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J. B. Clark, Esq., M.A., F.R.S.E., President, in the Chair.

On the Second Solutions of Lamé's Equation

$$\frac{d^2U}{du^2} = U\left\{n(n+1)pu + B\right\}.$$

By Lawrence Crawford, M.A., B.Sc.

1. I consider here the second solutions corresponding to the solutions of the above equation (when n is an integer) in finite terms for special values of B. If U_n be such a solution, and F_n the corresponding second solution, we know that

$$F_n = (2n+1)U_n \int_0^u \frac{du}{(U_n)^2}.$$

2. U_n may be of one of four types, n being even or odd. Consider first the case of n even, $2m$ say, then the first type is

$$U_n = (pu - a_1)(pu - a_2) \; . \; . \; . \; . \; (pu - a_m),$$

where all the a's are different and no one coincides with an e, as I have proved in a former paper. *

The F_n corresponding to this is then

$$(2n+1)U_n \int_0^u \frac{du}{(pu - a_1)^2 \; . \; . \; . \; . \; (pu - a_m)^2};$$

proceed to the consideration of this integral.

Let $\dfrac{1}{(pu - a_1)^2(pu - a_2)^2 \, . \, . \, (pu - a_m)^2} = \sum_{r=1}^{r=m}\left(\dfrac{A_r}{pu - a_r} + \dfrac{A'_r}{(pu - a_r)^2}\right)$,

then $$A'_r = \frac{1}{(a_r - a_1)^2(a_r - a_2)^2 \, . \, . \, . \, (a_r - a_m)^2}$$

and

$$A_r = \left[\frac{1}{p'u}\frac{d}{du}\left\{\frac{1}{(pu - a_1)^2 \, . \, . \, (pu - a_{r-1})^2(pu - a_{r+1})^2 \, . \, . \, (pu - a_m)^2}\right\}\right]_{pu = a_r}$$

$$= -\frac{2}{(a_r - a_1)^2 \, . \, . \, (a_r - a_m)^2}\left\{\frac{1}{a_r - a_1} + \frac{1}{a_r - a_2} + \, . \, . + \frac{1}{a_r - a_m}\right\}.$$

* "On the Factors of the Solutions in Finite Terms of Lamé's Equation," *Quarterly Journal of Pure and Applied Mathematics*, No. 114, 1897.

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By differentiation of $\frac{p'u}{pu-a}$, it is easy to prove that

$$\int\frac{du}{(pu-a)^2}=-\frac{1}{4a^3-g_2a-g_3}\cdot\frac{p'u}{pu-a}$$

$$+\int\frac{2(pu-a)du}{4a^3-g_2a-g_3}-\frac{6a^2-\frac{1}{2}g_2}{4a^3-g_2a-g_3}\int\frac{du}{pu-a},$$

$\therefore$ we find

$$\int_0^u\frac{du}{U_n{}^2}=\sum_{r=1}^{r=m}\left[-\frac{A'_r}{4a_r{}^3-g_2a_r-g_3}\cdot\frac{p'u}{pu-a_r}-\frac{2A'_r(\zeta u+a_ru)}{4a_r{}^3-g_2a_r-g_3}\right.$$

$$\left.+\left\{A_r-\frac{A'_r(6a_r{}^2-\frac{1}{2}g_2)}{4a_r{}^3-g_2a_r-g_3}\right\}\int_0^u\frac{du}{pu-a_r}\right].$$

I proceed now to prove that $A_r-\frac{A'_r(6a_r{}^2-\frac{1}{2}g_2)}{4a_r{}^3-g_2a_r-g_3}=0$,

noting that $A_r=\left[\frac{1}{p'u}\cdot\frac{d}{du}\left(\frac{\overline{pu-a_r}}{U_n{}^2}\right)\right]_{pu=a_r}$.

In the differential equation for U_n, put $U_n=R(pu-a_r)$, then

$$\frac{d^2U_n}{du^2}=(pu-a_r)\frac{d^2R}{du^2}+2\frac{dR}{du}p'u+Rp''u=\Big(n(n+1)pu+B\Big)(pu-a_r)R$$

$\therefore$ when $pu=a_r$, as R and therefore $\frac{d^2R}{du^2}$ is not then infinite,

$$\left[2\frac{dR}{du}p'u+Rp''u\right]_{pu=a_r}=0.$$

But

$$A_r=\left[\frac{1}{p'u}\,\frac{d}{du}\left(\frac{1}{R^2}\right)\right]_{pu=a_r}=\left[-\frac{2\frac{dR}{du}}{R^3p'u}\right]_{pu=a_r},\qquad A'_r=\left[\frac{1}{R^2}\right]_{pu=a_r},$$

$\left[2\frac{dR}{du}p'u+Rp''u\right]_{pu=a_r}=0$, and $\left[R^3p'^2u\right]_{pu=a_r}$ is not equal to 0,

$$\therefore\quad\left[\frac{2\frac{dR}{du}}{R^3p'u}+\frac{p''u}{R^2p'^2u}\right]_{pu=a_r}=0,$$

$$\textit{i.e.}\quad\left[A'_r\frac{p''u}{p'^2u}-A_r\right]_{pu=a_r}=0,$$

$$\textit{i.e.}\quad A_r-A'_r\cdot\frac{6a_r{}^2-\frac{1}{2}g_2}{4a_r{}^3-g_2a_r-g_3}=0$$

$\therefore$ all such terms as $\displaystyle\int_0^u \frac{du}{pu-a_r}$ do not appear in $F_n(u)$,

and
$$\int_0^u \frac{du}{U_n^2} = -\sum_{r=1}^{r=m} \frac{A'_r}{4a_r^3 - g_2 a_r - g_3}\left(2\zeta u + 2a_r u + \frac{p'u}{pu-a_r}\right)$$
$$= Cu + D\zeta u - p'u \,.\, \Sigma \frac{A'_r}{(4a_r^3 - g_2 a_r - g_3)(pu-a_r)}$$
$$= Cu + D\zeta u + \frac{p'u f(pu)}{U_n}$$

where C, D are constants, $f(pu)$ is an algebraic integral function of pu, the highest power involved being $p^{m-1}u$, and D is twice the coefficient of $p^{m-1}u$ in $f(pu)$,

$$\therefore \quad F_n = (2n+1)\{p'uf(pu) + U_n(Cu + D\zeta u)\}.$$

3. I shall work out now the second solution when U_n is of one of the types for n odd, having an irrational factor $\sqrt{pu-e}$. Then if $n = 2m+1$, $U_n = \sqrt{pu-e}(pu-a_1)(pu-a_2)\ \dots\ (pu-a_m)$, where all the a's are real and different and no one coincides with an e, as I have proved in the former paper already referred to.

Then
$$F_n = (2n+1)U_n\int_0^u \frac{du}{(pu-e)(pu-a_1)^2(pu-a_2)^2 \dots (pu-a_m)^2},$$

and
$$\frac{1}{(pu-e)(pu-a_1)^2 \dots (pu-a_m)^2} = \frac{C}{pu-e} + \sum_{r=0}^{r=m} \frac{A_r}{pu-a_r} + \sum_{r=0}^{r=m} \frac{A'_r}{(pu-a_r)^2},$$

where
$$C = \frac{1}{(e-a_1)^2(e-a_2)^2 \dots (e-a_m)^2}, \quad A'_r = \frac{1}{(a_r-e)(a_r-a_1)^2 \dots (a_r-a_m)^2},$$

and
$$A_r = \left[\frac{1}{p'u}\frac{d}{du}\left\{\frac{\overline{pu-a_r}^{\,2}}{U_n^2}\right\}\right]_{pu=a_r}.$$

By differentiating $\dfrac{p'u}{pu-e}$, it is found that

$$\int \frac{6e^2 - \frac{1}{2}g_2}{pu-e}du = 2\int (pu-e)du - \frac{p'u}{pu-e},$$

and with the result already quoted for $\int \frac{du}{(pu-a_r)^2}$, we have

$$\int_0^u \frac{du}{U_n^2} = -\frac{C}{6e^2-\frac{1}{2}g_2}\left\{2\zeta u + 2eu + \frac{p'u}{pu-e}\right\}$$

$$+\sum_{r=1}^{r=m}\left[-\frac{A_r'}{4a_r^3-g_2a_r-g_3}\cdot\frac{p'u}{pu-a_r}-\frac{2A'_r(\zeta u+a_ru)}{4a_r^3-g_2a_r-g_3}\right.$$

$$\left.+\int_0^u \frac{du}{pu-a_r}\left\{A_r-\frac{A'_r(6a_r^2-\frac{1}{2}g_2)}{4a_r^3-g_2a_r-g_3}\right\}\right]$$

Just as in the previous case, it follows that $A_r - \frac{A'_r(6a_r^2-\frac{1}{2}g_2)}{4a_r^3-g_2a_r-g_3}=0$

by the substitution in the differential equation of $R(pu-a_r)$ for U_n, hence

$$\int_0^u \frac{du}{U_n^2} = -\frac{C}{6e^2-\frac{1}{2}g_2}\left\{2(\zeta u+eu)+\frac{p'u}{pu-e}\right\}$$

$$-\sum_{r=1}^{r=m}\frac{A_r'}{4a_r^3-g_{2r}-g_3}\cdot\frac{p'u}{pu-a_r}-2\Sigma\frac{A_r'}{4a_r^3-g_2a_r-g_3}(\zeta u+a_ru)$$

$$=C'u+D\zeta u+\frac{p'uf(pu)}{(pu-e)(pu-a_1)\,.\,.\,(pu-a_m)},$$

where C', D are constants, $f(pu)$ an algebraic integral function of pu, the highest power involved being p^mu,

$$\therefore \qquad F_n=(2n+1)\left\{\frac{p'uf(pu)}{\sqrt{pu-e}}+(C'u+D\zeta u)U_n\right\}.$$

4. Similar work may be done for all cases, and the general form is $F_n=(2n+1)\left\{\frac{p'uf(pu)}{g(pu)}+(Cu+D\zeta u)U_n\right\}$, where $f(pu)$ is an algebraic integral function of pu, the highest power involved being pu to the power, when n is even, $n/2$ or $(n-2)/2$, according as U_n has no irrational factor or one, and when n is odd, $(n-1)/2$ or $(n-3)/2$, according as U_n has an irrational factor $\sqrt{pu-e}$ or the factors $\sqrt{(pu-e_1)(pu-e_2)(pu-e_3)}$, $g(pu)$ is the irrational factor, if any, in U_n, and C, D are constants, functions of the roots of the equation $U_n=0$, regarded as an equation in pu.

5. The forms for the second solutions are found in Halphen, *Fonctions Elliptiques*, Vol. II., pp. 483–5, but it is interesting to see that they can be worked out in this way by direct integration.