, then $\phi(s)\phi(r) = \phi(t)\phi(r)$, so $\phi(s) = \phi(t)$. This implies that ϕ is constant on the equivalence classes of $\approx$; hence, there exists a homomorphism $\phi''\colon \gamma_S(S) \to H$ such that $\phi'' \circ \gamma_S = \phi$. By Lemma 3.37, ϕ'' extends to a homomorphism $\phi'\colon G(S) \to H$ such that $\phi' \circ \gamma_S = \phi$. Uniqueness follows since the constructions $\phi \mapsto \phi''$ and $\phi'' \mapsto \phi'$ are unique, so if $\psi\colon S \to G(S)$ is another homomorphism with $\psi \circ \gamma_S = \phi$, then the restriction of ψ to $\gamma_S(S)$ must be equal to ϕ''.

If $\phi\colon S \to R$ is a homomorphism, then we can apply the above construction to $\gamma_R \circ \phi\colon S \to G(R)$ and thereby get the desired $\phi'\colon G(S) \to G(R)$. $\square$

Corollary 3.39. *Let S be left reversible and let G be a group. For every homomorphism $\phi\colon S \to G$ there then exists a $*$-homomorphism $\pi_\phi\colon C^*(I_l(S)) \to C^*(G)$ such that $\pi_\phi(\lambda_s) = \lambda_{\phi(s)}$ for each $s \in S$.*

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Proof. Consider a homomorphism $\phi\colon S \to G$. From Theorem 3.38 there exists a homomorphism $\phi'\colon G(S) \to G$ such that $\phi' \circ \gamma_S = \phi$. Then, $\phi' \circ \alpha_S\colon I_l(S) \to G$ satisfies $\phi'(\alpha_{I_l(S)}(\lambda_s)) = \lambda_{\phi(s)}$ for each $s \in S$, so the existence of π_ϕ follows from the universal property of $C^*(I_l(S))$. □

For example, if S is left reversible we may consider the quotient homomorphism $\alpha_S\colon I_l(S) \to G(S)$ and obtain a surjective $*$-homomorphism $\pi_S\colon C^*(I_l(S)) \to C^*(G(S))$. (Li also showed the existence of such a map from $C^*_s(S)$.) When $S = \mathbb{Z}^+$ this is the surjective part of the classical C^*-extension

$$0 \to K(\ell^2(\mathbb{Z}^+)) \to C^*_r(\mathbb{Z}^+) \to C(\mathbb{T}) \to 0,$$

where $K(\ell^2(\mathbb{Z}^+))$ are the compact operators on $\ell^2(\mathbb{Z}^+)$,

$$C^*_r(\mathbb{Z}^+) \simeq C^*_r(I_l(\mathbb{Z}^+)) \simeq C^*(I_l(\mathbb{Z}^+))$$

is the unique C^*-algebra generated by a single isometry, and

$$C(\mathbb{T}) \simeq C^*(\mathbb{Z}) \simeq C^*(G(\mathbb{Z}^+)).$$

By [**8**, Proposition 1.4], there also always exists a canonical $*$-homomorphism

$$\pi_{S,r}\colon C^*_r(I_l(S)) \to C^*_r(G(S)).$$

In general, it would be interesting to have a description of the kernel of π_S and $\pi_{S,r}$. Nica [**22**] gave some necessary and sufficient conditions for $C^*_r(S)$ to contain the compacts when (G, S) is a quasi-lattice ordered group.

3.5. Amenability and weak containment when S embeds into a group

In [**17**], Li showed that if S is left reversible and embeds into a group, and $J(S)$ is independent, then S is left amenable if and only if the canonical map $C^*_s(S) \to C^*_r(S)$ is an isomorphism. Note that to recover this formulation of the result from Li's statement, one has to use the fact that, when S embeds into a group, S is left reversible if and only if there is a character on $C^*_s(S)$. This is proved in [**17**, Lemma 4.6]. One also has to use that since S is left reversible, S is cancellative if and only if S embeds into a group. From [**19**], we know that an E-unitary inverse semigroup P has weak containment if and only if $G(P)$ is amenable. Hence, Theorem 3.31 and Corollary 3.34 give us the following.

Theorem 3.40. *A cancellative left reversible semigroup S is left amenable if and only if $I_l(S)$ has weak containment.*

This lets us recover Li's result.

Corollary 3.41. *Suppose that S is left reversible, that it embeds into a group, and that $J(S)$ is independent. S is then left amenable if and only if the canonical map $C^*_s(S) \to C^*_r(S)$ is an isomorphism.*

Proof. Theorems 3.31 and 3.22 together imply that h is an isomorphism. Proposition 3.26 shows that κ is an isomorphism. Theorem 3.40 shows that when S is left reversible, Λ is an isomorphism if and only if S is left amenable. The composition of κ, Λ and h is the canonical map $C_s^*(S) \to C_r^*(S)$. □

Corollary 3.42. *Suppose that S is cancellative, left reversible and satisfies Clifford's condition. The canonical map $C^*(S) \to C_r^*(S)$ is then an isomorphism if and only if S is left amenable.*

Proof. By Corollary 3.30, h is an isomorphism, and by Proposition 3.27 η is an isomorphism. Since S is left reversible, Theorem 3.40 implies that S is left amenable if and only if Λ is an isomorphism. The composition of η, Λ and h is the canonical map $C^*(S) \to C_r^*(S)$. □

Remark 3.43. Corollary 3.36 does *not* imply that left amenability of S is equivalent to weak containment of $I_l(S)$ for any left reversible S. Without $I_l(S)$ being E-unitary, one also has to prove that the inverse semigroup $H := \alpha_S^{-1}(1_{G(S)})$ has weak containment (see [**19**, Theorem 2.4 and Corollary 2.5]). Milan's results do, however, give us that weak containment of $I_l(S)$ implies left amenability of $G(S)$ and thus of S (for left reversible S).

When S is not left reversible, S cannot be left amenable, but $\Lambda_0\colon C_0^*(I_l(S)) \to C_{r,0}^*(I_l(S))$ can still be an isomorphism. Nica showed in [**22**] that his version of the full and reduced C^*-algebras for $\mathbb{F}_n^+$ are canonically isomorphic. Here, $\mathbb{F}_n^+$ is the free semigroup on n generators. This implies that Λ_0 is an isomorphism in this case. The semigroup $\mathbb{F}_n^+$ is easily seen to be not left reversible for $n \geqslant 2$.

Milan [**19**] developed a technique for determining weak containment of strongly E^*-unitary inverse semigroups P. Fixing an idempotent pure grading $\varphi\colon P \to G^0$, he defined

$$\begin{aligned} A_g &= \operatorname{span}\{p \in P\colon \varphi(p) = g\} \quad && \text{inside } \mathbb{C}P/\mathbb{C}0_P, \\ B_g &= \overline{A_g} && \text{inside } C_0^*(P). \end{aligned}$$

Milan then showed that $\{B_g\}_{g\in G}$ is a Fell bundle over G and that P has weak containment if and only if this Fell bundle is amenable. Milan stated this result for the universal grading of P, but the proof works for any idempotent pure grading.

In our setting, $I_l(S)$ is strongly E^*-unitary if and only if S embeds into a group G. Recalling the idempotent pure grading $\varphi\colon I_l(S)^0 \to G^0$ constructed in Proposition 3.19, one sees that the associated Fell bundle $\{B_g\}_{g\in G}$ is given by

$$B_g = \overline{\operatorname{span}\{\lambda_{t_1}^*\lambda_{s_1}\cdots\lambda_{t_n}^*\lambda_{s_n}\colon t_1^{-1}s_1\cdots t_n^{-1}s_n = g\}} \quad \text{in } C_0^*(I_l(S)). \tag{3.5}$$

Theorem 3.44. *Suppose that S embeds into a group G. Then, $\Lambda_0\colon C_0^*(I_l(S)) \to C_{r,0}^*(I_l(S))$ is an isomorphism if and only if the Fell bundle $\{B_g\}_{g\in G}$ defined in (3.5) is amenable.*

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Corollary 3.45. *Suppose that S embeds into a group and satisfies Clifford's condition. The canonical map $C^*(S) \to C_r^*(S)$ is then an isomorphism if and only if the Fell bundle $\{B_g\}_{g\in G}$ given by*

$$B_g = \overline{\text{span}\{\lambda_s\lambda_t^* \colon st^{-1} = g\}} \quad \text{in } C_0^*(I_l(S)) \tag{3.6}$$

is amenable.

Proof. For semigroups satisfying Clifford's condition, one can use Proposition 3.23 to deduce that the Fell bundles defined in (3.5) and (3.6) are the same. □

When (G, S) is a quasi-lattice ordered group, this expresses Nica's amenability of (G, S) in terms of the amenability of the Fell bundle defined in (3.6), and is, in view of Lemma 3.28, merely a restatement of [**19**, Proposition 5.2]. In the case where S is a finitely generated free semigroup, amenability of the Fell bundle defined in (3.6) may be deduced from [**10**]. However, the proof one would thereby get from Corollary 3.45, that $C^*(S) \to C_r^*(S)$ is an isomorphism, would not be simpler than Nica's original proof [**22**].

Li [**17**] showed that $C^*(S)$ and $C_r^*(S)$ are nuclear when S is countable, cancellative and right amenable. The last two conditions imply that S embeds into an amenable group. We show that S does not have to be countable to prove that $C_s^*(S)$ is nuclear.

Proposition 3.46. *Suppose that S embeds into an amenable group. Λ_0 is then an isomorphism, and $C_0^*(I_l(S))$ ($\simeq C_s^*(S)$) and $C_r^*(S)$ are nuclear.*

Proof. From [**9**, Theorem 4.7] we know that a Fell bundle over an amenable group satisfies the approximation property, and is thus amenable. Moreover, it was proved in [**1**] that a Fell bundle with nuclear unit fibre has nuclear cross-sectional C^*-algebra if it also satisfies the approximation property. The unit fibre in $\{B_g\}_{g\in G}$ is the closure of the span of $J(S)$ in $C_0^*(I_l(S))$, and is abelian. $C_r^*(S)$ is also nuclear since it is a quotient of $C_0^*(I_l(S))$ (see [**3**, Theorem 10.1.3]). □

Corollary 3.47. *Let S be cancellative and left reversible. S is then left amenable if and only if $C_0^*(I_l(S))$ ($\simeq C_s^*(S)$) is nuclear. If, in addition, $J(S)$ is independent, S is left amenable if and only if $C_r^*(S)$ is nuclear.*

Proof. One implication follows from Proposition 3.46, since a cancellative and left amenable S embeds into the amenable group $G(S)$. It remains to show that $C_0^*(I_l(S))$ being nuclear implies that S is left amenable. This was shown for $C_s^*(S)$ in [**17**, Proposition 4.17] with an argument analogous to the following. $C^*(G(S))$ is a quotient of $C_0^*(I_l(S))$ and is thus nuclear. This implies that $G(S)$ is amenable and that S is left amenable (see [**3**, Theorems 10.1.3 and 2.6.8]).

Assume that $J(S)$ is independent and that $C_r^*(S)$ is nuclear. By Theorem 3.22, $C_{0,r}^*(I_l(S)) \simeq C_r^*(S)$. Since $C_r^*(G(S))$ is a quotient of $C_{0,r}^*(I_l(S))$ it is also nuclear. This implies that $G(S)$ is amenable (see [**3**, Theorem 2.6.8]). □

Remark 3.48. Note that when S is left reversible, but not right cancellative, nuclearity of $C_0^*(I_l(S))$ still implies that S is left amenable.

Corollary 3.49. *Let R be a GCD domain. The canonical $*$-homomorphism $C^*(R \rtimes R^\times) \to C_r^*(R \rtimes R^\times)$ is then an isomorphism and $C^*(R \rtimes R^\times)$ is nuclear.*

Proof. As Li remarked in [**17**], $R \rtimes R^\times$ embeds into an amenable group, so Λ_0 is an isomorphism and $C_0^*(I_l(R \rtimes R^\times))$ is nuclear by Proposition 3.46. By Corollary 3.25, h is an isomorphism, and by Propositions 3.27 and 2.23, η is an isomorphism. □

Note that if one also assumes that R is a Dedekind domain, the previous corollary is weaker than the results given in [**7**, **17**], since not all rings of integers or Dedekind domains are GCD domains.

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