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A NOTE ON χ-BINDING FUNCTIONS AND LINEAR FORESTS

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Abstract

Let G and H be two vertex disjoint graphs. The *join* $G + H$ is the graph with $V(G + H) = V(G) + V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$. A (finite) linear forest is a graph consisting of (finite) vertex disjoint paths. We prove that for any finite linear forest F and any nonnull graph H, if $\{F, H\}$-free graphs have a χ-binding function $f(\omega)$, then $\{F, K_n + H\}$-free graphs have a χ-binding function $kf(\omega)$ for some constant k.

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1. Introduction

Throughout this paper, all graphs have finite vertex sets and no loops or parallel edges. We follow [1] for undefined notation and terminology.

We say that a graph G *contains* a graph H if some induced subgraph of G is isomorphic to H. A graph is H-*free* if it does not contain H. When $\mathcal{H}$ is a set of graphs, G is $\mathcal{H}$-*free* if G contains no graph of $\mathcal{H}$. A class of graphs $\mathcal{G}$ is called *hereditary* if every induced subgraph of any graph in $\mathcal{G}$ also belongs to $\mathcal{G}$. One important and well-studied class of hereditary graphs is the family of $\mathcal{H}$-free graphs.

Let G be a graph and X be a subset of $V(G)$. We use $G[X]$ to denote the subgraph of G induced by X, and call X a *clique* (*independent set*) if $G[X]$ is a complete graph. The *clique number* $\omega(G)$ of G is the maximum size taken over all cliques of G (we sometimes simply write $\omega(X)$ for $\omega(G[X])$). If $v \in V(G)$, we denote the set of neighbours of a vertex v by $N(v)$ or $N_G(v)$. For $X \subseteq V(G)$, let

$$N_G(X) = \{u \in V(G) \setminus X \mid u \text{ has a neighbour in } X\}.$$

(We omit the subscript G if there is no ambiguity.)

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Let G and H be two vertex disjoint graphs. The *union* $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The *join* $G + H$ is the graph with $V(G + H) = V(G) + V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$.

For a graph G, $\chi(G)$ denotes the chromatic number of G (we sometimes simply write $\chi(X)$ for $\chi(G[X])$). Erdős [9] showed that for any n, there exists a triangle-free graph with chromatic number at least n. Hence, in general, there exists no function of $\omega(G)$ that gives an upper bound on $\chi(G)$ for all graphs G. We denote by $\mathbb{N}$ the set of all positive integers. A class of graphs $\mathcal{G}$ is said to be *χ-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ (called a *χ-binding function*) such that $\chi(G) \le f(\omega(G))$ for every graph $G \in \mathcal{G}$; and the class is *polynomially χ-bounded* if f can be taken to be a polynomial.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for each induced subgraph H of G. Perfect graphs are a well-known hereditary χ-bounded graph class, that is, a class of graphs for which the identity function is a χ-binding function. A *hole* in a graph is an induced subgraph which is a cycle of length at least four, and a hole is *even* or *odd* according to whether its length is even or odd. An *antihole* of a graph G is an induced subgraph of G whose complement graph is a cycle of length at least four. Chudnovsky *et al.* [4] characterised perfect graphs as the class of {odd hole, odd antihole}-free graphs, a result known as the strong perfect graph theorem.

One important research direction in the area of χ-boundedness is to determine graph families $\mathcal{H}$ such that the class of $\mathcal{H}$-free graphs is χ-bounded, as well as finding the smallest possible χ-binding function for such a hereditary class of graphs. Gyárfás [15] and Sumner [32] independently reported the following conjecture.

CONJECTURE 1.1 [15, 32]. For every forest F, the class of F-free graphs is χ-bounded.

This conjecture remains open in general, though it has been proved for some very restricted trees (see, for example, [5, 15–18, 23, 25]).

For any positive integer t, we use P_t to denote a t-vertex path. It is known that P_3-free graphs are disjoint unions of complete graphs and P_4-free graphs are perfect [31]. From [14] (see also [13]), every P_5-free graph G with $\omega(G) \ge 3$ satisfies $\chi(G) \le 5 \cdot 3^{\omega(G)-3}$, and a recent result of Scott *et al.* [30] states that every P_5-free graph G satisfies $\chi(G) \le \omega(G)^{\log_2 \omega(G)}$. In general, Gyárfás [15] showed that $\chi(G) \le (t-1)^{\omega(G)-1}$ for all P_t-free graphs. This upper bound was improved to $\chi(G) \le (t-2)^{\omega(G)-1}$ in [14].

To support Conjecture 1.1, one approach is to continuously expand the known graph classes which are χ-bounded. We state three recent results of Chudnovsky *et al.* [6], Wu and Xu [34], and Schiermeyer and Randerath [22].

THEOREM 1.2 [6, Theorem 1.3]. *Let F be a forest. If F-free graphs are polynomially χ-bounded, then $\{F \cup P_4\}$-free graphs are polynomially χ-bounded.*

THEOREM 1.3 [34, Theorem 1.1]. *Let H be a connected graph or the union of a connected graph and an isolated vertex with $|V(H)| \ge 3$, and let G be a connected $\{P_5, K_1 + H\}$-free graph. If $\{P_5, H\}$-free graphs have a χ-binding function $f(\omega)$, then $\{P_5, K_1 + H\}$-free graphs have a χ-binding function $kf(\omega)$ for some constant k.*

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THEOREM 1.4 [22, Theorem 33]. *Let G be a $\{P_k, gem\}$-free graph for $k \geq 4$ with clique number $\omega(G) \geq 2$. Then, $\chi(G) \leq (k-2)(\omega(G)-1)$.*

We refer to a graph that contains at least one vertex as a *nonnull graph*. Using the idea of [22, Theorem 33], we generalise the results of Wu and Xu [34], and Schiermeyer and Randerath [22].

THEOREM 1.5. *For any finite linear forest F and any nonnull graph H, if $\{F, H\}$-free graphs have a χ-binding function $f(\omega)$, then $\{F, K_n + H\}$-free graphs have a χ-binding function $kf(\omega)$ for some constant k.*

We derive Theorem 1.5 from the following theorem.

THEOREM 1.6. *For any integers $n \geq 0$ and $t \geq 4$, if H is a nonnull graph and $\{P_t, H\}$-free graphs have a χ-binding function $f(\omega)$, then $\{P_t, K_n + H\}$-free graphs have a χ-binding function $(t-2)^{n+1} f(\omega)$.*

2. The main proof

The aim of this section is to prove Theorems 1.6 and 1.5. Following a proof idea in [22], we first establish a lemma which generalises a result of Schiermeyer and Randerath [22].

LEMMA 2.1. *Let $t \geq 4$ be an integer and G be a P_t-free graph with $\omega(G) \geq 2$. If there exists a function $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(x) \geq x$ and $\chi(N(v)) \leq \phi(\omega(G)-1)$ for every vertex v of G, then $\chi(G) \leq (t-2)\phi(\omega(G)-1)$.*

PROOF. We proceed by induction on t. It is known that P_4-free graphs are perfect. Therefore, $\chi(G) = \omega(G) \leq 2\omega(G) - 2 \leq 2\phi(\omega(G)-1)$ if $t = 4$. Now, for some fixed $t \geq 4$, suppose that $(t-2)\phi(\omega(G)-1)$ is a χ-binding function for all P_t-free graphs G. We will prove Lemma 2.1 holds for all P_{t+1}-free graphs to complete our proof.

Let G be a P_{t+1}-free graph. Without loss of generality, G is connected. Assuming that $\chi(G) > ((t+1)-2)\phi(\omega(G)-1)$, we shall reach a contradiction by constructing an induced $(t+1)$-vertex path P_{t+1} in G.

We define sets $V(G_i) \subseteq V(G_{i-1}) \subseteq \cdots \subseteq V(G_1) = V(G)$ and vertices $v_1 \in V(G_1)$, $v_2 \in V(G_2), \ldots, v_i \in V(G_i)$ for all i satisfying $1 \leq i \leq t-1$ with the following properties:

(1) G_i is a connected subgraph of G;
(2) $\chi(G_i) > (t-i)\phi(\omega(G)-1)$; and
(3) if $1 \leq j < i$ and $v \in V(G_i)$, then $v_j v$ is an edge of G if and only if $j = i-1$ and $v = v_i$.

Notice that $G_1 = G$ and $\chi(G_1) > (t-1)\phi(\omega(G)-1)$ as we have assumed. Let v_1 be any vertex of G_1. Assume that $G_1, G_2, \ldots, G_i$ and $v_1, v_2, \ldots, v_i$ are already defined for some $i \leq t-1$; moreover, properties (1)–(3) are satisfied. Define G_{i+1} and v_{i+1} as follows. Let A denote the set of neighbours of v_i in G_i. Let

$$B = V(G_i) \setminus (A \cup \{v_i\}).$$

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The graph $G[A]$ satisfies $\omega(A) \leq \omega(G) - 1$. Otherwise, adding v_i would give a clique of cardinality $\omega(G) + 1$. Furthermore, since $A = N_{G_i}(v_i)$,

$$\chi(A) \leq \phi(\omega(G) - 1).$$

Suppose first that $B \neq \emptyset$. Then, $\chi(G_i) \leq \chi(A) + \chi(B)$. It follows that

$$\begin{aligned}\chi(B) \geq \chi(G_i) - \chi(A) &> ((t+1) - 1 - i)\phi(\omega(G) - 1) - \phi(\omega(G) - 1)\\ &= (t - (i+1))\phi(\omega(G) - 1),\end{aligned}$$

which allows us to choose a connected component H of $G[B]$ satisfying $\chi(H) > (t - (i+1))\phi(\omega(G) - 1)$. Since G_i is connected by property (1), there exists a vertex $v_{i+1} \in A$ such that $V(H) \cup \{v_{i+1}\}$ induces a connected subgraph which we choose as G_{i+1}. Now it is easy to check that $G_1, G_2, \ldots, G_{i+1}$ and $v_1, v_2, \ldots, v_{i+1}$ satisfy the requirements in properties (1)–(3).

Suppose now that $B = \emptyset$. Then $\chi(G_i) \leq \phi(\omega(G) - 1)$, which in turn implies that $(t - i)\phi(\omega(G) - 1) < \chi(G_i) \leq \phi(\omega(G) - 1)$. It follows that $i = t$.

Since $A \neq \emptyset$ by properties (1) and (2) of G_i, v_{t+1} can be defined as any vertex of A, that is to say, $G[\{v_1, v_2, \ldots, v_{t+1}\}]$ is an induced $(t+1)$-vertex path P_{t+1} in G, which is a contradiction. This completes the proof of Lemma 2.1. □

Proof of Theorem 1.6. We proceed by induction on n. For a fixed integer $t \geq 4$, since $\{P_t, H\}$-free graphs have a χ-binding function $f(\omega)$, Theorem 1.6 holds when $n = 0$. We may assume that $\{P_t, K_{n-1} + H\}$-free graphs have a χ-binding function $(t-2)^n f(\omega)$. Now, let G be a $\{P_t, K_n + H\}$-free graph. Since G is $\{P_t, K_n + H\}$-free, $G[N(v)]$ is $\{P_t, K_{n-1} + H\}$-free for every vertex v of G. Therefore, there exists a function $(t-2)^n f : \mathbb{N} \to \mathbb{N}$ such that $(t-2)^n f(x) \geq x$ and $\chi(N(v)) \leq (t-2)^n f(\omega(G) - 1)$ for every vertex v of G. By Lemma 2.1, $\chi(G) \leq (t-2)(t-2)^n f(\omega(G) - 1) = (t-2)^{n+1} f(\omega(G) - 1)$. This proves Theorem 1.6. □

Using Theorem 1.6 as the induction base, we next prove Theorem 1.5 by induction on the number of paths contained in F.

Proof of Theorem 1.5. With the same arguments as in Theorem 1.6, it suffices to prove that $\{F, K_1 + H\}$-free graphs have a χ-binding function $kf(\omega)$ for some constant k.

Let G be an $\{F, K_1 + H\}$-free graph. Since F is a finite linear forest, we may assume that F consists of m vertex disjoint paths. We proceed by induction on m. If $m = 1$, by Theorem 1.6, we are done. Suppose Theorem 1.5 holds for any positive integer $m' < m$. Choose any path in F such that this path is a component of F, say P. Consequently, we assume that $|V(P)| = h$.

For each vertex $v \in V(P)$, the graph $G[N(v)]$ is $\{F, H\}$-free and thus $\chi(N(v)) \leq f(\omega(G))$. So, $\chi(N(V(P))) \leq hf(\omega(G))$. By the induction hypothesis, there exists an integer k' such that $\chi(G \setminus (V(P) \cup N(V(P)))) \leq k' f(\omega(G))$. Therefore, $\chi(G) \leq (k' + h) f(\omega(G))$. This proves Theorem 1.5. □

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3. Remarks

In most cases, proofs of χ-boundedness give fairly fast-growing functions, so it is interesting to ask: when do we get the stronger property of polynomial χ-boundedness? A provocative conjecture of Esperet [12] asserted that every χ-bounded hereditary class is polynomially χ-bounded, but this was recently disproved by Briański *et al.* [2]. So the question now is: which hereditary classes are polynomially χ-bounded? For any tree T, perhaps every T-free graph is polynomially χ-bounded. Scott *et al.* [28] proved that if T contains no P_5, then every T-free graph is polynomially χ-bounded. We refer to [6, 7, 19, 26–30] for some recent results and to [20, 22, 24] for some surveys about topics related to χ-boundedness.

Actually, if a χ-binding function is polynomial, it has another very important consequence. Graph classes with polynomial χ-binding functions satisfy the Erdős–Hajnal conjecture [10, 11].

CONJECTURE 3.1 (Erdős–Hajnal conjecture). For every graph H, there exists some $\epsilon > 0$ such that each H-free graph G has a clique or an independent set of size at least $|G|^{\epsilon}$.

The problem of finding a polynomial χ-binding function for the class of P_5-free graphs is still open, and the problem is open even for the class of $\{P_5, C_5\}$-free graphs (mentioned in [3]). The best known result is an exponential upper bound, $2^{\omega(G)-1}$, due to Chudnovsky and Sivaraman [8]). The following well-known problem is proposed by Schiermeyer [21].

PROBLEM 3.2 [21]. Are there polynomial functions f_{p_k} for $k \geq 5$ such that $\chi(G) \leq f_{p_k}(\omega(G))$ for every P_k-free graph G?

According to Theorem 1.6, we can directly derive the following result.

THEOREM 3.3. *For any integers $n \geq 0$ and $t \geq 4$, if H is a nonnull graph and $\{P_t, H\}$-free graphs have a polynomial χ-binding function $f(\omega)$, then $\{P_t, K_n + H\}$-free graphs have a polynomial χ-binding function $(t-2)^{n+1} f(\omega)$.*

Theorem 3.3 has some interesting corollaries. We use M_s to denote the disjoint union of s edges. A *friendship graph F_s* is the graph $K_1 + M_s$ (see Figure 1). We give a polynomial χ-binding function for $\{P_t, K_n + F_s\}$-free graphs. We first introduce the following result of Wagon [33].

LEMMA 3.4 [33]. *For every $s \in \mathbb{N}$, every M_s-free graph G satisfies $\chi(G) \leq \omega(G)^{2s-2}$.*

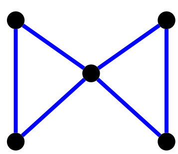

FIGURE 1. Graph F_2.

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Then we have the following corollary of Theorem 3.3.

COROLLARY 3.5. *Let $n \geq 0, s \geq 1$ and $t \geq 4$ be integers. Let G be a $\{P_t, K_n + F_s\}$-free graph. Then $\chi(G) \leq (t-2)^{n+1}(\omega(G)-1)^{2s-2}$.*

PROOF. Let H be a $\{P_t, F_s\}$-free graph and $\phi(x) = x^{2s-2}$. Since H is F_s-free, $H[N(v)]$ is M_s-free for any vertex v of H; moreover, $\omega(H[N(v)]) \leq \omega(H) - 1$. From Lemma 3.4,

$$\chi(N(v)) \leq \phi(\omega(H) - 1) = (\omega(H) - 1)^{2s-2}$$

for every vertex v of H. Therefore, from Lemma 2.1,

$$\chi(H) \leq (t-2)\phi(\omega(H) - 1) = (t-2)(\omega(H) - 1)^{2s-2}.$$

By Theorem 1.6, $\chi(G) \leq (t-2)^{n+1}(\omega(G)-1)^{2s-2}$ if G is a $\{P_t, K_n + F_s\}$-free graph. This completes the proof of Corollary 3.5. □

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