If $f(t)$ is continuous, take the subdivisions so fine that $M_s - m_s < \epsilon$. Then

$$|D| < \epsilon\,(b^{l+m+n} - a^{l+m+n})\,/\,(l+m+n),$$
$$\to 0 \text{ as } \epsilon \to 0.$$

If $f(t)$ is monotone or of bounded variation, take the subdivisions so fine that $t_s^{l+m+n} - t_{s-1}^{l+m+n} < \epsilon$. Then

$$(l+m+n)\,|D| < \epsilon \sum_{s=1}^{r} (M_s - m_s),$$
$$< \epsilon\,|f(b-0) - f(a+0)|, \text{ or } \epsilon K, \text{ say},$$

according as $f(t)$ is monotone or of bounded variation,

$$\to 0 \text{ as } \epsilon \to 0.$$

Hence the theorem is proved. The result clearly also holds for the limiting cases $a = 0$ or $b = \infty$ when $f(t)$ is not bounded in $a \leqq t \leqq b$ if I, J then exist as improper or infinite integrals, and $f(t)$ satisfies (1) or (2) in every closed sub-interval of $0 < t < \infty$.

THE UNIVERSITY,
MANCHESTER.

On Desargues Theorem

By J. H. M. WEDDERBURN.

The usual proofs of Desargues Theorem employ either metrical or analytical methods of projection from a point outside the plane; and if it is attempted to translate the analytical proof by the von Stuadt-Reye methods, the result is very long and there is trouble with coincidences. It is the object of this note to give a short geometrical proof which, in addition to the usual axioms of incidence and extension, uses only the assumption that a projectivity which leaves three points on a line unchanged also leaves all points on it unchanged. Degenerate cases are excluded as having no interest.

LEMMA 1. *If the triangles ABC, A′B′C′ are in perspective from O, and if B lies on A′C′, B′ on AC, then the triangles are coaxial.*

Let $X = (BC,\ B'C')$, $Y = (ACB',\ A'C'B)$, $Z = (AB,\ A'B')$, $D = (B'C',\ OAA')$, $F = (YZ,\ OBB')$, $E = (YZ,\ BC)$, $E' = (YZ,\ B'C'D)$; then $YZFE$ projects from B into $YAB'C$, which projects into

https://doi.org/10.1017/S0950184300000136 Published online by Cambridge University Press

$AYCB'$, which in turn projects from C' into $AA'OD$, which projects from B' into $YZFE'$, and hence $E=E'$ so that XYZ are collinear.

LEMMA 2. *If the triangles ABC, $A'B'C'$ are in perspective from O, and B' lies on AC and C' on AB but A' is not on BC, then the triangles are coaxial.*

In view of Lemma 1 we assume that B is not on $A'C'$. Let $X=(BC,\ B'C')$, $Y=(AC,\ A'C')$, $Z=(AB,\ A'B')$, $C''=(A'B,\ OC)$, $X''=(BC,\ B'C'')$, $Y''=(AC,\ A'C'')$, where C'' is distinct from C and C'; hence, by Lemma 1 applied to the triangles ABC and $A'B'C''$, the points $X''Y''Z$ are collinear. Also $AY''YC$ projects from A' into $OC''C'C$, which projects from B' into $BX''XC$, so that AB and $Y''X''$ meet in Z; hence XYZ are collinear.

LEMMA 3. *If the triangles ABC and $A'B'C'$ are in perspective from O, and A' lies on BC, B' on AC, and C' on AB, then the triangles are coaxial.*

Let $X=(BC,\ B'C')$, $Y=(CA,\ C'A')$, $Z=(AB,\ A'B')$, $C''=(A'B',\ OC')$, $A''=(B'C',\ OA)$, $B''=(C'A',\ OB)$; then C'' is distinct from O,C,C'. Since $OCC'C''$ projects from B' into $OAA''A'$ which projects from C' into $OBB'B''$, it follows that BC, $B'C'$, $B''C''$ meet in X; and similarly CA, $C'A'$, $C''A''$ meet in Y, and AB, $A'B'$, $A''B''$ in Z. Let $A''B''$ meet ACB' in P, and let $B''C''$ meet the same line in Q; P and Q are then distinct. Since $B''ZPA''$ projects from B' into $OA'AA''$; and this projects from Y into $OC'CC''$, which projects from B' into $B''XQC''$; it follows that ZX, $PQ=AC$, and $A''C''$ meet in Y and so XYZ are collinear.

LEMMA 4. *If the triangles ABC, and $A'B'C'$ are in perspective from O and B' lies on AC, then the triangles are coaxial.*

Let $X=(BC,\ B'C')$, $Y=(AC,\ A'C')$, $Z=(AB,\ A'B')$, $C''=(AB,\ OC)$, $A''=(C''Y,\ OA)$. We may suppose that A'', C'' are distinct from A', C', as otherwise one of the previous Lemmas applies. By Lemma 2 or 3 the triangles ABC and $A''B'C''$ are coaxial and so $X''=(BC,\ B'C'')$, $Y=(AC,\ A''C'')$, $Z''=(AB,\ A''B')$ are collinear. Also $ZAZ''B$ projects from B' into $A'AA''O$; and this projects from Y into $C'CC''O$, which projects from B' into $XCX''B$; hence ZX, AC, $Z''X''$ are concurrent in Y, that is, XYZ are collinear.

THEOREM. *If the triangles ABC and $A'B'C'$ are in perspective from O, they are coaxial.*

https://doi.org/10.1017/S0950184300000136 Published online by Cambridge University Press

Let $X = (BC,\ B'C')$, $Y = (AC,\ A'C')$, $Z = (AB,\ A'B')$, $B'' = (AC,\ OB')$, $X'' = (BC,\ B''C')$, $Z'' = (AB,\ A'B'')$; then, by Lemma 4 applied to the triangles ABC and $A'B''C'$, the points $X''YZ''$ are collinear. Also $ZABZ''$ projects from A' into $B'OBB''$, which projects from C' into $XCBX''$, and hence ZX, AC, $Z''X''$ are concurrent in Y, that is, XYZ are collinear.

PRINCETON, NEW JERSEY.

On a Chain of Circle Theorems.

By L. M. BROWN.

If P_1, P_2, P_3, P_4 *are four points on a circle* C, *and* P_{234} *is the orthocentre of triangle* $P_2 P_3 P_4$, P_{134} *the orthocentre of triangle* $P_1 P_3 P_4$ *and so on, then the quadrilateral* $P_{234} P_{134} P_{124} P_{123}$ *is congruent to the quadrilateral* $P_1 P_2 P_3 P_4$. This theorem seems to be due to Steiner (*Ges. Werke*, 1, p. 128; see H. F. Baker, *Introduction to Plane Geometry*, 1943, p. 332) and has appeared frequently since in collections of riders on the elementary circle theorems.

It is clear that $P_{234} P_{134} P_{124} P_{123}$ lie on a circle C_{1234} equal to the original circle C. But also angle $P_3 P_{134} P_4 = P_4 P_1 P_3 = P_4 P_2 P_3 = P_3 P_{234} P_4$ (with angles directed and equations modulo π), and hence $P_3 P_4 P_{134} P_{234}$ lie on a circle C_{34} equal to C, and which is in fact the mirror image of C in $P_3 P_4$. Similarly we obtain circles C_{12}, C_{13}, C_{14}, C_{23}, C_{24}, so that we have in all eight circles with four points on each. If any one of these be taken as the original circle, the same system of eight circles is obtained; if, *e.g.*, we begin with $P_3 P_4 P_{134} P_{234}$ on the circle C_{34}, the four orthocentres are P_1, P_2, P_{123}, P_{124} lying on C_{12} and the remaining circles are the images of C_{34} in the six sides of the quadrangle $P_3 P_4 P_{134} P_{234}$. Call this configuration K_4.

Let us now take a fifth point P_5 on C. Then any four of $P_1 P_2 P_3 P_4 P_5$ give a K_4. We have in fact five points $P_1 \ldots . P_5$, ten points $P_{123} \ldots . P_{345}$, a circle C, ten circles $C_{12} \ldots . C_{45}$ and five circles $C_{1234} \ldots . C_{2345}$. *Then the circles* $C_{1234} C_{1235} C_{1245} C_{1345} C_{2345}$ *all pass through a point* P_{12345}, *completing a system of* 16 *points and* 16 *circles, five points on each circle and five circles through each point.* We may show this by taking the circle C_{12}, *e.g.*, on which lie the five points $P_1 P_2 P_{123} P_{124} P_{125}$ and build up the K_4's obtained by taking these four at a time. Use a parallel notation and write $Q_1 = P_1$, $Q_2 = P_2$,