Can. J. Math. Vol. **49** (6), 1997 pp. 1117–1138.

THE VON NEUMANN ALGEBRA VN(G) OF A LOCALLY COMPACT GROUP AND QUOTIENTS OF ITS SUBSPACES

ZHIGUO HU

ABSTRACT. Let VN(G) be the von Neumann algebra of a locally compact group G. We denote by μ the initial ordinal with $|\mu|$ equal to the smallest cardinality of an open basis at the unit of G and $X = \{\alpha\,;\,\alpha < \mu\}$. We show that if G is nondiscrete then there exist an isometric $*$-isomorphism κ of $l^\infty(X)$ into VN(G) and a positive linear mapping π of VN(G) onto $l^\infty(X)$ such that $\pi\circ\kappa = \mathrm{id}_{l^\infty(X)}$ and κ and π have certain additional properties. Let UCB($\hat{G}$) be the C^*-algebra generated by operators in VN(G) with compact support and $F(\hat{G})$ the space of all $T \in$ VN(G) such that all topologically invariant means on VN(G) attain the same value at T. The construction of the mapping π leads to the conclusion that the quotient space UCB($\hat{G}$)$/F(\hat{G})\cap$UCB($\hat{G}$) has $l^\infty(X)$ as a continuous linear image if G is nondiscrete. When G is further assumed to be non-metrizable, it is shown that UCB($\hat{G}$)$/F(\hat{G})\cap$UCB($\hat{G}$) contains a linear isomorphic copy of $l^\infty(X)$. Similar results are also obtained for other quotient spaces.

1. **Introduction.** Let G be a locally compact group, $A(G)$ the Fourier algebra of G and VN(G) the von Neumann algebra generated by the left regular representation $\{\rho, L^2(G)\}$. With the action $u\cdot T$ defined by $\langle u\cdot T, v\rangle = \langle T, uv\rangle$ for $T \in$ VN(G), u, $v \in A(G)$, VN(G) forms an $A(G)$-module. First we list below some subalgebras and/or subspaces of VN(G) of our main interest in this paper.

$$\begin{aligned}
\mathrm{UCB}(\hat{G}) &= \text{the norm closure in VN}(G) \text{ of } \{T \in \mathrm{VN}(G)\,;\, \operatorname{supp} T \text{ is compact}\},\\
W(\hat{G}) &= \{T \in \mathrm{VN}(G)\,;\, \text{the map } u \longmapsto u\cdot T \text{ is weakly compact}\},\\
\mathcal{M}(\hat{G}) &= \text{the norm closure in VN}(G) \text{ of the measure algebra } M(G) \text{ of } G,\\
C^*_\rho(G) &= \text{the reduced } C^*\text{-algebra of } G,\\
F(\hat{G}) &= \{T \in \mathrm{VN}(G)\,;\, m(T) = \text{a fixed constant for all } m \in \mathrm{TIM}(\hat{G})\},\\
F_0(\hat{G}) &= \{T \in \mathrm{VN}(G)\,;\, m(T) = 0 \text{ for all } m \in \mathrm{TIM}(\hat{G})\},
\end{aligned}$$

where TIM($\hat{G}$) denotes the set of all topologically invariant means on VN(G).

As we know, when G is an abelian group with dual group $\hat{G}$, VN(G) is isometric algebra isomorphic to $L^\infty(\hat{G})$. In this situation, UCB($\hat{G}$), $W(\hat{G})$, $C^*_\rho(G)$ (= $C_0(\hat{G})$) are the spaces of uniformly continuous, weakly almost periodic continuous, continuous functions vanishing at ∞ on $\hat{G}$, respectively. Moreover, $\mathcal{M}(\hat{G})$ is the supremum norm closure of $B(\hat{G})$, where $B(\hat{G})$ is the Fourier-Stieltjes algebra of $\hat{G}$.

There are many results in the literature on these subalgebras and/or subspaces of VN(G). In particular, the following inclusive relations are well-known: $W(\hat{G}) \subseteq F(\hat{G})$

Received by the editors March 7, 1996; revised February 20, 1997.
This research is supported by an NSERC grant.
AMS subject classification: Primary: 22D25, 43A22, 43A30; secondary: 22D15, 43A07, 47D35.

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

(see Dunkl and Ramirez [4] and Granirer [8]), $C^*_\rho(G) \subseteq \mathcal{M}(\hat{G}) \subseteq W(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ (see Dunkl and Ramirez [4] and Granirer [11]), and $C^*_\rho(G) \subseteq F_0(\hat{G})$ if G is nondiscrete (see Dunkl and Ramirez [4] and Lau [19]). Granirer showed that $W(\hat{G}) \subseteq \mathrm{UCB}(\hat{G})$ if G is amenable (see [8]), and $C^*_\rho(G) = \mathcal{M}(\hat{G}) = \mathrm{UCB}(\hat{G}) \subseteq W(\hat{G})$ when G is discrete (see [9]). Also, when G is discrete, the equality $F(\hat{G}) = \mathrm{VN}(G)$ holds, because G is discrete if and only if $\mathrm{VN}(G)$ has a unique topologically invariant mean (see Lau and Losert [20] and Renaud [25]).

It is natural to ask whether the above inclusive relations are proper if G is nondiscrete. In this aspect, Granirer proved that the quotient space $\mathrm{UCB}(\hat{G})/W(\hat{G})$ is not norm separable if G is amenable and nondiscrete (see [8, Corollary 13]). In [1], Chou constructed a linear mapping π of $\mathrm{VN}(G)$ onto l^∞ such that π^* maps a big subset (having cardinality 2^c) of $(l^\infty)^*$ into $\mathrm{TIM}(\hat{G})$ when G is metrizable and nondiscrete. It follows that, under the same assumption on G, $\mathrm{VN}(G)/F(\hat{G})$ has l^∞ as a continuous linear image (*i.e.*, has l^∞ as a quotient) and $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ (and hence $\mathrm{UCB}(\hat{G})/W(\hat{G}) \cap \mathrm{UCB}(\hat{G})$) is not norm separable (see [1, Theorem 3.3 and Corollary 3.6]). More generally, we obtained in [17] the following: if G is nondiscrete, then both $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $\mathrm{VN}(G)/F(\hat{G})$ have the density character greater than $b(G)$, where the density character of a Banach space Y is the smallest cardinality such that there exists a norm dense subset of Y having that cardinality and $b(G)$ denotes the smallest cardinality of an open basis at the unit e of G (see [17, Corollary 6.2]). Granirer in [12] investigated quotient spaces of subspaces of $PM_p(G)$, the Banach dual space of the Figà-Talamanca-Gaudry-Herz algebra $A_p(G)$ of G ($1 < p < \infty$ and $A_2(G) = A(G)$). Among many other things, a special case of [12, Theorem 6] implies that $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ has l^∞ as a quotient if G is second countable and nondiscrete. Recently, Granirer improved this result by requiring only that G is metrizable nondiscrete (see [13, Corollary 7]).

The main purpose of this paper is to generalize and strengthen some of these results on the quotient Banach spaces of $\mathrm{UCB}(\hat{G})$ and $\mathrm{VN}(G)$. Here are some details on the organization of this paper.

Section 2 consists of some definitions and notations used throughout this paper.

For an initial ordinal μ, let X be the set of all ordinals less than μ and let $c_0(X)$ ($c(X)$) be the subspace of $l^\infty(X)$ consisting of all f in $l^\infty(X)$ such that $\lim_{\alpha \in X} f(\alpha) = 0$ ($\lim_{\alpha \in X} f(\alpha)$ exists). In Section 3, we characterize $c_0(X)$ and $c(X)$ for uncountable μ and then show that $l^\infty(X)/c_0(X)$ ($l^\infty(X)/c(X)$) contains an isometric (isomorphic) copy of $l^\infty(X)$.

Section 4 concerns itself with some projections in $\mathrm{VN}(G)$ when G is a σ-compact non-metrizable locally compact group. Let μ be the initial ordinal with $|\mu| = b(G)$ and let $X = \{\alpha\,;\, \alpha < \mu\}$. We unveil at first some new properties of the orthogonal net $(Q_\alpha)_{\alpha<\mu}$ of projections in $\mathrm{VN}(G)$ constructed in our [17]. Then we associate $c_0(X)$ with $F_0(\hat{G})$ ($c(X)$ with $F(\hat{G})$) in the following way: $f \in c_0(X)$ ($c(X)$) if and only if $\sum_{\alpha<\mu} f(\alpha) Q_\alpha \in F_0(\hat{G})$ ($F(\hat{G})$), where $\sum_{\alpha<\mu} f(\alpha) Q_\alpha$ denotes the w^*-limit of $\{\sum_{\alpha\in\tau} f(\alpha) Q_\alpha\,;\, \tau \subseteq X \text{ is finite}\}$ in $\mathrm{VN}(G)$ (Lemma 4.5). This association plays an important role in the attempt to establish certain isometric relations between some quotient spaces.

In Section 5, we improve Chou [1, Theorem 3.3] and our [17, Theorem 5.4] and obtain some strong isometric embedding results on some quotient spaces of $\mathrm{UCB}(\hat{G})$

and VN(G). Let G be a nondiscrete locally compact group, μ the initial ordinal with $|\mu| = b(G)$, and $X = \{\alpha\ ;\ \alpha < \mu\}$. We construct an isometric $*$-isomorphism κ of $l^\infty(X)$ into VN(G) and a bounded linear operator π of VN(G) onto $l^\infty(X)$ such that $\pi \circ \kappa = \mathrm{id}_{l^\infty(X)}$ and π^* embeds the big subset $\mathcal{F}(X)$ (having cardinality $2^{2^{|X|}}$) of $l^\infty(X)^*$ into TIM($\hat{G}$) (Theorem 5.1). The construction of this π leads to the conclusion that, for any nondiscrete locally compact group G, $\mathrm{VN}(G)/F(\hat{G})$ and $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ have $l^\infty(X)$ as a quotient (Corollary 5.3). Making use of the isometry κ, we further show that $\mathrm{UCB}(\hat{G})/F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $\mathrm{VN}(G)/F_0(\hat{G})$ ($\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $\mathrm{VN}(G)/F(\hat{G})$) contain an isometric copy of $l^\infty(X)/c_0(X)$ ($l^\infty(X)/c(X)$) if G is non-metrizable (Theorem 5.10).

Combining the embedding results in Sections 3 and 5, we obtain in Section 6 that $\mathrm{VN}(G)/F_0(\hat{G})$ and $\mathrm{UCB}(\hat{G})/F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ ($\mathrm{VN}(G)/F(\hat{G})$ and $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$) contain an isometric (isomorphic) copy of $l^\infty(X)$ if G is non-metrizable (Theorem 6.1). We also give some homomorphism results on other quotient spaces of UCB($\hat{G}$) and VN(G). In particular, $\mathrm{UCB}(\hat{G})/W(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $\mathrm{UCB}(\hat{G})/\mathcal{M}(\hat{G})$ have $l^\infty(X)$ as a quotient when G is nondiscrete (Theorem 6.3). Finally, we extend some of the previous results to spaces of operators in VN(G) with small support.

Let $d(G)$ be the smallest cardinality of a covering of G by compact sets. Note that if G is nondiscrete and if $d(G) \leq b(G)$ (*e.g.*, if G is nondiscrete and σ-compact) then VN(G) is isometric to a subspace of $l^\infty(X)$. Hence the isomorphism and homomorphism results of this paper on quotients of subspaces of VN(G) mean that these quotients are as big as they can be.

The author is indebted to Professor E. E. Granirer for providing the preprints of his papers [12] and [13] and to the referee for valuable suggestions.

2**. Definitions and notations.** Let **C** be the complex field. For a Banach space E over **C**, let E^* denote the Banach space of all bounded linear functionals on E. If $\phi \in E^*$, then the value of ϕ at an element x in E will be written as $\phi(x)$ or $\langle \phi, x\rangle$.

Let G be a locally compact group with unit element e and a fixed left Haar measure λ. The left invariant Haar integral associated with λ will be denoted by $\int_G \cdots dx$. For $1 \leq p \leq \infty$, let $\left(L^p(G), \|\cdot\|_p\right)$ be the usual Banach space associated with G and λ. With the inner product

$$(f, g) = \int_G f(x)\overline{g(x)}\, dx, \quad f, g \in L^2(G),$$

$L^2(G)$ becomes a Hilbert space.

Let VN(G) be the von Neumann algebra generated by the left regular representation $\{\rho, L^2(G)\}$ of G, *i.e.*, the closure of the linear span of $\{\rho(a)\ ;\ a \in G\}$ in the weak operator topology on $\mathcal{B}\big(L^2(G)\big)$, where $\mathcal{B}\big(L^2(G)\big)$ is the Banach algebra of all bounded linear operators on $L^2(G)$ and $[\rho(a)f](x) = f(a^{-1}x)$, $x \in G$, $f \in L^2(G)$.

Let $A(G)$ be the Fourier algebra of G, consisting of all functions of the form $f * \tilde{g}$, where $f, g \in L^2(G)$ and $\tilde{g}(x) = \overline{g(x^{-1})}$. If $\phi = f * \tilde{g} \in A(G)$, then ϕ can be regarded as an ultraweakly continuous functional on VN(G) defined by

$$\phi(T) = (Tf, g), \quad \text{for } T \in \mathrm{VN}(G).$$

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

Furthermore, as shown by P. Eymard in [5, pp. 210, 218], each ultraweakly continuous functional on $\mathrm{VN}(G)$ is of the form $f * \tilde{g}$ with $f, g \in L^2(G)$. Therefore, $A(G)$ is the predual of $\mathrm{VN}(G)$, *i.e.*, $A(G)^* = \mathrm{VN}(G)$. In particular, the w^*-topology (*i.e.*, the $\sigma\big(\mathrm{VN}(G), A(G)\big)$-topology) and the weak operator topology on $\mathrm{VN}(G)$ coincide. Also, $A(G)$ with pointwise multiplication and the norm

$$\|\phi\| = \sup\big\{|\phi(T)|\ ;\ T \in \mathrm{VN}(G) \text{ and } \|T\| \leq 1\big\}$$

forms a commutative Banach algebra.

There is a natural action of $A(G)$ on $\mathrm{VN}(G)$ given by

$$\langle u \cdot T, v\rangle = \langle T, uv\rangle, \quad \text{for } u, v \in A(G), T \in \mathrm{VN}(G).$$

Under this action, $\mathrm{VN}(G)$ becomes a Banach $A(G)$-module. For more details on the algebras $\mathrm{VN}(G)$ and $A(G)$, see Eymard [5].

An $m \in \mathrm{VN}(G)^*$ is called a *topologically invariant mean* on $\mathrm{VN}(G)$, if

(i) $\|m\| = \langle m, I\rangle = 1$, where $I = \rho(e)$ denotes the identity operator,

(ii) $\langle m, u \cdot T\rangle = \langle m, T\rangle$ for $T \in \mathrm{VN}(G)$ and $u \in A(G)$ with $u(e) = 1$.

Let $\mathrm{TIM}(\hat{G})$ be the set of all topologically invariant means on $\mathrm{VN}(G)$. It is known that $\mathrm{TIM}(\hat{G})$ is a non-empty w^*-compact convex subset of $\mathrm{VN}(G)^*$ and it is a singleton if and only if G is discrete (see Renaud [25] and Lau and Losert [20]). In [17], we obtained the exact cardinality $2^{2^{b(G)}}$ of $\mathrm{TIM}(\hat{G})$, where $b(G)$ is the smallest cardinality of an open basis at e when G is nondiscrete. Let $P_1(G) = \{u \in A(G)\ ;\ u$ is positive definite and $\|u\| = u(e) = 1\}$. A net $(\phi_\alpha)_{\alpha\in\Lambda}$ in $P_1(G)$ is said to be *topologically convergent to invariance* if $\lim_\alpha \|v\phi_\alpha - \phi_\alpha\| = 0$, for $v \in A(G)$ with $v(e) = 1$. Then any w^*-cluster point of $(\phi_\alpha)_{\alpha\in\Lambda}$ in $\mathrm{VN}(G)^*$ belongs to $\mathrm{TIM}(\hat{G})$.

Let $T \in \mathrm{VN}(G)$. We say that $x \in G$ is in the support of T, denoted by $\operatorname{supp} T$, if $\rho(x)$ is the ultraweak limit of operators of the form $u \cdot T$, $u \in A(G)$. An equivalent definition for $\operatorname{supp} T$ is that $x \in \operatorname{supp} T$ if and only if $u \cdot T = \mathbf{0}$ implies $u(x) = 0$ for all $u \in A(G)$ (see [5, Proposition 4.4] or [15, p. 119]).

Let $\mathrm{UCB}(\hat{G})$ denote the norm closure of $A(G) \cdot \mathrm{VN}(G)$ in $\mathrm{VN}(G)$. Then $\mathrm{UCB}(\hat{G})$ is a C^*-subalgebra and an $A(G)$-submodule of $\mathrm{VN}(G)$ (see [9]) which coincides with the norm closure of $\{T \in \mathrm{VN}(G)\ ;\ \operatorname{supp} T \text{ is compact}\}$. When G is an abelian group, $\mathrm{UCB}(\hat{G})$ is isometrically algebra isomorphic to the algebra of bounded uniformly continuous functions on the dual group $\hat{G}$ of G. For this reason, operators in $\mathrm{UCB}(\hat{G})$ are called uniformly continuous functionals on $A(G)$ (see [8]). The C^*-algebra $\mathrm{UCB}(\hat{G})$ and its relationship with other C^*-subalgebras of $\mathrm{VN}(G)$ have been studied by Granirer in [8] and [9] and by Lau in [19]. See Lau and Losert [20] for recent developments on this C^*-algebra and its dual space.

Chou used $F(\hat{G})$ to denote the space of all $T \in \mathrm{VN}(G)$ such that $m(T)$ equals a fixed constant $d(T)$ as m runs through $\mathrm{TIM}(\hat{G})$ and called $F(\hat{G})$ the space of topological almost convergent elements in $\mathrm{VN}(G)$. It is easy to check that $F(\hat{G})$ is a norm closed self-adjoint $A(G)$-submodule of $\mathrm{VN}(G)$. See Chou [1] for more information on $F(\hat{G})$. We denote by $F_0(\hat{G})$ the space of all $T \in F(\hat{G})$ such that $d(T) = 0$. $F_0(\hat{G})$ is also a norm closed self-adjoint $A(G)$-submodule of $\mathrm{VN}(G)$ and $F(\hat{G}) = \mathbf{C}I \oplus F_0(\hat{G})$.

Dunkl-Ramirez in [4] called $\{T \in \mathrm{VN}(G) ; u \longmapsto u \cdot T$ is a weakly compact operator of $A(G)$ into $\mathrm{VN}(G)\}$ the space of weakly almost periodic functionals of $A(G)$ and denoted it by $W(\hat{G})$. It turns out that $W(\hat{G})$ is a self-adjoint closed $A(G)$-submodule of $\mathrm{VN}(G)$ which coincides with the space of weakly almost periodic functions in $L^\infty(\hat{G})$ when G is abelian (see [4] for more details).

Let $M(G)$ denote the measure algebra of G, *i.e.*, the space of finite regular Borel measures on G with convolution as the multiplication. $M(G)$ can be considered as a subspace of $\mathrm{VN}(G)$ by

$$\langle \mu, u \rangle = \int_G \check{u}\, d\mu, \quad \text{for } u \in A(G),$$

where $\check{u}(x) = u(x^{-1})$, $x \in G$. Now $\|\mu\|_{\mathrm{VN}(G)} \leq \|\mu\|_{M(G)}$. In particular, if $f \in L^1(G)$, then $\langle f, u \rangle = \int_G f \check{u}\, dx$, $u \in A(G)$, and $\|f\|_{\mathrm{VN}(G)} \leq \|f\|_{L^1(G)}$. Let $\mathcal{M}(\hat{G})$ and $C^*_\rho(G)$ be the norm closures of $M(G)$ and $L^1(G)$ in $\mathrm{VN}(G)$, respectively. $C^*_\rho(G)$ is just the reduced C^*-algebra of G, *i.e.*, the norm closure of $\{\rho(f) ; f \in L^1(G)\}$ in $\mathcal{B}\big(L^2(G)\big)$, where $\rho(f)(h) = f * h$ for each $h \in L^2(G)$.

It is known that $W(\hat{G})$ has a unique topologically invariant mean (see [4] and [8]). In particular, this gives that $W(\hat{G}) \subseteq F(\hat{G})$. Also, $C^*_\rho(G) \subseteq \mathcal{M}(\hat{G}) \subseteq W(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ (see [4] and [11]) and $C^*_\rho(G) \subseteq F_0(\hat{G})$ if G is nondiscrete (see [4, Theorem 2.12] and [19, Proposition 4.2]). The inclusion $W(\hat{G}) \subseteq \mathrm{UCB}(\hat{G})$ was obtained by Granirer when G is amenable (see [8]). In the same paper, Granirer observed that if G is amenable then $\mathrm{UCB}(\hat{G}) = A(G) \cdot \mathrm{VN}(G)$. The converse is shown true by Chou for discrete groups and Lau and Losert for general case (see [20]).

Let E_1, E_2 be two Banach spaces. We say that E_2 contains an *isometric copy* of E_1 if there is a linear mapping $L \colon E_1 \longrightarrow E_2$ such that $\|Lx\| = \|x\|$ for all $x \in E_1$; E_2 contains an *isomorphic copy* of E_1 if there is a linear mapping $L \colon E_1 \longrightarrow E_2$ and some positive constants γ_1, γ_2 such that $\gamma_1\|x\| \leq \|Lx\| \leq \gamma_2\|x\|$ for all $x \in E_1$; E_2 has E_1 *as a quotient* if there is a bounded linear mapping from E_2 onto E_1.

A Banach space X is called *injective* if for any pair of Banach spaces $Y \subseteq Z$ and every bounded linear mapping T of Y into X there is a bounded linear mapping $\hat{T}$ of Z into X which extends T. Note that, if X is an infinite set, then $l^\infty(X)$ is injective (see [21, p. 105]).

If Y is a Banach space, we denote by $D(Y)$ the *density character* of Y, *i.e.*, the smallest cardinality such that there exists a norm dense subset of Y having that cardinality. $D(l^\infty(X)) = 2^{|X|}$ for any infinite set X.

3. **$l^\infty(X)$ and its subspaces and quotient spaces.** For any two sets A and B, $A \setminus B$ denotes their difference, 1_A denotes the characteristic function of A as a subset of the underlying set, 2^A is the set of all functions from A to $\{0, 1\}$, and $|A|$ is the cardinality of A. Then $|2^A| = 2^{|A|}$, the cardinality of the power set of A. So we also use 2^A to denote the power set of A.

If X is a set, let $l^\infty(X)$ be the Banach space of all bounded complex-valued functions on X with the supremum norm. When X is a directed set, we define two subspaces of X as following:

$$c_0(X) = \{f \in l^\infty(X) \,;\, \lim_{\alpha\in X} f(\alpha) = 0\},$$
$$c(X) = \{f \in l^\infty(X) \,;\, \lim_{\alpha\in X} f(\alpha) \text{ exists}\}.$$

Obviously, $c_0(X) \subseteq c(X)$ and $c(X) = \mathbf{C1} \oplus c_0(X)$, where $\mathbf{1}$ is the constant function of value one. When $X = \mathbf{N}$, the set of all positive integers, $l^\infty(X)$, $c_0(X)$ and $c(X)$ are l^∞, c_0 and c, respectively.

When α is an ordinal number, $|\alpha|$ means the cardinality of the set $\{\beta \,;\, \beta$ is an ordinal and $\beta < \alpha\}$. An ordinal α is called an *initial ordinal* if $|\alpha|$ is infinite and $\beta < \alpha$ implies $|\beta| < |\alpha|$ (see [26, p. 271]).

Let μ be an initial ordinal and let $X = \{\alpha \,;\, \alpha$ is an ordinal and $\alpha < \mu\}$. An element of $l^\infty(X)$ is called a *simple function* if it is of the form $\sum_{i=1}^n c_i 1_{E_i}$, where c_i is a constant and E_i is an interval in X, $i = 1, 2, \cdots, n$. Let

$$s(X) = \text{the norm closure of all simple functions in } l^\infty(X).$$

Then $s(X)$ is a closed subspace of $l^\infty(X)$ and $s(X) \subseteq c(X)$. If $X = \mathbf{N}$, then $s(X) = c(X) = c$. If $|\mu| > \aleph_0$, the first infinite cardinal number, then $s(X) \subsetneqq c(X)$ but $c \subseteq s(X)$ and $s(X)$ is not norm separable.

We give at first the following characterizations of $c_0(X)$ and $c(X)$ for a uncountable initial ordinal μ.

LEMMA 3.1. *Let μ be an initial ordinal with $|\mu| > \aleph_0$. Let $X = \{\alpha \,;\, \alpha$ is an ordinal and $\alpha < \mu\}$. Then*

(i) $c_0(X) = \{f \in l^\infty(X) \,;$ there exists an $\alpha_\circ < \mu$ such that $f(\alpha) = 0$ for all $\alpha_\circ \leq \alpha < \mu\}$,

(ii) $c(X) = \{f \in l^\infty(X) \,;$ there exists an $\alpha_\circ < \mu$ and a constant a such that $f(\alpha) = a$ for all $\alpha_\circ \leq \alpha < \mu\}$.

PROOF. Obviously, the set $\{f \in l^\infty(X) \,;$ there exists an $\alpha_\circ < \mu$ such that $f(\alpha) = 0$ for all $\alpha_\circ \leq \alpha < \mu\}$ is contained in $c_0(X)$.

Conversely, if $f \in c_0(X)$, then there exists a sequence $\alpha_1 < \alpha_2 < \cdots < \mu$ such that

$$|f(\alpha)| < \frac{1}{n}, \quad \text{for all } \alpha_n \leq \alpha < \mu, n = 1, 2, \ldots .$$

Let $[0, \alpha_n)$ denote the interval $\{\alpha \,;\, \alpha < \alpha_n\}$. Then $|[0, \alpha_n)| = |\alpha_n| < |\mu|$ for $n = 1, 2, \ldots$. By the König-Zermelo's inequality (see [26, p. 313]),

$$|\bigcup_{n=1}^\infty [0, \alpha_n)| \leq \sum_{n=1}^\infty |\alpha_n| < \prod_{n=1}^\infty |\mu| = |\mu|^{\aleph_0} = |\mu|,$$

since $|\mu| > \aleph_0$. Choose $\alpha_0 \in X \setminus \bigcup_{n=1}^{\infty}[0, \alpha_n)$. Then $\alpha_0 < \mu$ and $f(\alpha) = 0$ for all $\alpha_0 \leq \alpha < \mu$. Therefore, (i) is true.

(ii) follows from (i) since $c(X) = \mathbf{C1} \oplus c_0(X)$. ∎

For a compact topological space Ω, let $C(\Omega)$ be the Banach space of all continuous functions on Ω with the supremum norm. If X is a set (with the discrete topology), βX denotes the Stone-Čech compactification of X. Then $l^\infty(X)$ is isometrically isomorphic to $C(\beta X)$. Thus βX can be identified with the spectrum of $l^\infty(X)$, *i.e.*, the set of all nonzero multiplicative linear functionals on $l^\infty(X)$ with the Gelfand topology (see, say, [28, Proposition 4.5, p. 18]). In this way, each $x \in X$ is identified with the evaluation $\hat{x}$ on $l^\infty(X)$ at x, *i.e.*, $\hat{x}(f) = f(x)$ for $f \in l^\infty(X)$. On the other hand, βX can also be obtained by "fixing" the free ultrafilters on X, that is, $\beta X = \{\text{all ultrafilters on } X\}$ with $\{Z^* ; Z \subseteq X\}$ as a base for closed subsets of βX, where $Z^* = \{\phi \in \beta X ; Z \in \phi\}$ (see [6, pp. 86–87]). Now, every $x \in X$ corresponds to the fixed ultrafilter ϕ_x on X containing $\{x\}$, *i.e.*, $\phi_x = \{E ; x \in E \subseteq X\}$.

Making use of the Stone-Čech compactification of X, now we consider the embeddings of $l^\infty(X)$ into its quotient spaces.

LEMMA 3.2. *Let μ be an initial ordinal and let $X = \{\alpha ; \alpha$ is an ordinal and $\alpha < \mu\}$. Then*

(i) $l^\infty(X)/c_0(X)$ contains an isometric copy of $l^\infty(X)$,

(ii) $l^\infty(X)/c(X)$ contains an isomorphic copy of $l^\infty(X)$.

PROOF. When $X = \mathbf{N}$, this was shown by Granirer (see [10, p. 161]). In the following, we assume that $|\mu| > \aleph_0$. We now follow an argument of Granirer [10].

Since $|X \times X| = |X| = |\mu|$, we can write $X = \bigcup_{\alpha<\mu} A_\alpha$, where $|A_\alpha| = |X|$ and $A_\alpha \cap A_\beta = \emptyset$ for all $\alpha, \beta < \mu$ and $\alpha \neq \beta$. For any $\alpha < \mu$, A_α and X are cofinal, *i.e.*, $A_\alpha \cap [\beta, \mu) \neq \emptyset$ for all $\beta < \mu$, since μ is an initial ordinal and $|\mu| = |X| = |A_\alpha|$, where $[\beta, \mu)$ denotes the interval $\{\alpha ; \beta \leq \alpha < \mu\}$. Let

$$Y_0 = \{f \in l^\infty(X) ; f(A_0) = 0, f(A_\alpha) = c_\alpha, 0 < \alpha < \mu\},$$

i.e., the functions in $l^\infty(X)$ which are zero on A_0 and constant on each A_α. Then Y_0 is an isometric copy of $l^\infty(X)$.

Let X_0 be the closure in βX of the set $\{\varphi \in \beta X ; \varphi$ is a cluster point of the net $(\alpha)_{\alpha<\mu}$ in $\beta X\}$. If $f \in l^\infty(X)$, let $\bar{f} \in C(\beta X)$ be its unique extension and let $\tilde{f} = \bar{f}\,|_{X_0}$. Then the mapping $f \longmapsto \tilde{f}$ from Y_0 to $C(X_0)$ satisfies $\|f\|_\infty = \|\tilde{f}\|_{C(X_0)}$ for all $f \in Y_0$, since each $f \in Y_0$ is constant on each A_α. Thus, $C(X_0)$ contains an isometric copy Y_0 of $l^\infty(X)$.

To prove (i), we only have to show that $C(X_0)$ is isometric to $l^\infty(X)/c_0(X)$. If $\tilde{f} \in C(X_0)$, by Tietze's extension theorem, $\tilde{f}$ has an extension $\bar{f} \in C(\beta X)$. Let $f = \bar{f}|_X \in l^\infty(X)$. We define $L\colon C(X_0) \longrightarrow l^\infty(X)/c_0(X)$ by $L(\tilde{f}) = f + c_0(X)$. Then L is well-defined. Obviously, L is linear and onto. Observe that $\|\tilde{f}\| = \lim_\alpha \sup|f(\alpha)|$. Therefore, by Lemma 3.1, $\|\tilde{f}\| = \|f + c_0(X)\|$ for all $\tilde{f} \in C(X_0)$, *i.e.*, L is a linear isometry from $C(X_0)$ onto $l^\infty(X)/c_0(X)$.

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

Now, let us prove (ii). Let $\varphi_0 \in \beta X$ be a cluster point of the net $(\alpha)_{\alpha \in A_0}$, where A_0 is ordered by its natural way. Then $\varphi_0 \in X_0$. We define the projection $P: C(X_0) \longrightarrow \mathbf{C1}$ by $P\tilde{f} = \tilde{f}(\varphi_0)\mathbf{1}$. Let $Q = I - P$. Then $C(X_0) = \mathbf{C1} \oplus Q[C(X_0)]$. If $f \in Y_0$, then $Q\tilde{f} = \tilde{f} - \tilde{f}(\varphi_0)\mathbf{1} = \tilde{f}$, since $f(A_0) = 0$. Thus $\tilde{Y}_0 \subseteq Q[C(X_0)]$, where $\tilde{Y}_0 = \{\tilde{f} ; f \in Y_0\}$ which is isometric to $l^\infty(X)$. Let $L: C(X_0) \longrightarrow l^\infty(X)/c_0(X)$ be the linear isometry given in the previous paragraph. It is easy to see that $L(\mathbf{C1}) = c(X)/c_0(X)$. So $C(X_0)/\mathbf{C1}$ is isometric to $\big(l^\infty(X)/c_0(X)\big)/\big(c(X)/c_0(X)\big)$ which is isometric to $l^\infty(X)/c(X)$. But $Q[C(X_0)]$ is isomorphic to $C(X_0)/\mathbf{C1}$. Therefore, $l^\infty(X)/c(X)$ contains an isomorphic copy $\tilde{Y}_0$ of $l^\infty(X)$. The proof is completed. ∎

REMARK 3.3. (i) If $X = \mathbf{N}$, the set X_0 considered in the above proof is just $\beta\mathbf{N} \setminus \mathbf{N}$. But for uncountable X, $X_0 \underset{\neq}{\subset} \beta X \setminus X$.

(ii) We do not know whether $l^\infty(X)/c(X)$ contains an isometric copy of $l^\infty(X)$.

4. **Non-metrizable groups and orthogonal projections in** VN(G)**.** In this section, G will always be a σ-compact non-metrizable locally compact group. Let $b(G)$ be the smallest cardinality of an open basis at the unit element e of G. Trivially, we have $b(G) > \aleph_0$ (the first infinite cardinal number). Let μ be the initial ordinal with $|\mu| = b(G)$ and let $X = \{\alpha ; \alpha \text{ is an ordinal and } \alpha < \mu\}$.

In [17], we showed an important property of G concerning its local structure at e. Using this property, we constructed an orthogonal net of projections in VN(G) and a family of orthogonal nets in $P_1(G)$ which is topologically convergent to invariance. For convenience, we would like to collect some of our results in [17] here.

LEMMA 4.1 ([17, PROPOSITION 4.3]). *There exists a decreasing family $(N_\alpha)_{\alpha \le \mu}$ of normal subgroups of G (i.e., $\alpha \le \beta$ implies $N_\alpha \supseteq N_\beta$) such that*

(i) $N_0 = G$ and $N_\mu = \{e\}$;

(ii) N_α is compact for each $\alpha > 0$;

(iii) $N_\alpha / N_{\alpha+1}$ is metrizable but $N_{\alpha+1} \neq N_\alpha$ for all $\alpha < \mu$;

(iv) $N_\gamma = \bigcap_{\alpha<\gamma} N_\alpha$ for every limit ordinal $\gamma \le \mu$;

(v) $b(N_\alpha) = b(G)$ for all $\alpha < \mu$.

Furthermore, μ is minimal among all such families.

REMARK 4.2. (a) The main idea in constructing $(N_\alpha)_{\alpha \le \mu}$ is essentially the same as that used in Lau and Losert [20]. The net $(N_\alpha)_{\alpha \le \lambda}$ in [20] possesses property (i)–(iv). It is strengthened in [17] in the following two aspects: (1) the ordinal λ is totally determined by the local structure of G $\big(|\lambda| = b(G)\big)$; (2) $b(N_\alpha) = b(G)$ for all $\alpha < \mu$.

(b) Examining the proof of Lemma 4.1 (see [17]), we find that the family $(N_\alpha)_{\alpha \le \mu}$ can be chosen such that $\lambda(N_1) = 0$, where λ is the left Haar measure of G. This fact will be used later.

Due to the nature of $(N_\alpha)_{\alpha \le \mu}$, we can define a family $(P_\alpha)_{\alpha < \mu}$ of projections in VN(G). Let $P_0 = \mathbf{0} \in$ VN(G). For $0 < \alpha < \mu$, let $P_\alpha \in$ VN(G) be the central projection defined by convolution with the normalized Haar measure λ_α of N_α. More explicitly,

$P_\alpha: L^2(G) \longrightarrow L^2(G/N_\alpha)\big(\subseteq L^2(G)\big)$ is given by

$$(P_\alpha f)(x) = \int_{N_\alpha} f(t^{-1}x)\, d\lambda_\alpha(t), \quad f \in L^2(G), 0 < \alpha < \mu,$$

where $L^2(G/N_\alpha)$ is the subspace of $L^2(G)$ consisting of all functions in $L^2(G)$ which are constant on the cosets of N_α (see [5, (3.23)]).

Now $(P_\alpha)_{\alpha<\mu}$ is an increasing net of projections in $\mathrm{VN}(G)$, *i.e.*, $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for $\alpha < \beta < \mu$. Define

$$Q_\alpha = P_{\alpha+1} - P_\alpha, \quad \alpha < \mu.$$

Then $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $\mathrm{VN}(G)$, that is,

$$Q_\alpha Q_\beta = \begin{cases} Q_\alpha & \text{if } \alpha = \beta, \\ \mathbf{0} & \text{if } \alpha \neq \beta. \end{cases}$$

Let J be a set with $|J| = b(G)$ and let $\{U_j \, ; \, j \in J\}$ be an open basis at e. For each $j \in J$ and $\alpha < \mu$, we showed in [17] that there exists a $u^j_\alpha \in P_1(G)$ such that $\operatorname{supp} u^j_\alpha \subseteq U_j N_\alpha$ and

$$\langle Q_\beta, u^j_\alpha \rangle = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Direct $J \times X$ by $(i, \alpha) \leq (j, \beta)$ if and only if $U_j \subseteq U_i$ and $\alpha \leq \beta$.

LEMMA 4.3 ([17, LEMMA 5.2]). *The net $(u^j_\alpha)_{(j,\alpha)\in J\times X}$ has the following properties.*

(i) $u^j_\alpha \in P_1(G)$ *and* $\operatorname{supp} u^j_\alpha \subseteq U_j N_\alpha$ *for all* $(j, \alpha) \in J \times X$.

(ii) For each fixed $j \in J$, $(u^j_\alpha)_{\alpha\in X}$ *is a mutually orthogonal net in* $P_1(G)$, *i.e.*

$$\|u^j_\alpha - u^j_\beta\| = \|u^j_\alpha\| + \|u^j_\beta\| = 2, \quad \text{for all } \alpha, \beta < \mu \text{ with } \alpha \neq \beta.$$

(iii) $(u^j_\alpha)_{(j,\alpha)\in J\times X}$ *is topologically convergent to invariance.*

Let $\Lambda = \Lambda(X)$ be the set of all non-empty finite subsets of X directed by inclusion. Let $f \in l^\infty(X)$. For each $\tau \in \Lambda$, let $S_\tau = \sum_{\alpha\in\tau} f(\alpha) Q_\alpha$. Since $(Q_\alpha)_{\alpha<\mu}$ is an orthogonal net of projections in $\mathrm{VN}(G)$ and $f \in l^\infty(X)$, we have

$$\|S_\tau\| \leq \|f\|_\infty, \quad \text{for all } \tau \in \Lambda,$$

and the net $(S_\tau)_{\tau\in\Lambda}$ is convergent in the weak operator topology (or equivalently, the $\sigma\big(\mathrm{VN}(G), A(G)\big)$-topology) to an operator $T \in \mathrm{VN}(G)$ with $\|T\| \leq \|f\|_\infty$. We denote T by $\sum_{\alpha<\mu} f(\alpha) Q_\alpha$. Also, for any subset E of X, $\sum_{\alpha\in E} f(\alpha) Q_\alpha$ means $\sum_{\alpha<\mu} (f 1_E)(\alpha) Q_\alpha$, where 1_E is the characteristic function of E.

Here now, we present a few more properties of the orthogonal net $(Q_\alpha)_{\alpha<\mu}$ of projections in $\mathrm{VN}(G)$.

LEMMA 4.4.

(i) For each $\alpha < \mu$, $Q_\alpha \in \mathcal{M}(\hat{G}) \cap F_0(\hat{G})$.

(ii) Let $f \in l^\infty(X)$. *Then* $\|\sum_{\alpha<\mu} f(\alpha) Q_\alpha\| = \|f\|_\infty$.

(iii) $\sum_{\beta<\alpha} Q_\beta = P_\alpha$ *for all* $0 < \alpha < \mu$.

(iv) $\sum_{\alpha<\mu} Q_\alpha = I$ *(the identity operator in* $\mathrm{VN}(G)$*).*

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

PROOF. (i) Let $\alpha < \mu$. $Q_\alpha \in M(\hat{G})$ follows from the definition of Q_α. Let $m \in \mathrm{TIM}(\hat{G})$. Then, by [4, Theorem 12],

$$\langle m, Q_\alpha \rangle = \langle m, P_{\alpha+1} - P_\alpha \rangle = \lambda_{\alpha+1}\big(\{e\}\big) - \lambda_\alpha(\{e\}) = 0 - 0 = 0,$$

since N_β is nondiscrete for each $\beta < \mu$, where λ_β is the left Haar measure of N_β. Therefore, $Q_\alpha \in F_0(\hat{G})$.

(ii) We only have to prove that $\|f\|_\infty \leq \|\sum_{\alpha<\mu} f(\alpha)Q_\alpha\|$. Let $\beta \in X$. Take a $j \in J$. Then

$$\begin{aligned} |f(\beta)| &= |f(\beta)\langle Q_\beta, u_\beta^j \rangle| \\ &= \lim_{\tau\in\Lambda}\Big|\Big\langle \sum_{\alpha\in\tau} f(\alpha)Q_\alpha, u_\beta^j\Big\rangle\Big| \\ &= \Big|\Big\langle \sum_{\alpha<\mu} f(\alpha)Q_\alpha, u_\beta^j\Big\rangle\Big| \\ &\leq \Big\|\sum_{\alpha<\mu} f(\alpha)Q_\alpha\Big\| \|u_\beta^j\| \\ &= \Big\|\sum_{\alpha<\mu} f(\alpha)Q_\alpha\Big\|. \end{aligned}$$

Therefore, $\|f\|_\infty = \sup\{|f(\beta)|\ ; \beta < \mu\} \leq \|\sum_{\alpha<\mu} f(\alpha)Q_\alpha\|$.

(iii) For each $\alpha < \mu$, let

$$\begin{aligned} Y_\alpha &= P_\alpha\big[L^2(G)\big] = L^2(G/N_\alpha), \\ Z_\alpha &= Q_\alpha\big[L^2(G)\big] = (P_{\alpha+1} - P_\alpha)\big[L^2(G)\big]. \end{aligned}$$

Then $Y_1 = Z_0$ and $Y_{\alpha+1} = Y_\alpha \oplus Z_\alpha$ for all $\alpha < \mu$, where $\oplus$ denotes the direct sum of Hilbert spaces. If $\alpha_0 < \mu$ is a limit ordinal, then $Y_{\alpha_0} = \overline{\bigcup_{\alpha<\alpha_0} Y_\alpha}^{\|\cdot\|_2}$ by the Stone-Weierstrass theorem because of the fact $N_{\alpha_0} = \bigcap_{\alpha<\alpha_0} N_\alpha$ (by Lemma 4.1).

Let $0 < \alpha_0 < \mu$. Assume that, for all $0 < \alpha < \alpha_0$, $Y_\alpha = \oplus_{\beta<\alpha} Z_\beta$. If $\alpha_0 = \alpha + 1$, then $Y_{\alpha_0} = Y_\alpha \oplus Z_\alpha = (\oplus_{\beta<\alpha} Z_\beta) \oplus Z_\alpha = \oplus_{\beta<\alpha_0} Z_\beta$. Let $\alpha_0 < \mu$ be a limit ordinal. Obviously, $\oplus_{\beta<\alpha_0} Z_\beta \subseteq Y_{\alpha_0}$. But $Y_{\alpha_0} = \overline{\bigcup_{\alpha<\alpha_0} Y_\alpha}^{\|\cdot\|_2}$ and $\oplus_{\beta<\alpha_0} Z_\beta$ is closed in $L^2(G)$. By the assumption, we have $Y_{\alpha_0} = \oplus_{\beta<\alpha_0} Z_\beta$. By the transfinite induction, $Y_\alpha = \oplus_{\beta<\alpha} Z_\beta$ for all $0 < \alpha < \mu$. Therefore, $P_\alpha = \sum_{\beta<\alpha} Q_\beta$ for all $0 < \alpha < \mu$.

(iv) Similarly, $L^2(G) = \overline{\bigcup_{\alpha<\mu} Y_\alpha}^{\|\cdot\|_2}$ and $L^2(G) = \oplus_{\alpha<\mu} Z_\alpha$. Therefore, $I = \sum_{\alpha<\mu} Q_\alpha$. ∎

Recall that $F(\hat{G})$ $\big(F_0(\hat{G})\big)$ is the space of all $T \in \mathrm{VN}(G)$ such that $m(T)$ equals to a fixed constant $d(T)\big(m(T) = 0\big)$ for all $m \in \mathrm{TIM}(\hat{G})$. Then $F_0(\hat{G}) \subseteq F(\hat{G})$ and $F(\hat{G}) = \mathbf{C}I \oplus F_0(\hat{G})$. We associate the space $c_0(X)$ with $F_0(\hat{G})$ ($c(X)$ with $F(\hat{G})$) in the following lemma.

LEMMA 4.5. *Let $f \in l^\infty(X)$. Then*

(i) $f \in c_0(X)$ if and only if $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in F_0(\hat{G})$,

(ii) $f \in c(X)$ if and only if $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in F(\hat{G})$.

PROOF. ($\Rightarrow$) Let $f \in c_0(X)$. We may assume that $f \geq 0$. By Lemma 3.1, there exists an $\alpha_0 < \mu$ such that $f(\alpha) = 0$ for all $\alpha_0 \leq \alpha < \mu$. By Lemma 4.4,

$$0 \leq \sum_{\alpha<\mu} f(\alpha)Q_\alpha = \sum_{\alpha<\alpha_0} f(\alpha)Q_\alpha \leq \|f\|_\infty P_{\alpha_0}.$$

Let $m \in \mathrm{TIM}(\hat{G})$. Then

$$0 \leq \Big\langle m, \sum_{\alpha<\mu} f(\alpha)Q_\alpha \Big\rangle \leq \|f\|_\infty \langle m, P_{\alpha_0}\rangle.$$

But $\langle m, P_{\alpha_0}\rangle = \lambda_{\alpha_0}(\{e\})$ (by [4, Theorem 2.12]) and $\lambda_{\alpha_0}(\{e\}) = 0$ (since N_{α_0} is nondiscrete by Lemma 4.1). Therefore,

$$\Big\langle m, \sum_{\alpha<\mu} f(\alpha)Q_\alpha \Big\rangle = 0, \quad \text{for all } m \in \mathrm{TIM}(\hat{G}),$$

i.e., $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in F_0(\hat{G})$.

If $f \in c(X)$, say, $\lim_\alpha f(\alpha) = a$, then $g = f - a\mathbf{1} \in c_0(X)$ and hence $\sum_{\alpha<\mu} g(\alpha)Q_\alpha \in F_0(\hat{G})$. By Lemma 4.4,

$$\sum_{\alpha<\mu} (a\mathbf{1})(\alpha)Q_\alpha = a \sum_{\alpha<\mu} Q_\alpha = aI.$$

Therefore, $\sum_{\alpha<\mu} f(\alpha)Q_\alpha = \sum_{\alpha<\mu} g(\alpha)Q_\alpha + aI \in F(\hat{G})$.

($\Leftarrow$) Suppose that $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in F(\hat{G})$. Recall that the net $(u^j_\alpha)_{j,\alpha}$ is topologically convergent to invariance (Lemma 4.3). By Chou [1, Theorem 4.4], there exists a constant a such that $\lim_{j,\alpha} u^j_\alpha \cdot [\sum_{\beta<\mu} f(\beta)Q_\beta] = aI$ in norm ($a = d\big(\sum_{\beta<\mu} f(\beta)Q_\beta\big)$). Choose $v \in A(G)$ with $v(e) = 1$. Then

$$\begin{aligned} a = \langle aI, v\rangle &= \lim_{j,\alpha} \Big\langle u^j_\alpha \cdot \sum_{\beta<\mu} f(\beta)Q_\beta, v\Big\rangle \\ &= \lim_{j,\alpha} \Big\langle \sum_{\beta<\mu} f(\beta)Q_\beta, u^j_\alpha v\Big\rangle \\ &= \lim_{j,\alpha} \Big\langle \sum_{\beta<\mu} f(\beta)Q_\beta, u^j_\alpha\Big\rangle \\ &= \lim_{\alpha} f(\alpha), \end{aligned}$$

i.e., $f \in c(X)$ and $\lim_\alpha f(\alpha) = a$.

If $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in F_0(\hat{G})$, then $a = 0$ and hence $f \in c_0(X)$. ∎

REMARK 4.6. (i) In the proof of Lemma 4.4, by applying the orthogonal net $(Q_\alpha)_{\alpha<\mu}$ of projections in $\mathrm{VN}(G)$, we actually obtained a decomposition of $L^2(G)$, *i.e.*, $L^2(G)$ is the direct sum $\oplus_{\alpha<\mu} Q_\alpha[L^2(G)]$.

(ii) In [1], Chou called elements of $F(\hat{G})$ topological almost convergent. The concept "almost convergence" was originally introduced by Lorentz [22] for the sequence space l^∞. An equivalent condition for $f \in l^\infty$ to be almost convergent is that there exists a constant l such that $\lim_{n,p}\big[\frac{1}{p}\sum_{i=1}^{p} f(n+i)\big] = l$. Parallelly, we can extend this notion to

$l^\infty(X)$ in the following way: $f \in l^\infty(X)$ is almost convergent if there exists a constant l such that $\lim_{\alpha\in X, p\in\mathbf{N}}\Big[\frac{1}{p}\sum_{i=1}^{p} f(\alpha+i)\Big] = l$. The set of all such functions is denoted by $ac(X)$. Then $ac(X)$ is a closed subspace of $l^\infty(X)$, $c(X) \subseteq ac(X)$ but $c(X) \neq ac(X)$ (*e.g.*, let $f(\alpha) = 1$ if α is even and $f(\alpha) = 0$ if α is odd, then $f \in ac(X)$ (with $l = \frac{1}{2}$) but $f \notin c(X)$). In general, $ac(X)$ is much larger and more complicated than $c(X)$. For instance, we know that c is separable but $ac(\mathbf{N})$ is not separable. However, from Lemma 4.5 (also some results in next section), we see that, when we investigate the topological almost convergence in $\mathrm{VN}(G)$, the subspace of $l^\infty(X)$ corresponding to $F(\hat{G})$ is $c(X)$ rather than $ac(X)$.

5. **Results concerning isometric mappings.** Let G be a nondiscrete locally compact group. Let $b(G)$ be the smallest cardinality of an open basis at the unit element e of G. Let μ be the initial ordinal satisfying $|\mu| = b(G)$ and let

$$X = \{\alpha\ ;\ \alpha \text{ is an ordinal and } \alpha < \mu\}.$$

In [17], we defined a subset of $l^\infty(X)^*$ as following:

$$\mathcal{F}(X) = \Big\{\phi \in l^\infty(X)^*\ ;\ \|\phi\| = \phi(\mathbf{1}) = 1 \text{ and } \phi(f) = 0 \text{ if } f \in c_0(X)\Big\}.$$

If $X = \mathbf{N}$, $|\mathcal{F}(\mathbf{N})| = 2^{2^{\aleph_0}}$ since $\beta\mathbf{N}\setminus\mathbf{N} \subseteq \mathcal{F}(\mathbf{N})$ and $|\beta\mathbf{N}\setminus\mathbf{N}| = 2^{2^{\aleph_0}}$. We showed in [17] that $|\mathcal{F}(X)| = 2^{2^{|X|}}$ if $|\mu| > \aleph_0$ (see [17, Proposition 3.3]).

When G is metrizable and nondiscrete, Chou constructed a bounded linear mapping π of $\mathrm{VN}(G)$ onto l^∞ such that π^* embeds the large set $\mathcal{F}(\mathbf{N})$ into $\mathrm{TIM}(\hat{G})$ (see [1, Theorem 3.3]). In the case that G is non-metrizable, we built in [17] a family of bounded linear operators of $\mathrm{VN}(G)$ onto $l^\infty(X)$ and then obtained a one-one map $W\colon l^\infty(X)^* \longrightarrow 2^{\mathrm{VN}(G)^*}$ such that $W\big(l^\infty(X)^*\big) \subseteq 2^{\mathrm{TIM}(\hat{G})}$. The above results are substantially improved by the following theorem. For any nondiscrete locally compact group G, we will construct not only a sole bounded linear mapping π of $\mathrm{VN}(G)$ onto $l^\infty(X)$ satisfying $\pi^*\big(\mathcal{F}(X)\big) \subseteq \mathrm{TIM}(\hat{G})$ but also an isometric $*$-isomorphism κ of $l^\infty(X)$ into $\mathrm{VN}(G)$ such that $\pi\circ\kappa = \mathrm{id}_{l^\infty(X)}$.

THEOREM 5.1. *Let G be a nondiscrete locally compact group. Then there exists an isometric $*$-isomorphism κ of $l^\infty(X)$ into $\mathrm{VN}(G)$ and a positive linear mapping π of $\mathrm{VN}(G)$ onto $l^\infty(X)$ with $\|\pi\| = 1$ such that*

(a) $\pi\circ\kappa = \mathrm{id}_{l^\infty(X)}$ *and hence* $\pi^*\colon l^\infty(X)^* \longrightarrow \mathrm{VN}(G)^*$ *is isometric into and* $\kappa^*\colon \mathrm{VN}(G)^* \longrightarrow l^\infty(X)^*$ *is linear onto with* $\|\kappa^*\| = 1$;

(b) $\pi^*\big(\mathcal{F}(X)\big) \subseteq \mathrm{TIM}(\hat{G})$ *and* $\mathcal{F}(X) \subseteq \kappa^*\big(\mathrm{TIM}(\hat{G})\big)$.

PROOF. The existence of π for metrizable group G is due to Chou (see [1, Theorem 3.3]). In this case, we define $\kappa\colon l^\infty \longrightarrow \mathrm{VN}(G)$ by

$$\kappa(f) = \sum_{n=1}^{\infty} f(n)S(u_n), \quad f \in l^\infty,$$

where $(u_n)_{n\in\mathbf{N}}$ is a sequence in $P_1(G)$ which is topologically convergent to invariance and $\big(S(u_n)\big)_{n\in\mathbf{N}}$ is the same orthogonal sequence of projections in $\mathrm{VN}(G)$ as in [1]. Then κ and π have the required properties of the theorem.

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

In the following, we assume that G is non-metrizable. Assume at first that G is σ-compact. Let $(Q_\alpha)_{\alpha<\mu}$ and $(u_\alpha^j)_{(j,\alpha)\in J\times X}$ be the same as in Section 4. For each fixed $\alpha \in X$, consider the net $(u_\alpha^j)_{j\in J}$ in $P_1(G)$. Since $\|u_\alpha^j\| = 1$ for all $j \in J$, $(u_\alpha^j)_{j\in J}$ contains a $\sigma\big(\mathrm{VN}(G)^*, \mathrm{VN}(G)\big)$-convergent subnet $(u_\alpha^{j_\alpha})_{j_\alpha}$. Define $\kappa: l^\infty(X) \longrightarrow \mathrm{VN}(G)$ by

$$\kappa(f) = \sum_{\alpha<\mu} f(\alpha)Q_\alpha, \quad f \in l^\infty(X),$$

and $\pi: \mathrm{VN}(G) \longrightarrow l^\infty(X)$ by

$$\pi(T)(\alpha) = \lim_{j_\alpha}\langle T, u_\alpha^{j_\alpha}\rangle, \quad T \in \mathrm{VN}(G), \alpha \in X.$$

Clearly, κ is an isometric $*$-isomorphism (*i.e.*, κ is linear, multiplicative, $\kappa(\bar{f}) = \kappa(f)^*$ and hence $\|\kappa(f)\| = \|f\|$ for all $f \in l^\infty(X)$). π is linear, $\pi(I) = \mathbf{1}$ and $\pi(T) \geq \mathbf{0}$ if $T \geq \mathbf{0}$. If $T \in \mathrm{VN}(G)$ and $\alpha \in X$, $|\pi(T)(\alpha)| = \lim_{j_\alpha} |\langle T, u_\alpha^{j_\alpha}\rangle| \leq \|T\|$. Thus, $\|\pi\| = 1$. Also, from the properties of $(Q_\alpha)_\alpha$ and $(u_\alpha^j)_{j,\alpha}$, we see that $\pi \circ \kappa = \mathrm{id}_{l^\infty(X)}$. Therefore, π is onto.

To show (b), let $\phi \in \mathcal{F}(X)$. Then

$$1 = \langle \phi, \mathbf{1}\rangle = \big\langle \pi^*(\phi), I\big\rangle \leq \|\pi^*(\phi)\| = \|\phi\| = 1,$$

i.e., $\|\pi^*(\phi)\| = \langle \pi^*(\phi), I\rangle = 1$. If $T \in \mathrm{VN}(G)$ and $v \in A(G)$ with $v(e) = 1$, then

$$\begin{aligned}\lim_\alpha \pi(v\cdot T - T)(\alpha) &= \lim_\alpha \lim_{j_\alpha}\langle v\cdot T - T, u_\alpha^{j_\alpha}\rangle\\ &= \lim_\alpha \lim_{j_\alpha}\langle T, u_\alpha^{j_\alpha}v - u_\alpha^{j_\alpha}\rangle = 0,\end{aligned}$$

since $(u_\alpha^j)_{j,\alpha}$ is topologically convergent to invariance. By the definition of $\mathcal{F}(X)$,

$$\big\langle \pi^*(\phi), v\cdot T - T\big\rangle = \big\langle \phi, \pi(v\cdot T - T)\big\rangle = 0,$$

i.e., $\langle \pi^*(\phi), v\cdot T\rangle = \langle \pi^*(\phi), T\rangle$ for all $T \in \mathrm{VN}(G)$ and $v \in A(G)$ with $v(e) = 1$. We conclude that $\pi^*\big(\mathcal{F}(X)\big) \subseteq \mathrm{TIM}(\hat{G})$ and hence $\mathcal{F}(X) = \kappa^* \circ \pi^*\big(\mathcal{F}(X)\big) \subseteq \kappa^*\big(\mathrm{TIM}(\hat{G})\big)$.

In the general case (*i.e.*, G not necessarily σ-compact), let $G_\circ$ be a compactly generated open subgroup of G. Let $r: A(G) \longrightarrow A(G_\circ)$ be the restriction map and let $t: A(G_\circ) \longrightarrow A(G)$ be the extension map defined by $tv = \mathring{v}$, where $\mathring{v} = v$ on $G_\circ$ and 0 outside $G_\circ$. Then $r\circ t = \mathrm{id}_{A(G_\circ)}$, t is an isometry and $\|r\| \leq 1$ (see Eymard [5]). Therefore, r^* is isometric and t^* is onto. Granirer showed that $r^{**}\big(\mathrm{TIM}(\hat{G})\big) = \mathrm{TIM}(\widehat{G_\circ})$ and $t^{**}\big(\mathrm{TIM}(\widehat{G_\circ})\big) = \mathrm{TIM}(\hat{G})$. (see [7, pp. 118–119]). Note that now $G_\circ$ is also non-metrizable and $b(G_\circ) = b(G)$. We let $\kappa_\circ: l^\infty(X) \longrightarrow \mathrm{VN}(G_\circ)$ and $\pi_\circ: \mathrm{VN}(G_\circ) \longrightarrow l^\infty(X)$ be the mappings given in the previous paragraph. Define $\kappa = r^* \circ \kappa_\circ$ and $\pi = \pi_\circ \circ t^*$. Then κ and π satisfy the requirements. The proof is completed. ∎

REMARK 5.2. (i) The existence of κ and the injectivity of $l^\infty(X)$ (for the definition, see Section 2) guarantee the existence of a bounded linear mapping σ of $\mathrm{VN}(G)$ onto $l^\infty(X)$ with $\sigma \circ \kappa = \mathrm{id}_{l^\infty(X)}$. But, it is very difficult to see whether such σ is positive and

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

satisfies $\sigma^*\big(F(X)\big) \subseteq \mathrm{TIM}(\hat{G})$. Therefore, we have to explicitly construct the mapping π which possesses the desired properties.

(ii) It is worthwhile to point out that the inclusion $\kappa[l^\infty(X)] \subseteq \mathrm{UCB}(\hat{G})$ is actually true when G is non-metrizable. We need this fact later on. In fact, if G is σ-compact, then $\mathrm{supp}[\sum_{\alpha<\mu} f(\alpha)Q_\alpha] \subseteq N_1$ for all $f \in l^\infty(X)$, where N_1 is the same compact subgroup of G as in Lemma 4.1, and hence $\kappa[l^\infty(X)] \subseteq \mathrm{UCB}(\hat{G})$. Generally, let $G_\circ$ be a compactly generated open subgroup of G and let $r\colon A(G) \longrightarrow A(G_\circ)$ be the restriction map. Granirer showed that $r^*[\mathrm{UCB}(\widehat{G_\circ})] \subseteq \mathrm{UCB}(\hat{G})$ (see [8, p. 379]). From the proof of Theorem 5.1, now we also have $\kappa[l^\infty(X)] \subseteq \mathrm{UCB}(\hat{G})$.

Before we continue any further investigation on properties of the linear isometry κ, we first present several interesting consequences of Theorem 5.1.

COROLLARY 5.3. *Let G be a nondiscrete locally compact group. Then the quotient Banach spaces* $\mathrm{VN}(G)/F(\hat{G})$ *and* $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ *have* $l^\infty(X)$ *as a quotient.*

PROOF. Let π be the linear mapping of $\mathrm{VN}(G)$ onto $l^\infty(X)$ as in Theorem 5.1. From the proof of Theorem 5.1, we can see that $\pi[F_0(\hat{G})] \subseteq c_0(X)$. Hence, $\pi[F(\hat{G})] \subseteq c(X)$, since $F(\hat{G}) = \mathbf{C}I \oplus F_0(\hat{G})$ and $\pi(I) = \mathbf{1}$. Therefore, $\mathrm{VN}(G)/F(\hat{G})$ has $l^\infty(X)/c(X)$ as a quotient. Lemma 3.2 combined with the injectivity of $l^\infty(X)$ yields that the quotient Banach space $\mathrm{VN}(G)/F(\hat{G})$ has $l^\infty(X)$ as a quotient.

When G is metrizable, the fact that $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ has l^∞ as a quotient follows from Granirer [13, Corollary 7]. If G is non-metrizable, then $\kappa[l^\infty(X)] \subseteq \mathrm{UCB}(\hat{G})$ (by Remark 5.2(ii)). Thus $l^\infty(X) = \pi\circ\kappa[l^\infty(X)] \subseteq \pi[\mathrm{UCB}(\hat{G})]$, *i.e.*, $\pi[\mathrm{UCB}(\hat{G})] = l^\infty(X)$. So $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ has $l^\infty(X)/c(X)$ as a quotient. Consequently, the quotient Banach space $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ has $l^\infty(X)$ as a quotient. ∎

COROLLARY 5.4. *Let G be a non-metrizable locally compact group. Then the quotient Banach space* $\mathrm{UCB}(\hat{G})/C^*_\rho(G)$ *contains an isometric copy of* $l^\infty(X)$.

PROOF. We may assume that G is σ-compact.

Let $(N_\alpha)_{\alpha\le\mu}$, $(Q_\alpha)_{\alpha<\mu}$ and $(u^j_\alpha)_{j,\alpha}$ be the same as in Section 4. Let $\kappa\colon l^\infty(X) \longrightarrow \mathrm{VN}(G)$ be the linear isometry given by $\kappa(f) = \sum_{\alpha<\mu} f(\alpha)Q_\alpha$. By Remark 5.2(ii), $\kappa(f) \in \mathrm{UCB}(\hat{G})$ for all $f \in l^\infty(X)$. Define the linear mapping $L\colon l^\infty(X) \longrightarrow \mathrm{UCB}(\hat{G})/C^*_\rho(G)$ by $L(f) = \kappa(f) + C^*_\rho(G)$. Then $\|L(f)\| \le \|\kappa(f)\| = \|f\|$. On the other hand, for each $\beta < \mu$,

$$f(\beta) = \Big\langle \sum_{\alpha<\mu} f(\alpha)Q_\alpha, u^j_\beta \Big\rangle, \quad \text{for all } j.$$

According to Remark 4.2(b), we may assume that $\lambda(N_1) = 0$, where λ is the left Haar measure of G. If $\varphi \in L^1(G)$, for any fixed $\beta < \mu$,

$$|\langle \varphi, u^j_\beta\rangle| = \Big|\int_G \varphi(x) u^j_\beta(x^{-1})\,dx\Big| \le \int_{N_\beta U_j^{-1}} |\varphi(x)|\,dx.$$

Then $\lim_j |\langle \varphi, u^j_\beta\rangle| \le \lim_j \int_{N_\beta U_j^{-1}} |\varphi(x)|\,dx = 0$, since $\lim_j \lambda(N_\beta U_j^{-1}) = \lambda(N_\beta) \le \lambda(N_1) = 0$. Therefore,

$$|f(\beta)| = \lim_j \Big|\Big\langle \sum_{\alpha<\mu} f(\alpha)Q_\alpha + \varphi, u^j_\beta \Big\rangle\Big|$$

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

$$\begin{aligned} &\leq \Big\| \sum_{\alpha<\mu} f(\alpha)Q_\alpha + \varphi \Big\| \\ &= \|\kappa(f) + \varphi\|, \quad \text{for } \varphi \in L^1(G), \beta < \mu. \end{aligned}$$

Consequently, $\|f\| \leq \|\kappa(f)+\varphi\|$ for all $\varphi \in L^1(G)$, *i.e.*, $\|f\| \leq \|\kappa(f)+C^*_\rho(G)\| = \|L(f)\|$. It follows that $L: l^\infty(X) \longrightarrow \mathrm{UCB}(\hat{G})/C^*_\rho(G)$ is a linear isometry. ∎

COROLLARY 5.5. *Let G be a non-metrizable locally compact group. Then the quotient Banach space $\mathcal{M}(\hat{G})/C^*_\rho(G)$ contains an isometric copy of $s(X)$, where $s(X)$ is the subspace of $l^\infty(X)$ as defined in Section 3.*

In particular, $\mathcal{M}(\hat{G})/C^*_\rho(G)$ is not norm separable and contains an isometric copy of c.

PROOF. We may assume that G is σ-compact.

Let $(P_\alpha)_{\alpha<\mu}$ and $(Q_\alpha)_{\alpha<\mu}$ be the same as in Section 4. If $E \subseteq X$ is an interval and $f = 1_E$, by Lemma 4.4, $\sum_{\alpha<\mu} f(\alpha)Q_\alpha$ is of the form $P_\gamma - P_\beta$ or $I - P_\beta$ for some $0 \leq \beta < \gamma < \mu$. Then $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in \mathcal{M}(\hat{G})$ since $P_\alpha \in M(G)$ for each α. Hence, $\sum_{\alpha<\mu} f(\alpha)Q_\alpha \in \mathcal{M}(\hat{G})$ for all $f \in s(X)$ by the definition of $s(X)$. Let $K: s(X) \longrightarrow \mathcal{M}(\hat{G})/C^*_\rho(G)$ be the restriction to $s(X)$ of the linear isometry in Corollary 5.4. Then K is a linear isometry of $s(X)$ into $\mathcal{M}(\hat{G})/C^*_\rho(G)$. ∎

It is not hard to see that there exist $2^{\aleph_0}$ many infinite subsets I_γ of $\mathbf{N}$, $\gamma \in \Gamma$, $|\Gamma| = 2^{\aleph_0}$, such that $I_\gamma \cap I_{\gamma'}$ is finite if $\gamma \neq \gamma'$. This argument remains true for any uncountable initial ordinal μ if the generalized continuum hypothesis is assumed. More precisely, if μ is an initial ordinal with $|\mu| > \aleph_0$ and $X = \{\alpha ; \alpha < \mu\}$, there exist $2^{|X|}$ many subsets A_ω of X, $\omega \in \Omega$, $|\Omega| = 2^{|X|}$, such that $|A_\omega| = |X|$ and $|A_\omega \cap A_{\omega'}| < |X|$ if $\omega \neq \omega'$ (see [2, pp. 19, 288]). Now each A_ω and X are cofinal because $|A_\omega| = |X| = |\mu|$ and μ is an initial ordinal. Following an argument of Chou [1, p. 218], we can show that $\mathrm{TIM}(\hat{G})$ admits many extreme points by using the linear isometry κ in Theorem 5.1.

COROLLARY 5.6.. *Let G be a non-metrizable locally compact group. Then* $\mathrm{TIM}(\hat{G})$ *contains at least $2^{b(G)}$ many extreme points if the generalized continuum hypothesis is assumed.*

PROOF. We may assume that G is σ-compact. Let $(Q_\alpha)_{\alpha<\mu}$ and $(u^j_\alpha)_{j,\alpha}$ be the same as in Section 4. For each $\omega \in \Omega$, let $P_\omega = \sum_{\alpha\in A_\omega} Q_\alpha\big(= \kappa(1_{A_\omega})\big)$ and let $M_\omega = \{m \in \mathrm{TIM}(\hat{G}) ; m(P_\omega) = 1\}$.

It is easy to see that M_ω is w^*-compact and convex. M_ω is nonempty since each w^*-cluster point of $(u^j_\alpha)_{j\in J,\alpha\in A_\omega}$ belongs to M_ω. If $\omega \neq \omega'$, then $|A_\omega \cap A_{\omega'}| < |X|$ and hence, by the König's inequality, there is an $\alpha_0 < \mu$ such that $A_\omega \cap A_{\omega'} \subset [0, \alpha_0)$. Thus $\sum_{\alpha\in A_\omega\cap A_{\omega'}} Q_\alpha \leq \sum_{\alpha<\alpha_0} Q_\alpha = P_{\alpha_0}$ (by Lemma 4.4). For each $m \in \mathrm{TIM}(\hat{G})$,

$$0 \leq \Big\langle m, \sum_{\alpha\in A_\omega\cap A_{\omega'}} Q_\alpha \Big\rangle \leq \langle m, P_{\alpha_0}\rangle = \lambda_{\alpha_0}(e) = 0,$$

i.e., $\langle m, \sum_{\alpha\in A_\omega\cap A_{\omega'}} Q_\alpha\rangle = 0$. Therefore, $M_\omega \cap M_{\omega'} = \emptyset$ if $\omega \neq \omega'$.

By Krein-Milman theorem, each M_ω contains an extreme point. But extreme points of M_ω are also extreme in $\mathrm{TIM}(\hat{G})$. It follows that $\mathrm{TIM}(\hat{G})$ has at least $2^{|X|} = 2^{b(G)}$ many extreme points. ∎

REMARK 5.7. Chou showed the above corollary for metrizable nondiscrete locally compact groups without assuming the continuum hypothesis.

Now, we go back to the linear isometry κ. In order to establish certain isometric relations between quotient spaces of $\mathrm{VN}(G)$ (or $\mathrm{UCB}(\hat{G})$) and $l^\infty(X)$, a more precise quantitative understanding on κ is desired.

LEMMA 5.8. *Let G be a σ-compact non-metrizable locally compact group and let $\kappa\colon l^\infty(X) \longrightarrow \mathrm{VN}(G)$ be the same linear isometry as in Theorem 5.1. Then, for any $f \in l^\infty(X)$,*

$$(i)\ \ \|\kappa(f) + F_0(\hat{G})\| = \|\kappa(f) + F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| = \|f + c_0(X)\|\ ;$$
$$(ii)\ \ \|\kappa(f) + F(\hat{G})\| = \|\kappa(f) + F(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| = \|f + c(X)\|.$$

PROOF. Let $f \in l^\infty(X)$. If $h \in c_0(X)$, then $\kappa(h) \in F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ (by Lemma 4.5 and Remark 5.2(ii)). Since κ is an isometry,

$$\begin{aligned}\|f + h\| &= \|\kappa(f + h)\| = \|\kappa(f) + \kappa(h)\|\\ &\geq \|\kappa(f) + F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| \geq \|\kappa(f) + F_0(\hat{G})\|.\end{aligned}$$

Therefore, $\|f + c_0(X)\| \geq \|\kappa(f) + F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| \geq \|\kappa(f) + F_0(\hat{G})\|$.

Conversely, let $a = \lim_\alpha \sup |f(\alpha)|$. Then there exists a subnet $(\alpha_i)_i$ of $(\alpha)_{\alpha<\mu}$ such that $a = \lim_i |f(\alpha_i)|$. Let $(u^j_\alpha)_{j,\alpha}$ be the same net in $P_1(G)$ as in Section 4. Let $m \in \mathrm{VN}(G)^*$ be a w^*-cluster point of $(u^j_{\alpha_i})_{j,i}$. Then $m \in \mathrm{TIM}(\hat{G})$ (since $(u^j_{\alpha_i})_{j,i}$ is also topologically convergent to invariance by Lemma 4.3) and

$$\begin{aligned}|\langle m, \kappa(f)\rangle| &= \lim_{j,i} |\langle \kappa(f), u^j_{\alpha_i}\rangle|\\ &= \lim_{j,i} \Big|\Big\langle \sum_{\alpha<\mu} f(\alpha) Q_\alpha, u^j_{\alpha_i}\Big\rangle\Big|\\ &= \lim_i |f(\alpha_i)| = a.\end{aligned}$$

Thus, for any $T \in F_0(\hat{G})$,

$$\|\kappa(f) + T\| \geq |\langle m, \kappa(f) + T\rangle| = |\langle m, \kappa(f)\rangle| = a.$$

But, by the definition of a, for any $\epsilon > 0$, there exists an $\alpha_0 < \mu$ such that $|f(\alpha)| \leq a + \epsilon$ for all $\alpha_0 < \alpha < \mu$. Let $h = -f 1_{[0,,\alpha_0]}$. Then $h \in c_0(X)$ and $f + h = f 1_{(\alpha_0,\mu)}$. So $\|f + h\| \leq a + \epsilon$. Therefore,

$$\|\kappa(f) + T\| \geq a \geq \|f + h\| - \epsilon \geq \|f + c_0(X)\| - \epsilon.$$

Since $\epsilon > 0$ and $T \in F_0(\hat{G})$ are arbitrary, we get that $\|f + c_0(X)\| \leq \|\kappa(f) + F_0(\hat{G})\|$. Therefore, (i) holds.

Similarly, we have $\|f + c(X)\| \geq \|\kappa(f) + F(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| \geq \|\kappa(f) + F(\hat{G})\|$.

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

Let $T \in F(\hat{G})$. Then there exists a constant a such that $T \in aI + F_0(\hat{G})$. Notice that $\kappa(\mathbf{1}) = I$ (Lemma 4.4). According to the above proof, we have

$$\begin{aligned} \|f + c(X)\| &= \|f + C\mathbf{1} + c_0(X)\| \\ &\leq \|(f + a\mathbf{1}) + c_0(X)\| \\ &= \|\kappa(f + a\mathbf{1}) + F_0(\hat{G})\| \\ &= \|\kappa(f) + aI + F_0(\hat{G})\| \\ &\leq \|\kappa(f) + T\|. \end{aligned}$$

It follows that $\|f + c(X)\| \leq \|\kappa(f) + F(\hat{G})\|$. The proof is completed. ∎

Let $G_\circ$ be an open subgroup of G and let $r: A(G) \longrightarrow A(G_\circ)$ be the restriction map. Then r is onto and r^* is isometric (see Eymard [5]). Granirer showed that $r^*[\mathrm{UCB}(\widehat{G_\circ})] \subseteq \mathrm{UCB}(\hat{G})$ and $r^{**}[\mathrm{TIM}(\hat{G})] = \mathrm{TIM}(\widehat{G_\circ})$ (see [8, p. 379]). Therefore, $r^*\big(F_0(\widehat{G_\circ})\big) \subseteq F_0(\hat{G})$, $r^*[F_0(\widehat{G_\circ}) \cap \mathrm{UCB}(\widehat{G_\circ})] \subseteq F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})$, $r^*\big(F(\widehat{G_\circ})\big) \subseteq F(\hat{G})$, and $r^*[F(\widehat{G_\circ}) \cap \mathrm{UCB}(\widehat{G_\circ})] \subseteq F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$. Furthermore, we can show that r^* induces linear isometries on quotient spaces.

LEMMA 5.9. *(i) Let $T \in \mathrm{VN}(G_\circ)$. Then*
(a) $\|r^*(T) + F_0(\hat{G})\| = \|T + F_0(\widehat{G_\circ})\|$;
(b) $\|r^*(T) + F(\hat{G})\| = \|T + F(\widehat{G_\circ})\|$.
(ii) Let $T \in \mathrm{UCB}(\widehat{G_\circ})$. Then
(a) $\|r^*(T) + F_0(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| = \|T + F_0(\widehat{G_\circ}) \cap \mathrm{UCB}(\widehat{G_\circ})\|$;
(b) $\|r^*(T) + F(\hat{G}) \cap \mathrm{UCB}(\hat{G})\| = \|T + F(\widehat{G_\circ}) \cap \mathrm{UCB}(\widehat{G_\circ})\|$.

PROOF. We only give a proof of (i). (ii) can be proved analogously.

Let $T \in \mathrm{VN}(G_\circ)$. If $S \in F_0(\widehat{G_\circ})$, then $r^*(S) \in F_0(\hat{G})$. Thus,

$$\begin{aligned} \|T + S\| &= \|r^*(T + S)\| = \|r^*(T) + r^*(S)\| \\ &\geq \|r^*(T) + F_0(\hat{G})\|. \end{aligned}$$

Therefore, $\|T + F_0(\widehat{G_\circ})\| \geq \|r^*(T) + F_0(\hat{G})\|$.

Conversely, let (u_i) be a net in $P_1(G)$ which is topologically convergent to invariance. Let $T_1 \in F_0(\hat{G})$. By Chou [1, Theorem 4.4], $\lim_i u_i \cdot T_1 = \mathbf{0}$ in norm. Then we have

$$\begin{aligned} \|r^*(T) + T_1\| &\geq \limsup_i \|u_i \cdot r^*(T) + u_i \cdot T_1\| \\ &= \limsup_i \|u_i \cdot r^*(T)\| \\ &= \limsup_i \|r^*[(ru_i) \cdot T]\| \\ &= \limsup_i \|(ru_i) \cdot T\|. \end{aligned}$$

For each i, $(ru_i) \cdot T - T \in F_0(\widehat{G_\circ})$. So,

$$\begin{aligned} \|T + F_0(\widehat{G_\circ})\| &\leq \|T + (ru_i) \cdot T - T\| \\ &= \|(ru_i) \cdot T\| \quad \text{for all } i. \end{aligned}$$

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

Therefore,

$$\|T + F_0(\widehat{G_\circ})\| \leq \limsup_i \|(ru_i)\cdot T\| \leq \|r^*(T) + T_1\|$$

for all $T_1 \in F_0(\hat{G})$. Consequently,

$$\|T + F_0(\widehat{G_\circ})\| \leq \|r^*(T) + F_0(\hat{G})\|.$$

Therefore, $\|r^*(T) + F_0(\hat{G})\| = \|T + F_0(\widehat{G_\circ})\|$, *i.e.*, (a) holds.

Similarly, we have $\|T + F(\widehat{G_\circ})\| \geq \|r^*(T) + F(\hat{G})\|$.

Let $T_2 \in F(\hat{G})$. Then $T_2 - aI \in F_0(\hat{G})$ for some constant a. Notice that $r^*(I_\circ) = I$, where $I_\circ$ is the identity in $\mathrm{VN}(G_\circ)$. By the above proved equality, we have

$$\begin{aligned} \|r^*(T) + T_2\| &= \|r^*(T + aI_\circ) + (T_2 - aI)\| \\ &\geq \|r^*(T + aI_\circ) + F_0(\hat{G})\| \\ &= \|(T + aI_\circ) + F_0(\widehat{G_\circ})\| \\ &\geq \|T + F(\widehat{G_\circ})\|. \end{aligned}$$

It follows that $\|r^*(T) + F(\hat{G})\| \geq \|T + F(\widehat{G_\circ})\|$. This concludes the proof. ∎

We are now ready to give one of the main results in this section.

THEOREM 5.10. *Let G be a non-metrizable locally compact group. Then*

(a) the quotient Banach spaces $\mathrm{VN}(G)/F_0(\hat{G})$ *and* $\mathrm{UCB}(\hat{G})/F_0(\hat{G})\cap\mathrm{UCB}(\hat{G})$ *contain an isometric copy of $l^\infty(X)/c_0(X)$;*

(b) the quotient Banach spaces $\mathrm{VN}(G)/F(\hat{G})$ *and* $\mathrm{UCB}(\hat{G})/F(\hat{G})\cap\mathrm{UCB}(\hat{G})$ *contain an isometric copy of $l^\infty(X)/c(X)$.*

PROOF. If G is σ-compact, Lemma 5.8 implies (a) and (b).

Generally, let $G_\circ$ be a compactly generated open subgroup of G. Then $G_\circ$ is also non-metrizable and $b(G_\circ) = b(G)$. Now (a) and (b) follow from Lemmas 5.8 and 5.9. ∎

6**. Isomorphism and homomorphism results and some remarks.** Let G be a nondiscrete locally compact group. Let μ be the initial ordinal with $|\mu| = b(G)$ and let $X = \{\alpha\ ;\ \alpha \text{ is an ordinal and } \alpha < \mu\}$. Combining the embedding results in Theorem 5.10 and Lemma 3.2, we have

THEOREM 6.1. *Let G be a non-metrizable locally compact group. Then*

(a) the quotient Banach spaces $\mathrm{VN}(G)/F_0(\hat{G})$ *and* $\mathrm{UCB}(\hat{G})/F_0(\hat{G})\cap\mathrm{UCB}(\hat{G})$ *contain an isometric copy of $l^\infty(X)$;*

(b) the quotient Banach spaces $\mathrm{VN}(G)/F(\hat{G})$ *and* $\mathrm{UCB}(\hat{G})/F(\hat{G})\cap\mathrm{UCB}(\hat{G})$ *contain an isomorphic copy of $l^\infty(X)$.*

REMARK 6.2. Among other results of [13] on $PM_p(G)$, the dual Banach space of the Figà-Talamanca-Gaudry-Herz algebra $A_p(G)$ of G ($1 < p < \infty$ and $A_2(G) = A(G)$), Granirer [13, Corollary 7] implies that $\mathrm{UCB}(\hat{G})/F(\hat{G})\cap\mathrm{UCB}(\hat{G})$ has l^∞ as a quotient if G is metrizable nondiscrete. A result of Chou [1, Theorem 3.3] yields that $\mathrm{VN}(G)/F(\hat{G})$ has

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

l^∞ as a quotient when G is metrizable nondiscrete. Here, in fact, Theorem 6.1 generalizes their results to non-metrizable groups and the conclusions are also strictly stronger.

We know that

$$C^*_\rho(G) \subseteq \mathcal{M}(\hat{G}) \subseteq W(\hat{G}) \subseteq F(\hat{G}), \text{ and } \mathcal{M}(\hat{G}) \subseteq \mathrm{UCB}(\hat{G}).$$

These inclusions and Corollary 5.3 lead to the following homomorphism results.

THEOREM 6.3. *Let G be a nondiscrete locally compact group. Then*
(i) the quotient Banach spaces $\mathrm{VN}(G)/W(\hat{G})$, $\mathrm{VN}(G)/\mathcal{M}(\hat{G})$, and $\mathrm{VN}(G)/C^*_\rho(G)$ have $l^\infty(X)$ as a quotient;
(ii) the quotient Banach spaces $\mathrm{UCB}(\hat{G})/W(\hat{G}) \cap \mathrm{UCB}(\hat{G})$, $\mathrm{UCB}(\hat{G})/\mathcal{M}(\hat{G})$, and $\mathrm{UCB}(\hat{G})/C^*_\rho(G)$ have $l^\infty(X)$ as a quotient.

COROLLARY 6.4. *Let G be an amenable nondiscrete locally compact group. Then the quotient Banach space $\mathrm{UCB}(\hat{G})/W(\hat{G})$ has $l^\infty(X)$ as a quotient.*

PROOF. $W(\hat{G}) \subseteq \mathrm{UCB}(\hat{G})$ when G is amenable (see [8, Proposition 1]). ■

Recall that, for a Banach space Y, $D(Y)$ denotes the density character of Y, *i.e.*, the smallest cardinality such that there exists a norm dense subset of Y having that cardinality. It is known that $D\big(l^\infty(X)\big) = 2^{|X|}$ for any infinite set X. Also, if Y has Z as a quotient, then $D(Y) \geq D(Z)$. Therefore, by Corollary 5.3, we have the following.

COROLLARY 6.5. *Let G be a nondiscrete locally compact group. Then*
(i) $D[\mathrm{VN}(G)/F(\hat{G})] \geq 2^{b(G)}$;
(ii) $D[\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})] \geq 2^{b(G)}$.

Let $u \in P_1(G)$ and let

$$u^\perp = \{T \in \mathrm{VN}(G)\ ;\ u \cdot T = \mathbf{0}\}.$$

If $T \in u^\perp$ and $m \in \mathrm{TIM}(\hat{G})$, then $m(T) = m(u \cdot T) = 0$. Hence, $u^\perp \subseteq F_0(\hat{G})$. The format of the following corollary is due to Granirer.

COROLLARY 6.6. *Let G be a nondiscrete locally compact group. Let $u \in P_1(G)$ and let Y be a subspace of $\mathrm{VN}(G)$ such that $\mathrm{UCB}(\hat{G})$ is contained in the norm closure of $W(\hat{G}) + Y + u^\perp$. Then $D(Y) \geq 2^{b(G)}$.*

PROOF. Let Z be the norm closure of $F(\hat{G}) + Y$ in $\mathrm{VN}(G)$. Then $\mathrm{UCB}(\hat{G}) \subseteq Z$ since $W(\hat{G}) + u^\perp \subseteq F(\hat{G})$.

Let $Y_\circ$ be a dense subset of Y such that $|Y_\circ| = D(Y)$ and let $\{u_i\}_{i\in I} \subseteq \mathrm{UCB}(\hat{G})$ be such that $\{u_i + F(\hat{G}) \cap \mathrm{UCB}(\hat{G})\}_{i\in I}$ is dense in $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $|I| = D[\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})]$. For each $i \in I$, since $u_i \in Z$, there exist sequences $(f_i^n)_n$ in $F(\hat{G})$ and $(y_i^n)_n$ in $Y_\circ$ such that

$$\|u_i - (f_i^n + y_i^n)\| < \frac{1}{n}, \quad n \in \mathbf{N}.$$

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

If $i, j \in I$ and $i \neq j$, then $u_i - u_j \notin F(\hat{G})$ and hence

$$\|(u_i - u_j) - (f_i^n - f_j^n)\| \geq \|(u_i - u_j) + F(\hat{G})\| > 0, \quad \text{for all } n \in \mathbf{N}.$$

Therefore, the mapping from I into $Y_\circ^{\aleph_0}$, given by $i \longmapsto (y_i^n)_n$ is one-to-one. So $|I| \leq |Y_\circ^{\aleph_0}| = D(Y)^{\aleph_0}$. But $|I| \geq 2^{b(G)}$ (Corollary 6.5) and $D(Y) > \aleph_0$ ([8, Theorem 12]). Consequently, $D(Y) = D(Y)^{\aleph_0} \geq 2^{b(G)}$. ∎

REMARK 6.7. (i) Since $l^\infty(X)$ contains an isometric copy of l^∞, Corollary 5.3, 5.4, Theorem 6.1, 6.3, and Corollary 6.4 remain true if $l^\infty(X)$ is replaced by l^∞.

(ii) We showed in [17] that both $\mathrm{UCB}(\hat{G})/F(\hat{G}) \cap \mathrm{UCB}(\hat{G})$ and $\mathrm{VN}(G)/F(\hat{G})$ have the density character greater than $b(G)$ if G is nondiscrete (see [17, Corollary 6.2]). Corollary 6.5 improves the estimate on the density characters of these two quotient spaces.

(iii) Under the same assumptions of Corollary 6.6, Granirer showed that Y is not norm separable if G is nondiscrete (see [8, Theorem 12]). We improved this in [18, Theorem 5.4.3]: $D(Y) > b(G)$ if G is nondiscrete. The conclusion is strengthened further by Corollary 6.6.

(iv) The cardinality estimate in Corollary 6.5 and 6.6 cannot be improved since if G is nondiscrete and if $d(G) \leq b(G)$ (*e.g.*, if G is nondiscrete and σ-compact) then $\mathrm{VN}(G)$ is isometric to a subspace of $l^\infty(X)$, where $d(G)$ is the smallest cardinality of a covering of G by compact sets.

Finally, we want to extend the results obtained so far to spaces of operators in $\mathrm{VN}(G)$ with small support. First, we need the following preparations.

DEFINITION 6.8. Let $\aleph > 0$ be a cardinal. A nonempty subset B of G is called a $G_\aleph$-set if B is an intersection of $\aleph$ many open subsets of G.

If Y is a closed subspace of $\mathrm{VN}(G)$ and E is a closed subset of G, we denote by Y_E the space of all operators in Y with support contained in E.

Let G be a σ-compact non-metrizable locally compact group and let $(N_\alpha)_{\alpha \leq \mu}$ and $(Q_\alpha)_{\alpha<\mu}$ be the same nets as in Section 4. Let ν be an initial ordinal with $\nu < \mu$. Then $\nu + \alpha < \mu$ for all $\alpha < \mu$ and $\nu + \alpha = \nu + \beta$ if and only if $\alpha = \beta$ (see [26]). For $\alpha < \mu$, let $Q'_\alpha = Q_{\nu+\alpha}$. Then $(Q'_\alpha)_{\alpha<\mu}$ is also an orthogonal net of projections in $\mathrm{VN}(G)$ with $\operatorname{supp} Q'_\alpha \subseteq N_\nu$ for all $\alpha < \mu$. We point out that Lemma 4.4 and 4.5 remain true if $(Q_\alpha)_{\alpha<\mu}$ is replaced by $(Q'_\alpha)_{\alpha<\mu}$ and parts (iii) and (iv) of Lemma 4.4 are replaced by the following (iii)′ and (iv)′, respectively:

(iii)′ $\sum_{\beta<\alpha} Q'_\beta = P_{\nu+\alpha} - P_\nu$, for all $0 < \alpha < \mu$,

(iv)′ $\sum_{\alpha<\mu} Q'_\alpha = I - P_\nu$.

Let $E \subseteq G$ be a closed set which contains a $G_\aleph$-set B with $\aleph < b(G)$ and $e \in B$. Since $b(G) > \aleph_0$, we may assume that $\aleph$ is infinite. If ν is the initial ordinal with $|\nu| = \aleph$, then, from the proof of Lemma 4.1 (see [17]), we see that the net $(N_\alpha)_{\alpha\leq\mu}$ can be chosen such that $N_\nu \subseteq B \subseteq E$. Therefore, $\operatorname{supp}[\sum_{\alpha<\mu} f(\alpha)Q'_\alpha] \subseteq E$, *i.e.*, $\sum_{\alpha<\mu} f(\alpha)Q'_\alpha \in \mathrm{UCB}(\hat{G})_E$

for all $f \in l^\infty(X)$. If we define

$$\kappa'(f) = \sum_{\alpha<\mu} f(\alpha)Q'_\alpha, \quad f \in l^\infty(X),$$
$$\pi'(T)(\alpha) = \pi(T)(\nu+\alpha), \quad T \in \mathrm{VN}(G), \alpha \in X,$$

then κ' is also a linear isometry of $l^\infty(X)$ into $\mathrm{VN}(G)$ and π' is a bounded linear mapping of $\mathrm{VN}(G)$ onto $l^\infty(X)$. Also, notice that $P_\nu \in F_0(\hat{G})$. Examining the proofs of the previous results on quotient spaces, it is seen that all the subspaces Y of $\mathrm{VN}(G)$ there (including $C^*_\rho(G)$, $\mathcal{M}(\hat{G})$, $W(\hat{G})$, $F_0(\hat{G})$, $F(\hat{G})$, $\mathrm{UCB}(\hat{G})$, and $\mathrm{VN}(G)$) can be replaced by Y_E if G is a non-metrizable locally compact group.

Note that if G is metrizable, then any $G_\aleph$-set ($\aleph < b(G)$) is open in G and hence E contains a $G_\aleph$-set if and only if $\mathrm{int}(E) \neq \emptyset$, where $\mathrm{int}(E)$ denotes the interior of E. A particular case of Granirer [13, Corollary 7] implies that $\mathrm{UCB}(\hat{G})_E / [F(\hat{G})_E \cap \mathrm{UCB}(\hat{G})_E]$ and $\mathrm{VN}(G)_E / F(\hat{G})_E$ have l^∞ as a quotient if G is metrizable nondiscrete and $e \in \mathrm{int}(E)$, *i.e.*, in this case, Corollary 5.3 also holds if $\mathrm{VN}(G)$, $\mathrm{UCB}(\hat{G})$, and $F(\hat{G})$ are replaced by $\mathrm{VN}(G)_E$, $\mathrm{UCB}(\hat{G})_E$, and $F(\hat{G})_E$, respectively.

As a consequence of the above discussion on non-metrizable groups, combining with Granirer's result, we conclude the following.

THEOREM 6.9. *Let E be a closed subset of G which contains a $G_\aleph$-set B with $\aleph < b(G)$ and $e \in B$. Then Theorem 5.10, 6.1, 6.3, Corollary 5.3, 5.4, 5.5, 6.4, and 6.5 remain true if all the subspaces Y of* $\mathrm{VN}(G)$ *there are replaced by Y_E.*

For any fixed $x \in G$, let L_x be the left translation on $A(G)$ by x (*i.e.*, $u \longmapsto_x u$, $u \in A(G)$). Then L^*_x is a linear isometry of $\mathrm{VN}(G)$ onto itself. It can be shown that $L^*_x(Y_E) = Y_{xE}$, where $Y = C^*_\rho(G)$, $\mathcal{M}(\hat{G})$, $W(\hat{G})$, or $\mathrm{UCB}(\hat{G})$. Therefore, for these spaces, the restriction $e \in B$ in the above theorem can be released.

COROLLARY 6.10. *Let E be a closed subset of G containing a $G_\aleph$-set in G with $\aleph < b(G)$. Then Theorem 6.3, Corollary 5.4, 5.5, and 6.4 are true if all the subspaces Y of* $\mathrm{VN}(G)$ *there are replaced by Y_E.*

REMARK 6.11. Granirer in [12] and [13] investigated operators in $PM_p(G)$ ($1 < p < \infty$) with thin support. In particular, [13, Corollary 6 and 7] imply that $\mathrm{VN}(G)_E / F(\hat{G})_E$ and $\mathrm{UCB}(\hat{G})_E / [F(\hat{G})_E \cap \mathrm{UCB}(\hat{G})_E]$ have l^∞ as a quotient if E is first countable at e and one of the following two conditions is satisfied:

(1) $\mathbf{R}$ (or $\mathbf{T}$) is a closed subgroup of G, $S \subset \mathbf{R}$ (or $\mathbf{T}$) is a symmetric set such that $e \in aSb \subseteq E$ for some $a, b \in G$;

(2) $e \in \mathrm{int}_{aHb}(E)$ for some $a, b \in G$ and some nondiscrete subgroup H of G.

Notice that if G is non-metrizable and E is a set as in Theorem 6.9, then E is not first countable at e but it satisfies (2). In fact, let ν be the initial ordinal with $|\nu| = \aleph$ and $G_\circ$ a compactly generated open subgroup of G. Then a non-metrizable subgroup N_ν of $G_\circ$ (as in Lemma 4.1) can be chosen such that $e \in N_\nu \subseteq B \subseteq E$. Therefore, Theorem 6.9 extends Granirer's result to non-metrizable E with l^∞ replaced by $l^\infty(X)$ and condition (2) by $e \in B \subseteq E$ for some $G_\aleph$-set B with $\aleph < b(G)$.

https://doi.org/10.4153/CJM-1997-055-5 Published online by Cambridge University Press

REFERENCES

1. C. Chou, *Topological invariant means on the von Neumann algebra* VN(G). Trans. Amer. Math. Soc. **273**(1982), 207–229.
2. W. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*. Springer-Verlag, New York-Heidelberg-Berlin, 1974.
3. M. M. Day, *Amenable semigroups*. Illinois J. Math. **1**(1957), 509–544.
4. C. Dunkl and D. Ramirez, *Weakly almost periodic functionals on the Fourier algebra*. Trans. Amer. Math. Soc. **185**(1973), 501–514.
5. P. Eymard, *L'algèbre de Fourier d'un groupe locallement compact*. Bull. Soc. Math. France **92**(1964), 181–236.
6. L. Gillman and M. Jerison, *Rings Of Continuous Functions.* Van Nostrand, Princeton, 1960.
7. E. E. Granirer, *Properties of the set of topological invariant means on P. Eymard's* W^**-algebra* VN(G). Indag. Math. **36**(1974), 116–121.
8. ———, *Weakly almost periodic and uniformly continuous functionals on the Fourier algebra of any locally compact group*. Trans. Amer. Math. Soc. **189**(1974), 371–382.
9. ———, *Density theorems for some linear subspaces and some* C^**-subalgebras of* VN(G). Istituto Nazionale di Alta Mathematica, Symposia Mathematica **22**(1977), 61–70.
10. ———, *Geometric and topological properties of certain* w^* *compact convex subsets of double duals of Banach spaces, which arise from the study of invariant means*. Illinois J. Math. **30**(1986), 148–174.
11. ———, *On some spaces of linear functionals on the algebras* $A_p(G)$ *for locally compact groups*. Colloq. Math. **52**(1987), 119–132.
12. ———, *On convolution operators with small support which are far from being convolution by a bounded measure*. Colloq. Math. **67**(1994), 33–60.
13. ———, *Day points for quotients of the Fourier algebra* $A(G)$*, extreme nonergodicity of their duals and extreme non Arens regularity*. Illinois J. Math. **40**(1996), 402–419.
14. F. P. Greenleaf, *Invariant Means On Topological Groups*. Van Nostrand, New York, 1969.
15. C. Herz, *Harmonic synthesis for subgroups*. Annales de l'Institut Fourier (Grenoble) **23**(1973), 91–123.
16. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I*. Springer-Verlag, New York-Heidelberg-Berlin, 1979.
17. Z. Hu, *On the set of topologically invariant means on the von Neumann algebra* VN(G). Illinois J. Math. **39**(1995), 463–490.
18. ———, *Locally compact groups and invariant means on their von Neumann algebras*. Ph.D. dissertation.
19. A. T. Lau, *Uniformly continuous functionals on the Fourier algebra of any locally compact group*. Trans. Amer. Math. Soc. **251**(1979), 39–59.
20. A. T. Lau and V. Losert, *The* C^**-algebra generated by operators with compact support on a locally compact group*. J. Funct. Anal. **112**(1993), 1–30.
21. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces, Vol. I*. Springer, 1977.
22. G. G. Lorentz, *A contribution to the theory of divergent series*. Acta Math. **80**(1948), 167–190.
23. A. L. T. Paterson, *Amenability*. Math. Surveys Monographs **29**, Amer. Math. Soc., Providence, Rhode Island, 1988.
24. J. P. Pier, *Amenable Locally Compact Groups*. John Wiley and Sons, New York, 1984.
25. P. E. Renaud, *Invariant means on a class of von Neumann algebras*. Trans. Amer. Math. Soc. **170**(1972), 285–291.
26. J. E. Rubin, *Set Theory For The Mathematician*. Holden-Day, San Francisco-Cambridge-London-Amsterdam, 1967.
27. S. Sakai, C^**-Algebras And* W^**-Algebras*. Springer-Verlag, Berlin-Heidelberg-New York, 1971.
28. M. Takesaki, *Theory Of Operator Algebras I*. Springer-Verlag, New York-Heidelberg-Berlin, 1979.

Department of Mathematics and Statistics
University of Windsor
Windsor, Ontario
Canada